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Convergence Follows From Cesàro Summability in the Case of Slowly Decreasing or Slowly Oscillating Double Sequences in Certain Senses

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Abstract. Let $(u_{\mu\nu})$ be a double sequence of real or complex numbers which is $(C, 1, 1)$ summable to a finite limit. We obtain some Tauberian conditions of slow decreasing or oscillating types in terms of the generator sequences in certain senses under which P-convergence of a double sequence $(u_{\mu\nu})$ follows from its $(C, 1, 1)$ summability. We give Tauberian theorems in which Tauberian conditions are of Hardy and Landau types as special cases of our results. We present some Tauberian conditions in terms of the de la Vallée Poussin means of double sequences under which P-convergence of a double sequence $(u_{\mu\nu})$ follows from its $(C, 1, 1)$ summability. Moreover, we give analogous results for $(C, 1, 0)$ and $(C, 0, 1)$ summability methods.

1. Introduction

There are many definitions of convergence for double sequences. However, convergence in Pringsheim's sense (or P-convergence) is the most commonly used definition of convergence for double sequences. The reason why this definition of convergence is preferred in the theory of double sequences is that it allows a sequence to convergence depending on a condition. Since this definition is better suited for the study of double sequences than the others, it is not surprising that many researchers interested in convergence of double sequences benefit from this definition in their studies.

By the early 1900s while the definition of convergence in Pringsheim's sense emerged, the extension of the summability theory for the single sequences to the multiple sequences was in its infancy. After the concept of double sequence was studied by Hardy [9] and Bromwich [3] in detail, studies on this newly defined concept had gained a tremendous momentum. One of the researchers who carried on some works on summability of double sequences, Robison [15] proved that any bounded double sequence is transformed by a regular transformation of Cesàro type into a another bounded double sequence. Agnew [1] obtained the extension of certain theorems on transformations of double sequences. Afterwards, Knopp [10] found out some Tauberian results for $(C, 1, 1)$ summable double sequences generalizing conditions which were came up with for single sequences by Tauber [18]. Móricz [11] put forward some Tauberian theorems for double sequences which P-convergence follows from $(C, 1, 1)$ summability under necessary and sufficient conditions and slow decrease conditions in certain senses. Totur [19] examined some conditions needed

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for $(C, 1, 1)$ summable double sequences to be convergent by using different approach. Önder and Çanak [13] attained some Tauberian conditions in terms of slow oscillation and slow decrease in certain senses, under which convergence of a double sequence in Pringsheim's sense follows from its statistical $(C, 1, 1)$ summability. In addition to these, a considerable number of researchers have attended to the question of summability of double sequences and related topics in recent years; these include especially Fındık et al. [8], Belen [2], Chen and Hsu [6], Edely and Mursaleen [7], Savaş [16], Totur and Çanak [20], etc.

In order that a summability method may be useful, it should be regular for some class of double sequences. It is known that the $(C, 1, 1)$ summability method is regular for class of bounded double sequences, which means that every P-convergent sequence of this class is $(C, 1, 1)$ summable to same limit. In the present paper, we are interested in the converse conclusion from the $(C, 1, 1)$ summability to P-convergence, which hold only under some additional condition so-called a Tauberian condition imposed on the double sequence. Such results concerning the $(C, 1, 1)$ summability of double sequences has been investigated before, for example by Knopp [10] and Totur [19]. Our main purpose in this paper is to re-examine the question of how P-convergence is obtained from the $(C, 1, 1)$ summability of double sequences using an approach based on generator sequences in certain senses which are defined differently from that adopted by Knopp [10] and Totur [19]. To motivate this, in § 2 we recall basic definitions and notations with respect to double sequences and its Cesàro means in certain senses. Later on, we introduce generator sequences, the Kronecker identities, emerging depends on these sequences, and de la Vallée Poussin means for double sequences in certain senses. In § 3, we firstly present some lemmas to be benefitted in the proofs of main results of its relevant section for double sequences. In the sequel, we establish a Tauberian theorem for double sequences that P-convergence follows from the $(C, 1, 1)$ summability under conditions of slow decrease of generator sequence $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ in certain senses and additional condition on $(u_{\mu\nu})$ and we present some corollaries related to this theorem. And also, we examine some conditions needed for the $(C, 1, 1)$ summable double sequences to be convergent. In § 4 and § 5 in parallel with § 3, we attain some Tauberian results for the $(C, 1, 0)$ and $(C, 0, 1)$ summable double sequences, respectively.

2. Preliminaries

In this section, we begin with basic definitions and notations with respect to double sequences and its Cesàro means in certain senses needed throughout this paper. In the sequel, we mention about how relations exist between described notions and we construct some examples concerning statements which hold for single sequences but not for double sequences. Besides we familiarize the generator sequences, Kronecker identities, emerging depends on these sequences, and de la Vallée Poussin means for double sequences in certain senses. We end this section by introducing concepts of slow decrease and slow oscillation in certain senses and we state how a transition exists between them in the wake of defining of these concepts.

A double sequence $u = (u_{\mu\nu})$ is a function u from $\mathbb{N} \times \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers) into the set $\mathbb{K}$ ($\mathbb{K}$ is the set of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers). The real or complex number $u_{\mu\nu}$ denotes the value of the function at a point $(\mu, \nu) \in \mathbb{N} \times \mathbb{N}$ and is called the (μ, ν)-term of the double sequence.
We denote the set of all double sequences of real and complex numbers by $w^2(\mathbb{R})$ and $w^2(\mathbb{C})$, respectively.

A double sequence $(u_{\mu\nu})$ is said to be convergent in Pringsheim's sense (or P-convergent) to ℓ if for every $\epsilon > 0$ there exists a positive integer $\nu_0(\epsilon)$ such that $|u_{\mu\nu} - \ell| < \epsilon$ whenever $\mu, \nu \geq \nu_0$ (see [14]). The number ℓ is called the Pringsheim limit of u and we denote by $P - \lim_{\mu,\nu\to\infty} u_{\mu\nu} = \ell$.

We say that a double sequence $(u_{\mu\nu})$ converges to ℓ if $(u_{\mu\nu})$ tends to ℓ as both μ and ν tend to infinity independently of one another. We denote the set of all P-convergent double sequences of real and complex numbers by $c^2(\mathbb{R})$ and $c^2(\mathbb{C})$, respectively.

We note that convergence mentioned throughout this paper is convergence in Pringsheim's sense.

A double sequence $(u_{\mu\nu})$ is bounded if there exists a positive number M such that $|u_{\mu\nu}| < M$ for all μ and ν.
We denote the set of all bounded double sequences of real and complex numbers by $\ell^2_\infty(\mathbb{R})$ and $\ell^2_\infty(\mathbb{C})$, respectively.

We note that a P-convergent double sequence may not be bounded contrary to the case in single sequences. For instance, the sequence $(u_{\mu\nu})$ defined by

$$u_{\mu\nu} = \begin{cases} 3^{\nu} & \text{if } \mu = 1;\ \nu = 0,1,2,\dots, \\ 3^{\mu+3} & \text{if } \nu = 4;\ \mu = 0,1,2,\dots, \\ 0 & \text{otherwise} \end{cases}$$

is P-convergent, but it is unbounded.

For a double sequence $(u_{\mu\nu})$, we define its $(C,1,1)$ means by

$$\sigma^{11}_{\mu\nu}(u) := \frac{1}{(\mu+1)(\nu+1)} \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} u_{ij} \tag{1}$$

for all nonnegative integers μ, ν (see [11]).
The $(C,1,0)$ and $(C,0,1)$ means of $(u_{\mu\nu})$ are defined by

$$\sigma^{10}_{\mu\nu}(u) := \frac{1}{\mu+1} \sum_{i=0}^{\mu} u_{i\nu} \quad \text{and} \quad \sigma^{01}_{\mu\nu}(u) := \frac{1}{\nu+1} \sum_{j=0}^{\nu} u_{\mu j}$$

for all nonnegative integers μ, ν, respectively.
A double sequence $(u_{\mu\nu})$ is said to be $(C,1,1)$ summable to a finite number ℓ if $(\sigma^{11}_{\mu\nu}(u))$ converges to the same number in Pringsheim's sense. Similarly, the $(C,1,0)$ and $(C,0,1)$ summable sequences are defined.

We note that a P-convergent double sequence need not be $(C,1,1)$ summable. For instance, the sequence $(u_{\mu\nu})$ defined by

$$u_{\mu\nu} = \begin{cases} w_{\mu} & \text{if } \nu = 0;\ \mu = 0,1,2,\dots, \\ w_{\nu} & \text{if } \mu = 0;\ \nu = 0,1,2,\dots, \\ 0 & \text{otherwise}, \end{cases}$$

where $(w_{\nu}) = \left(\sum_{k=0}^{\nu} (-1)^{k+1} k \right)$, is convergent to 0. On the other hand, one can check that

$$u_{\mu\nu} = \begin{cases} -\frac{\mu}{2} & \text{if } \nu = 0;\ \mu = 2k,\ k = 0,1,2,\dots, \\ \frac{\mu+1}{2} & \text{if } \nu = 0;\ \mu = 2k+1,\ k = 0,1,2,\dots, \\ -\frac{\nu}{2} & \text{if } \mu = 0;\ \nu = 2q,\ q = 0,1,2,\dots, \\ \frac{\nu+1}{2} & \text{if } \mu = 0;\ \nu = 2q+1,\ q = 0,1,2,\dots, \\ 0 & \text{otherwise.} \end{cases}$$

So, we have from the definition of $(C,1,1)$ means that

$$\sigma^{11}_{\mu\nu}(u) = \begin{cases} 0 & \text{if } \mu, \nu \text{ are even}, \\ \frac{1}{2(\mu+1)} & \text{if } \mu \text{ is even}, \nu \text{ is odd}, \\ \frac{1}{2(\nu+1)} & \text{if } \mu \text{ is odd}, \nu \text{ is even}, \\ \frac{1}{4} & \text{if } \mu,\nu \text{ are odd.} \end{cases}$$

Since the limit

$$\lim_{\mu,\nu\to\infty} \sigma^{11}_{\mu\nu}(u) = \lim_{\mu,\nu\to\infty} \frac{1}{(\mu+1)(\nu+1)} \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} u_{ij} = \begin{cases} \frac{1}{4} & \text{if } \mu,\nu \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$

the sequence $(\sigma^{11}_{\mu\nu}(u))$ is not convergent and hence $(u_{\mu\nu})$ is not $(C,1,1)$ summable.

In addition to this, a P-convergent double sequence is $(C, 1, 1)$ summable to its P-limit under the boundedness condition of double sequence (see [1]). However, the converse of this statement is not always true. In other words, a double sequence which is $(C, 1, 1)$ summable and bounded may not be P-convergent. An example indicating this case was constructed by Mursaleen and Edely [12].

Throughout this paper, the symbols $u_{\mu\nu} = O_L(1)$, $u_{\mu\nu} = O(1)$ and $u_{\mu\nu} = o(1)$ represent that $(u_{\mu\nu})$ is bounded below, bounded and P-convergent to zero as $\mu, \nu \to \infty$, respectively.

For a double sequence $(u_{\mu\nu})$, we define

$$\begin{aligned}
\Delta_{11}u_{\mu\nu} &= \Delta_{10}\Delta_{01}u_{\mu\nu} = \Delta_{10}\left(\Delta_{01}u_{\mu\nu}\right) = \Delta_{01}\left(\Delta_{10}u_{\mu\nu}\right) \\
&= u_{\mu\nu} - u_{\mu,\nu-1} - u_{\mu-1,\nu} + u_{\mu-1,\nu-1}, \\
\Delta_{10}u_{\mu\nu} &= u_{\mu\nu} - u_{\mu-1,\nu}, \\
\Delta_{01}u_{\mu\nu} &= u_{\mu\nu} - u_{\mu,\nu-1}
\end{aligned}$$

for all integers $\mu, \nu \geq 1$.

The Kronecker identities for a sequence $(u_{\mu\nu})$ are defined by

$$u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u) = \frac{1}{\mu+1}\sum_{i=1}^{\mu} i\Delta_{10}u_{i\nu} =: V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u)$$

and

$$u_{\mu\nu} - \sigma^{01}_{\mu\nu}(u) = \frac{1}{\nu+1}\sum_{j=1}^{\nu} j\Delta_{01}u_{\mu j} =: V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u)$$

for nonnegative integers μ, ν.

The double sequence $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is the $(C, 1, 0)$ mean of the sequence $(\mu\Delta_{10}u_{\mu\nu})$ and called the generator sequence of $(u_{\mu\nu})$ in the sense $(1, 0)$. In harmony with this defining, the double sequence $(V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u))$ is the $(C, 0, 1)$ mean of the sequence $(\nu\Delta_{01}u_{\mu\nu})$ and called the generator sequence of $(u_{\mu\nu})$ in the sense $(0, 1)$ (see [10]).

More generally, the double Kronecker identity for a sequence $(u_{\mu\nu})$ are defined by means of the generator sequences $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ and $(V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u))$ as follows:

$$u_{\mu\nu} - \sigma^{11}_{\mu\nu}(u) = V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u)$$

where

$$V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u) := V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u) + V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u) - \frac{1}{(\mu+1)(\nu+1)}\sum_{i=1}^{\mu}\sum_{j=1}^{\nu} ij\Delta_{11}u_{ij}$$

for nonnegative integers μ, ν.

The double sequence $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is called the generator sequence of $(u_{\mu\nu})$ in the sense $(1, 1)$ (cf. [2]).

In addition, the $(C, 1, 1)$ means of integer order $\alpha \geq 0$ of sequences $(u_{\mu\nu})$ and $\left(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u)\right)$ are defined by

$$\sigma^{11^{(\alpha)}}_{\mu\nu}(u) := \begin{cases} \dfrac{1}{(\mu+1)(\nu+1)}\displaystyle\sum_{i=0}^{\mu}\sum_{j=0}^{\nu}\sigma^{11^{(\alpha-1)}}_{ij}(u) & \text{if } \alpha \geq 1 \\ u_{\mu\nu} & \text{if } \alpha = 0 \end{cases}$$

and

$$V^{11^{(\alpha)}}_{\mu\nu}(\Delta_{11}u) := \begin{cases} \dfrac{1}{(\mu+1)(\nu+1)}\displaystyle\sum_{i=0}^{\mu}\sum_{j=0}^{\nu}V^{11^{(\alpha-1)}}_{ij}(\Delta_{11}u) & \text{if } \alpha \geq 1 \\ V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u) & \text{if } \alpha = 0 \end{cases}$$

respectively.
In parallel with these, the $(C, 1, 0)$ means of integer order $\alpha \geq 0$ of sequences $(u_{\mu\nu})$ and $\left(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u)\right)$ are defined by

$$\sigma^{10^{(\alpha)}}_{\mu\nu}(u) := \begin{cases} \dfrac{1}{\mu+1}\displaystyle\sum_{i=0}^{\mu} \sigma^{10^{(\alpha-1)}}_{i\nu}(u) & \text{if } \alpha \geq 1 \\ u_{\mu\nu} & \text{if } \alpha = 0 \end{cases}$$

and

$$V^{10^{(\alpha)}}_{\mu\nu}(\Delta_{10}u) := \begin{cases} \dfrac{1}{\mu+1}\displaystyle\sum_{i=0}^{\mu} V^{10^{(\alpha-1)}}_{i\nu}(\Delta_{10}u) & \text{if } \alpha \geq 1 \\ V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u) & \text{if } \alpha = 0 \end{cases}$$

respectively. Similarly, the $(C, 0, 1)$ means of integer order $\alpha \geq 0$ of sequences $(u_{\mu\nu})$ and $\left(V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u)\right)$ can be defined.
Throughout this paper, we will use the notation $\sigma^{11}_{\mu\nu}$ and $V^{11^{(1)}}_{\mu\nu}$ instead of $\sigma^{11^{(1)}}_{\mu\nu}(u)$ and $V^{11^{(1)}}_{\mu\nu}(\Delta_{11}u)$ for the sake of convenience.

The de la Vallée Poussin means of a double sequence $(u_{\mu\nu})$ in sense $(1, 1)$ are defined by

$$\tau^{>,11}_{\mu\nu}(u) := \frac{1}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \sum_{i=\mu+1}^{\lambda_\mu} \sum_{j=\nu+1}^{\lambda_\nu} u_{ij}, \qquad \lambda > 1 \tag{2}$$

and

$$\tau^{<,11}_{\mu\nu}(u) := \frac{1}{(\mu - \lambda_\mu)(\nu - \lambda_\nu)} \sum_{i=\lambda_\mu+1}^{\mu} \sum_{j=\lambda_\nu+1}^{\nu} u_{ij}, \qquad 0 < \lambda < 1 \tag{3}$$

for sufficiently large nonnegative integers μ, ν. Here, we denote the integral part of $\lambda\mu$ by $\lambda_\mu := [\lambda\mu]$.
In parallel with these, the de la Vallée Poussin means of a double sequence $(u_{\mu\nu})$ in sense $(1, 0)$ are defined by

$$\tau^{>,10}_{\mu\nu}(u) := \frac{1}{\lambda_\mu - \mu} \sum_{i=\mu+1}^{\lambda_\mu} u_{i\nu}, \qquad \lambda > 1 \tag{4}$$

and

$$\tau^{<,10}_{\mu\nu}(u) := \frac{1}{\mu - \lambda_\mu} \sum_{i=\lambda_\mu+1}^{\mu} u_{i\nu}, \qquad 0 < \lambda < 1 \tag{5}$$

for sufficiently large nonnegative integers μ, ν. Similarly, the de la Vallée Poussin means of a double sequence $(u_{\mu\nu})$ in sense $(0, 1)$ can be defined.

At present, we define analogous concepts of Schmidt's slow decrease and slow oscillation conditions for sequences $(u_{\mu\nu})$ of real and complex numbers in certain senses (see [17]). In the wake of defining of these concepts, we mention about how a transition exists between them.
Let $\lambda_\mu := [\lambda\mu]$ denote the integral part of $\lambda\mu$. We say that a double sequence $(u_{\mu\nu})$ of real numbers is slowly decreasing in sense $(1, 1)$ if

$$\lim_{\lambda\to 1^+} \liminf_{\mu,\nu\to\infty} \min_{\substack{\mu+1\leq i\leq\lambda_\mu \\ \nu+1\leq j\leq\lambda_\nu}} \left(u_{ij} - u_{\mu\nu}\right) \geq 0; \tag{6}$$

that is, for each $\epsilon > 0$ there exist $\nu_0 = \nu_0(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$u_{ij} - u_{\mu\nu} \geq -\epsilon \text{ whenever } \nu_0 < \mu < i \leq \lambda_\mu \text{ and } \nu_0 < \nu < j \leq \lambda_\nu.$$

Condition (6) is equivalent to

$$\lim_{\lambda\to 1^-} \liminf_{\mu,\nu\to\infty} \min_{\substack{\lambda_\mu < i \leq \mu \\ \lambda_\nu < j \leq \nu}} \left(u_{\mu\nu} - u_{ij}\right) \geq 0; \tag{6'}$$

that is, for each $\epsilon > 0$ there exist $\nu_0 = \nu_0(\epsilon)$ and $0 < \lambda = \lambda(\epsilon) < 1$ such that

$$u_{\mu\nu} - u_{ij} \geq -\epsilon \text{ whenever } \nu_0 < \lambda_\mu < i \leq \mu \text{ and } \nu_0 < \lambda_\nu < j \leq \nu.$$

We say that a double sequence $(u_{\mu\nu})$ of complex numbers is slowly oscillating in sense $(1,1)$ if

$$\lim_{\lambda\to 1^+} \limsup_{\mu,\nu\to\infty} \max_{\substack{\mu+1 \leq i \leq \lambda_\mu \\ \nu+1 \leq j \leq \lambda_\nu}} \left|u_{ij} - u_{\mu\nu}\right| = 0; \tag{7}$$

that is, for each $\epsilon > 0$ there exist $\nu_0 = \nu_0(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$\left|u_{ij} - u_{\mu\nu}\right| \leq \epsilon \text{ whenever } \nu_0 < \mu < i \leq \lambda_\mu \text{ and } \nu_0 < \nu < j \leq \lambda_\nu.$$

Condition (7) is equivalent to

$$\lim_{\lambda\to 1^-} \liminf_{\mu,\nu\to\infty} \min_{\substack{\lambda_\mu < i \leq \mu \\ \lambda_\nu < j \leq \nu}} \left|u_{\mu\nu} - u_{ij}\right| = 0; \tag{7'}$$

that is, for each $\epsilon > 0$ there exist $\nu_0 = \nu_0(\epsilon)$ and $0 < \lambda = \lambda(\epsilon) < 1$ such that

$$|u_{\mu\nu} - u_{ij}| \leq \epsilon \text{ whenever } \nu_0 < \lambda_\mu < i \leq \mu \text{ and } \nu_0 < \lambda_\nu < j \leq \nu.$$

It can be seen from (7) that a P-convergent double sequence of complex numbers is slowly oscillating in sense $(1,1)$, but converse of this is not true in general. An example indicating this situation was constructed by Çakalli and Patterson [4].
We say that a double sequence $(u_{\mu\nu})$ of real numbers is slowly decreasing in sense $(1,0)$ if

$$\lim_{\lambda\to 1^+} \liminf_{\mu,\nu\to\infty} \min_{\mu+1 \leq i \leq \lambda_\mu} (u_{i\nu} - u_{\mu\nu}) \geq 0, \tag{8}$$

or equivalently,

$$\lim_{\lambda\to 1^-} \liminf_{\mu,\nu\to\infty} \min_{\lambda_\mu < i \leq \mu} (u_{\mu\nu} - u_{i\nu}) \geq 0, \tag{8'}$$

besides it is said to be slowly decreasing in the strong sense $(1,0)$ if (8) is satisfied with

$$\min_{\substack{\mu+1 \leq i \leq \lambda_\mu \\ \nu+1 \leq j \leq \lambda_\nu}} \left(u_{ij} - u_{\mu j}\right) \quad \text{instead of} \quad \min_{\mu+1 \leq i \leq \lambda_\mu} \left(u_{i\nu} - u_{\mu\nu}\right). \tag{9}$$

We say that a double sequence $(u_{\mu\nu})$ of complex numbers is slowly oscillating in sense $(1,0)$ if

$$\lim_{\lambda\to 1^+} \limsup_{\mu,\nu\to\infty} \max_{\mu+1 \leq i \leq \lambda_\mu} |u_{\mu\nu} - u_{i\nu}| = 0, \tag{10}$$

or equivalently,

$$\lim_{\lambda\to 1^-} \limsup_{\mu,\nu\to\infty} \max_{\lambda_\mu < i \leq \mu} |u_{\mu\nu} - u_{i\nu}| = 0, \tag{10'}$$

besides it is said to be slowly oscillating in the strong sense $(1,0)$ if (10) is satisfied with

$$\max_{\substack{\mu+1\leq i\leq\lambda_\mu\\ \nu+1\leq j\leq\lambda_\nu}}\left|u_{ij}-u_{\mu j}\right| \quad \text{instead of} \quad \max_{\mu+1\leq i\leq\lambda_\mu}\left|u_{i\nu}-u_{\mu\nu}\right|. \tag{11}$$

Similarly, conditions of slow decrease and slow oscillation of double sequence $(u_{\mu\nu})$ of real and complex numbers in sense $(0,1)$ and the strong sense $(0,1)$ can be defined.

We note that if a sequence $(u_{\mu\nu})$ is slowly decreasing in sense $(1,0)$ and slowly decreasing in the strong sense $(0,1)$, then $(u_{\mu\nu})$ is slowly decreasing in sense $(1,1)$. In harmony with this statement, we can say that if a sequence $(u_{\mu\nu})$ is slowly decreasing in sense $(0,1)$ and slowly decreasing in the strong sense $(1,0)$, then $(u_{\mu\nu})$ is slowly decreasing in sense $(1,1)$, as well.
In fact, assume that $(u_{\mu\nu})$ is slowly decreasing in sense $(0,1)$ and in the strong sense $(1,0)$ without loss of generality. For all large enough μ and ν, that is, $\mu,\nu\geq\nu_0$ and $\lambda>1$, we have

$$\min_{\substack{\mu+1\leq i\leq\lambda_\mu\\ \nu+1\leq j\leq\lambda_\nu}}\left(u_{ij}-u_{\mu\nu}\right) = \min_{\substack{\mu+1\leq i\leq\lambda_\mu\\ \nu+1\leq j\leq\lambda_\nu}}\left(u_{ij}-u_{\mu j}+u_{\mu j}-u_{\mu\nu}\right) \geq \min_{\substack{\mu+1\leq i\leq\lambda_\mu\\ \nu+1\leq j\leq\lambda_\nu}}\left(u_{ij}-u_{\mu j}\right)+\min_{\nu+1\leq j\leq\lambda_\nu}\left(u_{\mu j}-u_{\mu\nu}\right). \tag{12}$$

Taking the lim inf and the limit of both sides of (12) as $\mu,\nu\to\infty$ and $\lambda\to 1^+$ respectively, we obtain that the terms on the right-hand side of (12) are greater than 0. Thus, we arrive that $(u_{\mu\nu})$ is slowly decreasing in sense $(1,1)$.

Similarly, if a sequence $(u_{\mu\nu})$ is slowly oscillating in sense $(1,0)$ and slowly oscillating in the strong sense $(0,1)$, then $(u_{\mu\nu})$ is slowly oscillating in sense $(1,1)$. In harmony with this statement, we can say that if a sequence $(u_{\mu\nu})$ is slowly oscillating in sense $(0,1)$ and slowly oscillating in the strong sense $(1,0)$, then $(u_{\mu\nu})$ is slowly oscillating in sense $(1,1)$, as well.
In fact, assume that $(u_{\mu\nu})$ is slowly oscillating in sense $(1,0)$ and in the strong sense $(0,1)$ without loss of generality. For all large enough μ and ν, that is, $\mu,\nu\geq\nu_0$ and $\lambda>1$, we have

$$\max_{\substack{\mu+1\leq i\leq\lambda_\mu\\ \nu+1\leq j\leq\lambda_\nu}}\left|u_{ij}-u_{\mu\nu}\right| = \max_{\substack{\mu+1\leq i\leq\lambda_\mu\\ \nu+1\leq j\leq\lambda_\nu}}\left|u_{ij}-u_{i\nu}+u_{i\nu}-u_{\mu\nu}\right| \leq \max_{\substack{\mu+1\leq i\leq\lambda_\mu\\ \nu+1\leq j\leq\lambda_\nu}}\left|u_{ij}-u_{i\nu}\right|+\max_{\nu+1\leq j\leq\lambda_\nu}\left|u_{i\nu}-u_{\mu\nu}\right|. \tag{13}$$

Taking the lim sup and the limit of both sides of (13) as $\mu,\nu\to\infty$ and $\lambda\to 1^+$ respectively, we obtain that the terms on the right-hand side of (13) are equal to 0. Thus, we arrive that $(u_{\mu\nu})$ is slowly decreasing in sense $(1,1)$.

3. Some Results for the $(C,1,1)$ Summable Double Sequences

This section essentially consists of two parts. In the first part, we present some lemmas to be benefited in the proofs of main results of this section for double sequences. In the second part, we discuss various Tauberian conditions which pave the way for a Tauberian conclusion from the $(C,1,1)$ summability to convergence for double sequences. In the sequel, we end this section by some corollaries.

3.1. Lemmas

In this subsection, we express and prove the following assertions to be utilized in the proofs of main results of this section for double sequences. The following lemma presents two representations of difference between the general terms of double sequences $(u_{\mu\nu})$ and $(\sigma^{11}_{\mu\nu}(u))$ by the aid of the de la Vallée Poussin means of the sequence $(u_{\mu\nu})$ in sense $(1,1)$.

Lemma 3.1. *Let $u=(u_{\mu\nu})$ be a double sequence.*

(i) For $\lambda > 1$ and sufficiently large μ, ν, we have

$$\begin{aligned} u_{\mu\nu} - \sigma^{11}_{\mu\nu}(u) &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\mu + 1)(\nu + 1)} \left(\sigma^{11}_{\lambda_\mu,\lambda_\nu} - \tau^{>,11}_{\mu\nu}\right) - \frac{\lambda_\mu + 1}{\mu + 1} \left(\sigma^{11}_{\lambda_\mu,\nu} - \tau^{>,11}_{\mu\nu}\right) \\ &- \frac{\lambda_\nu + 1}{\nu + 1} \left(\sigma^{11}_{\mu,\lambda_\nu} - \tau^{>,11}_{\mu\nu}\right) - \left(\tau^{>,11}_{\mu\nu} - u_{\mu\nu}\right). \end{aligned} \tag{14}$$

(ii) For $0 < \lambda < 1$ and sufficiently large μ, ν, we have

$$\begin{aligned} u_{\mu\nu} - \sigma^{11}_{\mu\nu}(u) &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\mu + 1)(\nu + 1)} \left(\sigma^{11}_{\lambda_\mu,\lambda_\nu} - \tau^{<,11}_{\mu\nu}\right) - \frac{\lambda_\mu + 1}{\mu + 1} \left(\sigma^{11}_{\lambda_\mu,\nu} - \tau^{<,11}_{\mu\nu}\right) \\ &- \frac{\lambda_\nu + 1}{\nu + 1} \left(\sigma^{11}_{\mu,\lambda_\nu} - \tau^{<,11}_{\mu\nu}\right) + \left(u_{\mu\nu} - \tau^{<,11}_{\mu\nu}\right). \end{aligned} \tag{15}$$

Proof. (*i*) For $\lambda > 1$, we have

$$\begin{aligned} \tau^{>,11}_{\mu\nu}(u) &= \frac{1}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \sum_{i=\mu+1}^{\lambda_\mu} \sum_{j=\nu+1}^{\lambda_\nu} u_{ij} \\ &= \frac{1}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \left[\left(\sum_{i=0}^{\lambda_\mu} - \sum_{i=0}^{\mu}\right)\left(\sum_{j=0}^{\lambda_\nu} - \sum_{j=0}^{\nu}\right)\right] u_{ij} \\ &= \frac{1}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \left[\sum_{i=0}^{\lambda_\mu}\sum_{j=0}^{\lambda_\nu} - \sum_{i=0}^{\lambda_\mu}\sum_{j=0}^{\nu} - \sum_{i=0}^{\mu}\sum_{j=0}^{\lambda_\nu} + \sum_{i=0}^{\mu}\sum_{j=0}^{\nu}\right] u_{ij} \\ &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \sigma^{11}_{\lambda_\mu,\lambda_\nu} - \frac{(\lambda_\mu + 1)(\nu + 1)}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \sigma^{11}_{\lambda_\mu,\nu} \\ &- \frac{(\mu + 1)(\lambda_\nu + 1)}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \sigma^{11}_{\mu,\lambda_\nu} + \frac{(\mu + 1)(\nu + 1)}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \sigma^{11}_{\mu,\nu} \end{aligned} \tag{16}$$

for sufficiently large μ, ν. It follows from equation (16) that

$$\begin{aligned} -\sigma^{11}_{\mu\nu}(u) &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\mu + 1)(\nu + 1)} \sigma^{11}_{\lambda_\mu,\lambda_\nu} - \frac{\lambda_\mu + 1}{\mu + 1} \sigma^{11}_{\lambda_\mu,\nu} - \frac{\lambda_\nu + 1}{\nu + 1} \sigma^{11}_{\mu,\lambda_\nu} \\ &- \frac{(\lambda_\mu - \mu)(\lambda_\nu - \nu)}{(\mu + 1)(\nu + 1)} \tau^{>,11}_{\mu\nu} \\ &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\mu + 1)(\nu + 1)} \sigma^{11}_{\lambda_\mu,\lambda_\nu} - \frac{\lambda_\mu + 1}{\mu + 1} \sigma^{11}_{\lambda_\mu,\nu} - \frac{\lambda_\nu + 1}{\nu + 1} \sigma^{11}_{\mu,\lambda_\nu} \\ &- \left[\frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\mu + 1)(\nu + 1)} - \frac{\lambda_\mu + 1}{\mu + 1} - \frac{\lambda_\nu + 1}{\nu + 1} + 1\right] \tau^{>,11}_{\mu\nu} \\ &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\mu + 1)(\nu + 1)} \left(\sigma^{11}_{\lambda_\mu,\lambda_\nu} - \tau^{>,11}_{\mu\nu}\right) - \frac{\lambda_\mu + 1}{\mu + 1} \left(\sigma^{11}_{\lambda_\mu,\nu} - \tau^{>,11}_{\mu\nu}\right) \\ &- \frac{\lambda_\nu + 1}{\nu + 1} \left(\sigma^{11}_{\mu,\lambda_\nu} - \tau^{>,11}_{\mu\nu}\right) - \tau^{>,11}_{\mu\nu} \end{aligned} \tag{17}$$

for sufficiently large μ, ν. If we add the term $u_{\mu\nu}$ to both sides of equality (17), we complete the proof of (*i*).
(*ii*) This is similar to the proof of part (*i*) of Lemma 3.1. □

In the next lemma proved by Totur [19], the difference between the general terms of double sequences $(u_{\mu\nu})$ and $(\sigma^{11}_{\mu\nu}(u))$ interprets differently from the statement given in Lemma 3.1.

Lemma 3.2. *([11], Lemma 1) Let $u = (u_{\mu\nu})$ be a double sequence.*

(i) For $\lambda > 1$ and sufficiently large μ, ν, we have

$$\begin{aligned} u_{\mu\nu} - \sigma^{11}_{\mu\nu}(u) &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \left(\sigma^{11}_{\lambda_\mu,\lambda_\nu} - \sigma^{11}_{\lambda_\mu,\nu} - \sigma^{11}_{\mu,\lambda_\nu} + \sigma^{11}_{\mu\nu}\right) \\ &+ \frac{\lambda_\mu + 1}{\lambda_\mu - \mu} \left(\sigma^{11}_{\lambda_\mu,\nu} - \sigma^{11}_{\mu\nu}\right) + \frac{\lambda_\nu + 1}{\lambda_\nu - \nu} \left(\sigma^{11}_{\mu,\lambda_\nu} - \sigma^{11}_{\mu\nu}\right) \\ &- \frac{1}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \sum_{i=\mu+1}^{\lambda_\mu} \sum_{j=\nu+1}^{\lambda_\nu} \left(u_{ij} - u_{\mu\nu}\right). \end{aligned}$$

(ii) For $0 < \lambda < 1$ and sufficiently large μ, ν, we have

$$\begin{aligned} u_{\mu\nu} - \sigma^{11}_{\mu\nu}(u) &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\mu - \lambda_\mu)(\nu - \lambda_\nu)} \left(\sigma^{11}_{\mu\nu} - \sigma^{11}_{\lambda_\mu,\nu} - \sigma^{11}_{\mu,\lambda_\nu} + \sigma^{11}_{\lambda_\mu,\lambda_\nu}\right) \\ &+ \frac{\lambda_\mu + 1}{\mu - \lambda_\mu} \left(\sigma^{11}_{\mu\nu} - \sigma^{11}_{\lambda_\mu,\nu}\right) + \frac{\lambda_\nu + 1}{\nu - \lambda_\nu} \left(\sigma^{11}_{\mu\nu} - \sigma^{11}_{\mu,\lambda_\nu}\right) \\ &+ \frac{1}{(\mu - \lambda_\mu)(\nu - \lambda_\nu)} \sum_{i=\lambda_\mu+1}^{\mu} \sum_{j=\lambda_\nu+1}^{\nu} \left(u_{\mu\nu} - u_{ij}\right). \end{aligned}$$

To be also evaluated as a result of Lemma 3.2, the below-mentioned representations were obtained by Totur [19] during the demonstration of the de la Vallée Poussin means of double sequence $(u_{\mu\nu})$ being $(C, 1, 1)$ summable to ℓ are convergent to same number.

Lemma 3.3. *([19], Lemma 4) Let $u = (u_{\mu\nu})$ be a double sequence.*

(i) For $\lambda > 1$ and sufficiently large μ, ν, we have

$$\begin{aligned} \tau^{>,11}_{\mu\nu}(u) - \sigma^{11}_{\mu\nu}(u) &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\lambda_\mu - \mu)(\lambda_\nu - \nu)} \left(\sigma^{11}_{\lambda_\mu,\lambda_\nu} - \sigma^{11}_{\lambda_\mu,\nu} - \sigma^{11}_{\mu,\lambda_\nu} + \sigma^{11}_{\mu\nu}\right) \\ &+ \frac{\lambda_\mu + 1}{\lambda_\mu - \mu} \left(\sigma^{11}_{\lambda_\mu,\nu} - \sigma^{11}_{\mu\nu}\right) + \frac{\lambda_\nu + 1}{\lambda_\nu - \nu} \left(\sigma^{11}_{\mu,\lambda_\nu} - \sigma^{11}_{\mu\nu}\right). \end{aligned}$$

(ii) For $0 < \lambda < 1$ and sufficiently large μ, ν, we have

$$\begin{aligned} \tau^{<,11}_{\mu\nu}(u) - \sigma^{11}_{\mu\nu}(u) &= \frac{(\lambda_\mu + 1)(\lambda_\nu + 1)}{(\mu - \lambda_\mu)(\nu - \lambda_\nu)} \left(\sigma^{11}_{\lambda_\mu,\lambda_\nu} - \sigma^{11}_{\lambda_\mu,\nu} - \sigma^{11}_{\mu,\lambda_\nu} + \sigma^{11}_{\mu\nu}\right) \\ &+ \frac{\lambda_\mu + 1}{\mu - \lambda_\mu} \left(\sigma^{11}_{\mu\nu} - \sigma^{11}_{\lambda_\mu,\nu}\right) + \frac{\lambda_\nu + 1}{\nu - \lambda_\nu} \left(\sigma^{11}_{\mu\nu} - \sigma^{11}_{\mu,\lambda_\nu}\right). \end{aligned}$$

In [5], Çanak proved that a generator sequence is convergent under some suitable conditions. Inspiring this theorem given for the single sequences, we indicate under which conditions a double generator sequence of $(u_{\mu\nu})$ in sense $(1, 1)$ is P-convergent.

Lemma 3.4. *For a double sequence $u = (u_{\mu\nu})$ of real numbers, let the assumptions*

$$\lim_{\lambda\to 1^+}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{>,11}_{\mu\nu}\right)\geq 0,\quad \lim_{\lambda\to 1^+}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)\geq 0,$$
$$\lim_{\lambda\to 1^+}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)\leq 0 \tag{18}$$

and

$$\lim_{\lambda\to 1^-}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{<,11}_{\mu\nu}\right)\leq 0,\quad \lim_{\lambda\to 1^-}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{<,11}_{\mu\nu}\right)\leq 0,$$
$$\lim_{\lambda\to 1^-}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{<,11}_{\mu\nu}\right)\geq 0 \tag{19}$$

hold. If the conditions

$$\lim_{\lambda\to 1^+}\liminf_{\mu,\nu\to\infty}\left(\tau^{>,11}_{\mu\nu}-u_{\mu\nu}\right)\geq 0 \tag{20}$$

and

$$\lim_{\lambda\to 1^-}\liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\tau^{<,11}_{\mu\nu}\right)\geq 0 \tag{21}$$

are satisfied, then the generator sequence $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is P-convergent to 0.

Proof. Assume that conditions (18)-(21) are satisfied. In order to prove that $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is P-convergent to 0, we examine difference between the general terms of sequences $(u_{\mu\nu})$ and $(\sigma^{11}_{\mu\nu}(u))$ in two cases $\lambda > 1$ and $0 < \lambda < 1$. We firstly consider the case $\lambda > 1$. If we take the lim sup of both sides of identity (14) as $\mu,\nu\to\infty$, then we obtain that for each $\lambda > 1$

$$\begin{aligned}
\limsup_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\sigma^{11}_{\mu\nu}(u)\right) &\leq \limsup_{\mu,\nu\to\infty}\frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\mu+1)(\nu+1)}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)\\
&+ \limsup_{\mu,\nu\to\infty}\frac{\lambda_\mu+1}{\mu+1}\limsup_{\mu,\nu\to\infty}\left(-\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{>,11}_{\mu\nu}\right)\right)\\
&+ \limsup_{\mu,\nu\to\infty}\frac{\lambda_\nu+1}{\nu+1}\limsup_{\mu,\nu\to\infty}\left(-\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)\right)\\
&+ \limsup_{\mu,\nu\to\infty}\left(-\left(\tau^{>,11}_{\mu\nu}-u_{\mu\nu}\right)\right)\\
&= \lambda^2\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)-\lambda\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{>,11}_{\mu\nu}\right)\\
&- \lambda\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)-\liminf_{\mu,\nu\to\infty}\left(\tau^{>,11}_{\mu\nu}-u_{\mu\nu}\right)
\end{aligned}$$

because of that

$$\lim_{\mu\to\infty}\frac{\lambda_\mu+1}{\mu+1}=\lambda \quad\text{and}\quad \lim_{\nu\to\infty}\frac{\lambda_\nu+1}{\nu+1}=\lambda. \tag{22}$$

If we take the limit of both sides of the last inequality as $\lambda\to 1^+$, we get

$$\begin{aligned}
\limsup_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\sigma^{11}_{\mu\nu}(u)\right) &\leq \lim_{\lambda\to 1^+}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)\\
&- \lim_{\lambda\to 1^+}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{>,11}_{\mu\nu}\right)\\
&- \lim_{\lambda\to 1^+}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)\\
&- \lim_{\lambda\to 1^+}\liminf_{\mu,\nu\to\infty}\left(\tau^{>,11}_{\mu\nu}-u_{\mu\nu}\right).
\end{aligned}$$

From assumptions in (18) and (20), it follows that

$$\limsup_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\sigma^{11}_{\mu\nu}(u)\right)\le 0. \tag{23}$$

On the other hand, we consider the case $0<\lambda<1$. If we take the lim inf of both sides of identity (15) as $\mu,\nu\to\infty$, then we obtain that for each $0<\lambda<1$

$$\begin{aligned}
\liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\sigma^{11}_{\mu\nu}(u)\right) &\ge \liminf_{\mu,\nu\to\infty}\frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\mu+1)(\nu+1)}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{<,11}_{\mu\nu}\right)\\
&+ \liminf_{\mu,\nu\to\infty}\frac{\lambda_\mu+1}{\mu+1}\liminf_{\mu,\nu\to\infty}\left(-\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{<,11}_{\mu\nu}\right)\right)\\
&+ \liminf_{\mu,\nu\to\infty}\frac{\lambda_\nu+1}{\nu+1}\liminf_{\mu,\nu\to\infty}\left(-\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{<,11}_{\mu\nu}\right)\right)\\
&+ \liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\tau^{<,11}_{\mu\nu}\right)\\
&= \lambda^2\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{<,11}_{\mu\nu}\right)-\lambda\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{<,11}_{\mu\nu}\right)\\
&- \lambda\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{<,11}_{\mu\nu}\right)+\liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\tau^{<,11}_{\mu\nu}\right)
\end{aligned}$$

because of limits in (22). If we take the limit of both sides of last inequality as $\lambda\to 1^-$, we get

$$\begin{aligned}
\liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\sigma^{11}_{\mu\nu}(u)\right) &\ge \lim_{\lambda\to1^-}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{<,11}_{\mu\nu}\right)\\
&- \lim_{\lambda\to1^-}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{<,11}_{\mu\nu}\right)\\
&- \lim_{\lambda\to1^-}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{<,11}_{\mu\nu}\right)\\
&+ \lim_{\lambda\to1^-}\liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\tau^{<,11}_{\mu\nu}\right).
\end{aligned}$$

From assumptions in (19) and (21), it follows that

$$\liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\sigma^{11}_{\mu\nu}(u)\right)\ge 0. \tag{24}$$

If we combine inequalities (23) with (24), we conclude

$$\lim_{\mu,\nu\to\infty}\left(u_{\mu\nu}-\sigma^{11}_{\mu\nu}(u)\right)=0$$

which means by the double Kronecker identity that $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is P- convergent to 0. □

3.2. Main Results

In this subsection, we prove a Tauberian theorem for double sequences that P-convergence follows from the $(C,1,1)$ summability under the conditions of slow decrease of the generator sequence $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ in certain senses and additional condition on $(u_{\mu\nu})$ and we present some corollaries related to this theorem. In the sequel, we end this part by giving some Tauberian conditions for the $(C,1,1)$ summability method.

Theorem 3.5. *Let a bounded double sequence $(u_{\mu\nu})$ be $(C,1,1)$ summable to a number ℓ. If its generator sequence $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is slowly decreasing in sense $(0,1)$ (or $(1,0)$) and slowly decreasing in the strong sense $(1,0)$ (or $(0,1)$), then $(u_{\mu\nu})$ is P- convergent to ℓ.*

Proof. Suppose that a bounded double sequence $(u_{\mu\nu})$ is $(C,1,1)$ summable to ℓ and $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is slowly decreasing in sense $(0,1)$ and slowly decreasing in the strong sense $(1,0)$ without loss of generality. In order to prove that $(u_{\mu\nu})$ is P-convergent to same number, we firstly indicate that $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is P-convergent to 0. Since $(\sigma^{11}_{\mu\nu}(u))$ is P-convergent to ℓ and the $(C,1,1)$ summability method is regular under the boundedness condition of the sequence $(u_{\mu\nu})$, we obtain that $(\sigma^{11^{(2)}}_{\mu\nu}(u))$ is also convergent to same number. It follows from the double Kronecker identity that $(V^{11^{(1)}}_{\mu\nu}(\Delta_{11}u))$ is P-convergent to 0. For $\lambda > 1$, if we replace $u_{\mu\nu}$ by $V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u)$ in Lemma 3.2 (*i*), we obtain

$$\begin{aligned}
V^{11^{(0)}}_{\mu\nu} - V^{11^{(1)}}_{\mu\nu} &= \frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\left(V^{11^{(1)}}_{\lambda_\mu,\lambda_\nu} - V^{11^{(1)}}_{\lambda_\mu,\nu} - V^{11^{(1)}}_{\mu,\lambda_\nu} + V^{11^{(1)}}_{\mu\nu}\right) \\
&+ \frac{\lambda_\mu+1}{\lambda_\mu-\mu}\left(V^{11^{(1)}}_{\lambda_\mu,\nu} - V^{11^{(1)}}_{\mu\nu}\right) + \frac{\lambda_\nu+1}{\lambda_\nu-\nu}\left(V^{11^{(1)}}_{\mu,\lambda_\nu} - V^{11^{(1)}}_{\mu\nu}\right) \\
&- \frac{1}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\sum_{i=\mu+1}^{\lambda_\mu}\sum_{j=\nu+1}^{\lambda_\nu}\left(V^{11^{(0)}}_{ij} - V^{11^{(0)}}_{\mu\nu}\right) \\
&= \frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\left(V^{11^{(1)}}_{\lambda_\mu,\lambda_\nu} - V^{11^{(1)}}_{\lambda_\mu,\nu} - V^{11^{(1)}}_{\mu,\lambda_\nu} + V^{11^{(1)}}_{\mu\nu}\right) \\
&+ \frac{\lambda_\mu+1}{\lambda_\mu-\mu}\left(V^{11^{(1)}}_{\lambda_\mu,\nu} - V^{11^{(1)}}_{\mu\nu}\right) + \frac{\lambda_\nu+1}{\lambda_\nu-\nu}\left(V^{11^{(1)}}_{\mu,\lambda_\nu} - V^{11^{(1)}}_{\mu\nu}\right) \\
&- \frac{1}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\sum_{i=\mu+1}^{\lambda_\mu}\sum_{j=\nu+1}^{\lambda_\nu}\left(V^{11^{(0)}}_{ij} - V^{11^{(0)}}_{\mu j}\right) \\
&- \frac{1}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\sum_{i=\mu+1}^{\lambda_\mu}\sum_{j=\nu+1}^{\lambda_\nu}\left(V^{11^{(0)}}_{\mu j} - V^{11^{(0)}}_{\mu\nu}\right) \\
&\le \frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\left(V^{11^{(1)}}_{\lambda_\mu,\lambda_\nu} - V^{11^{(1)}}_{\lambda_\mu,\nu} - V^{11^{(1)}}_{\mu,\lambda_\nu} + V^{11^{(1)}}_{\mu\nu}\right) \\
&+ \frac{\lambda_\mu+1}{\lambda_\mu-\mu}\left(V^{11^{(1)}}_{\lambda_\mu,\nu} - V^{11^{(1)}}_{\mu\nu}\right) + \frac{\lambda_\nu+1}{\lambda_\nu-\nu}\left(V^{11^{(1)}}_{\mu,\lambda_\nu} - V^{11^{(1)}}_{\mu\nu}\right) \\
&- \min_{\substack{\mu<i\le\lambda_\mu \\ \nu<j\le\lambda_\nu}}\left(V^{11^{(0)}}_{ij} - V^{11^{(0)}}_{\mu j}\right) - \min_{\nu<j\le\lambda_\nu}\left(V^{11^{(0)}}_{\mu j} - V^{11^{(0)}}_{\mu\nu}\right)
\end{aligned} \tag{25}$$

for sufficiently large enough μ,ν. Considering that $(V^{11^{(1)}}_{\mu\nu}(\Delta_{11}u))$ is P-convergent to 0,

$$\lim_{\mu\to\infty}\frac{\lambda_\mu+1}{\lambda_\mu-\mu} = \frac{\lambda}{\lambda-1} \quad \text{and} \quad \lim_{\nu\to\infty}\frac{\lambda_\nu+1}{\lambda_\nu-\nu} = \frac{\lambda}{\lambda-1},$$

if we take the lim sup of both sides of inequality (25) as $\mu,\nu\to\infty$, then we reach

$$\limsup_{\mu,\nu\to\infty}\left(V^{11^{(0)}}_{\mu\nu} - V^{11^{(1)}}_{\mu\nu}\right) \le -\liminf_{\mu,\nu\to\infty}\min_{\substack{\mu<i\le\lambda_\mu \\ \nu<j\le\lambda_\nu}}\left(V^{11^{(0)}}_{ij} - V^{11^{(0)}}_{\mu j}\right) - \liminf_{\mu,\nu\to\infty}\min_{\nu<j\le\lambda_\nu}\left(V^{11^{(0)}}_{\mu j} - V^{11^{(0)}}_{\mu\nu}\right).$$

If we take the limit of both sides of last inequality as $\lambda\to1^+$, then we find

$$\limsup_{\mu,\nu\to\infty}\left(V^{11^{(0)}}_{\mu\nu} - V^{11^{(1)}}_{\mu\nu}\right) \le -\lim_{\lambda\to1^+}\liminf_{\mu,\nu\to\infty}\min_{\substack{\mu<i\le\lambda_\mu \\ \nu<j\le\lambda_\nu}}\left(V^{11^{(0)}}_{ij} - V^{11^{(0)}}_{\mu j}\right) - \lim_{\lambda\to1^+}\liminf_{\mu,\nu\to\infty}\min_{\nu<j\le\lambda_\nu}\left(V^{11^{(0)}}_{\mu j} - V^{11^{(0)}}_{\mu\nu}\right) \le 0 \tag{26}$$

due to the fact that $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is slowly decreasing in sense $(0,1)$ and slowly decreasing in the strong sense $(1,0)$. Following a similar procedure to above for $0<\lambda<1$, if we replace $u_{\mu\nu}$ by $V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u)$ in Lemma 3.2 (*ii*), we obtain

$$\begin{aligned}
V^{11^{(0)}}_{\mu\nu} - V^{11^{(1)}}_{\mu\nu} &= \frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\mu-\lambda_\mu)(\nu-\lambda_\nu)}\left(V^{11^{(1)}}_{\mu\nu} - V^{11^{(1)}}_{\lambda_\mu,\nu} - V^{11^{(1)}}_{\mu,\lambda_\nu} + V^{11^{(1)}}_{\lambda_\mu,\lambda_\nu}\right)\\
&+ \frac{\lambda_\mu+1}{\mu-\lambda_\mu}\left(V^{11^{(1)}}_{\mu\nu} - V^{11^{(1)}}_{\lambda_\mu,\nu}\right) + \frac{\lambda_\nu+1}{\nu-\lambda_\nu}\left(V^{11^{(1)}}_{\mu\nu} - V^{11^{(1)}}_{\mu,\lambda_\nu}\right)\\
&+ \frac{1}{(\mu-\lambda_\mu)(\nu-\lambda_\nu)}\sum_{i=\lambda_\mu+1}^{\mu}\sum_{j=\lambda_\nu+1}^{\nu}\left(V^{11^{(0)}}_{\mu\nu} - V^{11^{(0)}}_{ij}\right)\\
&= \frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\mu-\lambda_\mu)(\nu-\lambda_\nu)}\left(V^{11^{(1)}}_{\mu\nu} - V^{11^{(1)}}_{\lambda_\mu,\nu} - V^{11^{(1)}}_{\mu,\lambda_\nu} + V^{11^{(1)}}_{\lambda_\mu,\lambda_\nu}\right)\\
&+ \frac{\lambda_\mu+1}{\mu-\lambda_\mu}\left(V^{11^{(1)}}_{\mu\nu} - V^{11^{(1)}}_{\lambda_\mu,\nu}\right) + \frac{\lambda_\nu+1}{\nu-\lambda_\nu}\left(V^{11^{(1)}}_{\mu\nu} - V^{11^{(1)}}_{\mu,\lambda_\nu}\right)\\
&+ \frac{1}{(\mu-\lambda_\mu)(\nu-\lambda_\nu)}\sum_{i=\lambda_\mu+1}^{\mu}\sum_{j=\lambda_\nu+1}^{\nu}\left(V^{11^{(0)}}_{\mu\nu} - V^{11^{(0)}}_{\mu j}\right)\\
&+ \frac{1}{(\mu-\lambda_\mu)(\nu-\lambda_\nu)}\sum_{i=\lambda_\mu+1}^{\mu}\sum_{j=\lambda_\nu+1}^{\nu}\left(V^{11^{(0)}}_{\mu j} - V^{11^{(0)}}_{ij}\right)\\
&\geq \frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\mu-\lambda_\mu)(\nu-\lambda_\nu)}\left(V^{11^{(1)}}_{\mu\nu} - V^{11^{(1)}}_{\lambda_\mu,\nu} - V^{11^{(1)}}_{\mu,\lambda_\nu} + V^{11^{(1)}}_{\lambda_\mu,\lambda_\nu}\right)\\
&+ \frac{\lambda_\mu+1}{\mu-\lambda_\mu}\left(V^{11^{(1)}}_{\mu\nu} - V^{11^{(1)}}_{\lambda_\mu,\nu}\right) + \frac{\lambda_\nu+1}{\nu-\lambda_\nu}\left(V^{11^{(1)}}_{\mu\nu} - V^{11^{(1)}}_{\mu,\lambda_\nu}\right)\\
&+ \min_{\lambda_\nu<j\leq\nu}\left(V^{11^{(0)}}_{\mu\nu} - V^{11^{(0)}}_{\mu j}\right) + \min_{\substack{\lambda_\mu<i\leq\mu\\ \lambda_\nu<j\leq\nu}}\left(V^{11^{(0)}}_{\mu j} - V^{11^{(0)}}_{ij}\right) \qquad (27)
\end{aligned}$$

for sufficiently large enough μ,ν. Considering that $(V^{11^{(1)}}_{\mu\nu}(\Delta_{11}u))$ is P-convergent to 0,

$$\lim_{\mu\to\infty}\frac{\lambda_\mu+1}{\mu-\lambda_\mu} = \frac{\lambda}{1-\lambda} \quad\text{and}\quad \lim_{\nu\to\infty}\frac{\lambda_\nu+1}{\nu-\lambda_\nu} = \frac{\lambda}{1-\lambda},$$

if we take the lim inf of both sides of inequality (27) as $\mu,\nu\to\infty$, then we reach

$$\liminf_{\mu,\nu\to\infty}\left(V^{11^{(0)}}_{\mu\nu} - V^{11^{(1)}}_{\mu\nu}\right) \geq \liminf_{\mu,\nu\to\infty}\min_{\lambda_\nu<j\leq\nu}\left(V^{11^{(0)}}_{\mu\nu} - V^{11^{(0)}}_{\mu j}\right) + \liminf_{\mu,\nu\to\infty}\min_{\substack{\lambda_\mu<i\leq\mu\\ \lambda_\nu<j\leq\nu}}\left(V^{11^{(0)}}_{\mu j} - V^{11^{(0)}}_{ij}\right).$$

If we take the limit of both sides of last inequality as $\lambda\to1^-$, then we find

$$\liminf_{\mu,\nu\to\infty}\left(V^{11^{(0)}}_{\mu\nu} - V^{11^{(1)}}_{\mu\nu}\right) \geq \lim_{\lambda\to1^-}\liminf_{\mu,\nu\to\infty}\min_{\lambda_\nu<j\leq\nu}\left(V^{11^{(0)}}_{\mu\nu} - V^{11^{(0)}}_{\mu j}\right) + \lim_{\lambda\to1^-}\liminf_{\mu,\nu\to\infty}\min_{\substack{\lambda_\mu<i\leq\mu\\ \lambda_\nu<j\leq\nu}}\left(V^{11^{(0)}}_{\mu j} - V^{11^{(0)}}_{ij}\right) \geq 0 \qquad (28)$$

due to the fact that $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is slowly decreasing in sense $(0,1)$ and slowly decreasing in the strong sense $(1,0)$. If we combine inequalities (26) with (28), we conclude

$$\lim_{\mu,\nu\to\infty}V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u) = \lim_{\mu,\nu\to\infty}V^{11^{(1)}}_{\mu\nu}(\Delta_{11}u).$$

Because $(V^{11^{(1)}}_{\mu\nu}(\Delta_{11}u))$ is P-convergent to 0, $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is also P-convergent to 0, which means from the double Kronecker identity that $(u_{\mu\nu})$ is P-convergent to ℓ. □

In consideration of Theorem 3.5, we can state the following corollary.

Corollary 3.6. *Let a bounded double sequence $(u_{\mu\nu})$ be $(C,1,1)$ summable to a number ℓ. If conditions*

$$\mu\Delta_{10}V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u) \geq -C$$

and

$$\nu\Delta_{01}V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u) \geq -C$$

are satisfied for some $C \geq 0$, then $(u_{\mu\nu})$ is P-convergent to ℓ.

Analogous results for double sequences of complex numbers can be formulated as follow.

Theorem 3.7. *Let a bounded double sequence $(u_{\mu\nu})$ be $(C,1,1)$ summable to a number ℓ. If its generator sequence $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is slowly oscillating in sense $(0,1)$ (or $(1,0)$) and slowly oscillating in the strong sense $(1,0)$ (or $(0,1)$), then $(u_{\mu\nu})$ is P-convergent to ℓ.*

In consideration of Theorem 3.7, we can state the following corollary.

Corollary 3.8. *Let a bounded double sequence $(u_{\mu\nu})$ be $(C,1,1)$ summable to a number ℓ. If conditions*

$$|\mu\Delta_{10}V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u)| \leq M$$

and

$$|\nu\Delta_{01}V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u)| \leq M$$

are satisfied for some $M \geq 0$, then $(u_{\mu\nu})$ is P-convergent to ℓ.

Before finishing this subsection, we examine some conditions needed for the $(C,1,1)$ summable double sequences to be convergent.

Theorem 3.9. *Let a double sequence $(u_{\mu\nu})$ be $(C,1,1)$ summable to a number ℓ. If conditions*

$$\lim_{\lambda\to 1^+}\liminf_{\mu,\nu\to\infty}\left(\tau^{>,11}_{\mu\nu} - u_{\mu\nu}\right) \geq 0 \tag{29}$$

and

$$\lim_{\lambda\to 1^-}\liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu} - \tau^{<,11}_{\mu\nu}\right) \geq 0 \tag{30}$$

are satisfied, then $(u_{\mu\nu})$ is P-convergent to ℓ.

Proof. Suppose that a double sequence $(u_{\mu\nu})$ is $(C,1,1)$ summable to ℓ and conditions (29) and (30) are satisfied. In order to prove that $(u_{\mu\nu})$ is P-convergent to same number, it is enough to indicate that conditions in (18) and (19) are verified. For $\lambda > 1$, we have from Lemma 3.3 (i)

$$\begin{aligned}\sigma^{11}_{\lambda_\mu,\nu} - \tau^{>,11}_{\mu\nu} \quad &= \quad \frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\left(\sigma^{11}_{\lambda_\mu,\nu} - \sigma^{11}_{\lambda_\mu,\lambda_\nu} - \sigma^{11}_{\mu\nu} + \sigma^{11}_{\mu,\lambda_\nu}\right)\\ &+ \quad \frac{\mu+1}{\lambda_\mu-\mu}\left(\sigma^{11}_{\mu\nu} - \sigma^{11}_{\lambda_\mu,\nu}\right) + \frac{\lambda_\nu+1}{\lambda_\nu-\nu}\left(\sigma^{11}_{\mu\nu} - \sigma^{11}_{\mu,\lambda_\nu}\right).\end{aligned} \tag{31}$$

for sufficiently large μ,ν. Considering that $(\sigma^{11}_{\mu\nu}(u))$ is P-convergent to ℓ and

$$\lim_{\mu\to\infty}\frac{\lambda_\mu+1}{\lambda_\mu-\mu} = \frac{\lambda}{\lambda-1} \quad \text{and} \quad \lim_{\mu\to\infty}\frac{\mu+1}{\lambda_\mu-\mu} = \frac{1}{\lambda-1}, \tag{32}$$

if we take the lim inf of both sides of equality (31) as $\mu,\nu\to\infty$, then we reach

$$\begin{aligned}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{>,11}_{\mu\nu}\right) &\geq \left(\frac{\lambda}{\lambda-1}\right)^2\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\sigma^{11}_{\mu\nu}+\sigma^{11}_{\mu,\lambda_\nu}\right)\\ &+\frac{1}{\lambda-1}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu\nu}-\sigma^{11}_{\lambda_\mu,\nu}\right)\\ &+\frac{\lambda}{\lambda-1}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu\nu}-\sigma^{11}_{\mu,\lambda_\nu}\right).\end{aligned}$$

If we take the limit of both sides of last inequality as $\lambda\to1^+$, then we find

$$\lim_{\lambda\to1^+}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\tau^{>,11}_{\mu\nu}\right)\geq 0.$$

In the same vein, for $\lambda>1$, we have from Lemma 3.3 (*i*)

$$\begin{aligned}\sigma^{11}_{\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu} &= \frac{(\lambda_\mu+1)(\lambda_\nu+1)}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\left(\sigma^{11}_{\lambda_\mu,\nu}-\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\sigma^{11}_{\mu\nu}+\sigma^{11}_{\mu,\lambda_\nu}\right)\\ &+\frac{\lambda_\mu+1}{\lambda_\mu-\mu}\left(\sigma^{11}_{\mu\nu}-\sigma^{11}_{\lambda_\mu,\nu}\right)+\frac{\nu+1}{\lambda_\nu-\nu}\left(\sigma^{11}_{\mu\nu}-\sigma^{11}_{\mu,\lambda_\nu}\right)\end{aligned}\tag{33}$$

for sufficiently large μ,ν. Considering that $(\sigma^{11}_{\mu\nu}(u))$ is P-convergent to ℓ and limits in (32) exist, if we take the lim inf of both sides of equality (33) as $\mu,\nu\to\infty$, then we reach

$$\begin{aligned}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right) &\geq \left(\frac{\lambda}{\lambda-1}\right)^2\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\sigma^{11}_{\mu\nu}+\sigma^{11}_{\mu,\lambda_\nu}\right)\\ &+\frac{\lambda}{\lambda-1}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu\nu}-\sigma^{11}_{\lambda_\mu,\nu}\right)\\ &+\frac{1}{\lambda-1}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu\nu}-\sigma^{11}_{\mu,\lambda_\nu}\right).\end{aligned}$$

If we take the limit of both sides of last inequality as $\lambda\to1^+$, then we find

$$\lim_{\lambda\to1^+}\liminf_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)\geq 0.$$

In addition to what is attained above, for $\lambda>1$, we have from Lemma 3.3 (*i*)

$$\begin{aligned}\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu} &= \frac{(\mu+1)(\lambda_\nu+1)}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\left(\sigma^{11}_{\mu,\lambda_\nu}-\sigma^{11}_{\lambda_\mu,\lambda_\nu}\right)\\ &+\frac{(\lambda_\mu+1)(\nu+1)}{(\lambda_\mu-\mu)(\lambda_\nu-\nu)}\left(\sigma^{11}_{\lambda_\mu,\nu}-\sigma^{11}_{\mu\nu}\right)\\ &+\frac{\nu+1}{\lambda_\nu-\nu}\left(\sigma^{11}_{\mu\nu}-\sigma^{11}_{\lambda_\mu,\lambda_\nu}\right)\end{aligned}\tag{34}$$

sufficiently large μ,ν. Considering that $(\sigma^{11}_{\mu\nu}(u))$ is P-convergent to ℓ and limits in (32) exist, if we take the lim sup of both sides of equality (34) as $\mu,\nu\to\infty$, then we reach

$$\begin{aligned}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right) &\leq \frac{\lambda}{(\lambda-1)^2}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu,\lambda_\nu}-\sigma^{11}_{\lambda_\mu,\lambda_\nu}\right)\\ &+\frac{\lambda}{(\lambda-1)^2}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\nu}-\sigma^{11}_{\mu\nu}\right)\\ &+\frac{1}{\lambda-1}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\mu\nu}-\sigma^{11}_{\lambda_\mu,\lambda_\nu}\right).\end{aligned}$$

If we take the limit of both sides of last inequality as $\lambda \to 1^+$, then we find

$$\lim_{\lambda\to 1^+}\limsup_{\mu,\nu\to\infty}\left(\sigma^{11}_{\lambda_\mu,\lambda_\nu}-\tau^{>,11}_{\mu\nu}\right)\leq 0.$$

Thus, we can declare that conditions in (18) are verified. Following a similar procedure to above for $0<\lambda<1$, we can observe that conditions in (19) are also verified. In that case, we obtain from Lemma 3.4 that $(V^{11^{(0)}}_{\mu\nu}(\Delta_{11}u))$ is P-convergent to 0. Therefore, we conclude from the double Kronecker identity that $(u_{\mu\nu})$ is P-convergent to ℓ. □

4. Some Results for the $(C,1,0)$ Summable Double Sequences

This section essentially consists of two parts. In the first part, we present some lemmas to be benefited in the proofs of main results of this section for double sequences. In the second part, we discuss various Tauberian conditions which pave the way for a Tauberian conclusion from the $(C,1,0)$ summability to convergence for double sequences. In the sequel, we end this section by some corollaries.

4.1. Lemmas

In this subsection, we express and prove the following assertions to be utilized in the proofs of main results of this section for double sequences. The following lemma presents two representations of difference between the general terms of double sequences $(u_{\mu\nu})$ and $(\sigma^{10}_{\mu\nu}(u))$ by means of the de la Vallée Poussin means of the sequence $(u_{\mu\nu})$ in sense $(1,0)$.

Lemma 4.1. *Let $u=(u_{\mu\nu})$ be a double sequence.*

(i) For $\lambda>1$ and sufficiently large μ,ν, we have

$$u_{\mu\nu}-\sigma^{10}_{\mu\nu}(u)=-\frac{\lambda_\mu+1}{\mu+1}\left(\sigma^{10}_{\lambda_\mu,\nu}-\tau^{>,10}_{\mu\nu}\right)-\left(\tau^{>,10}_{\mu\nu}-u_{\mu\nu}\right). \tag{35}$$

(ii) For $0<\lambda<1$ and sufficiently large μ,ν, we have

$$u_{\mu\nu}-\sigma^{10}_{\mu\nu}(u)=-\frac{\lambda_\mu+1}{\mu+1}\left(\sigma^{10}_{\lambda_\mu,\nu}-\tau^{<,10}_{\mu\nu}\right)+\left(u_{\mu\nu}-\tau^{<,10}_{\mu\nu}\right). \tag{36}$$

Proof. (i) For $\lambda>1$, we have

$$\begin{aligned}\tau^{>,10}_{\mu\nu}(u) &= \frac{1}{\lambda_\mu-\mu}\sum_{i=\mu+1}^{\lambda_\mu}u_{i\nu}=\frac{1}{\lambda_\mu-\mu}\left(\sum_{i=0}^{\lambda_\mu}-\sum_{i=0}^{\mu}\right)u_{i\nu}\\ &= \frac{\lambda_\mu+1}{\lambda_\mu-\mu}\sigma^{10}_{\lambda_\mu,\nu}-\frac{\mu+1}{\lambda_\mu-\mu}\sigma^{10}_{\mu,\nu}\end{aligned} \tag{37}$$

for sufficiently large μ,ν. It follows from equation (37) that

$$\begin{aligned}-\sigma^{10}_{\mu\nu}(u) &= -\frac{\lambda_\mu+1}{\mu+1}\sigma^{10}_{\lambda_\mu,\nu}+\frac{\lambda_\mu-\mu}{\mu+1}\tau^{>,10}_{\mu\nu}\\ &= -\frac{\lambda_\mu+1}{\mu+1}\sigma^{10}_{\lambda_\mu,\nu}+\left[\frac{\lambda_\mu+1}{\mu+1}-1\right]\tau^{>,10}_{\mu\nu}\\ &= -\frac{\lambda_\mu+1}{\mu+1}\left(\sigma^{10}_{\lambda_\mu,\nu}-\tau^{>,10}_{\mu\nu}\right)-\tau^{>,10}_{\mu\nu}\end{aligned} \tag{38}$$

for sufficiently large μ,ν. If we add the term $u_{\mu\nu}$ to both sides of equality (38), we complete the proof of (i).
(ii) This is similar to the proof of part (i) of Lemma 4.1. □

In the next lemma being a consequence of Lemma 3.2, the difference between the general terms of double sequences $(u_{\mu\nu})$ and $(\sigma^{10}_{\mu\nu}(u))$ interprets differently from the statement given in Lemma 4.1.

Lemma 4.2. *([11], Lemma 2) Let $u = (u_{\mu\nu})$ be a double sequence.*

(i) For $\lambda > 1$ and sufficiently large μ, ν, we have

$$u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u) = \frac{\lambda_\mu + 1}{\lambda_\mu - \mu}\left(\sigma^{10}_{\lambda_\mu,\nu} - \sigma^{10}_{\mu\nu}\right) - \frac{1}{\lambda_\mu - \mu}\sum_{i=\mu+1}^{\lambda_\mu}\left(u_{i\nu} - u_{\mu\nu}\right).$$

(ii) For $0 < \lambda < 1$ and sufficiently large μ, ν, we have

$$u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u) = \frac{\lambda_\mu + 1}{\mu - \lambda_\mu}\left(\sigma^{10}_{\mu\nu} - \sigma^{10}_{\lambda_\mu,\nu}\right) + \frac{1}{\mu - \lambda_\mu}\sum_{i=\lambda_\mu+1}^{\mu}\left(u_{\mu\nu} - u_{i\nu}\right).$$

Lemma 4.3. *([19], Lemma 4) Let $u = (u_{\mu\nu})$ be a double sequence.*

(i) For $\lambda > 1$ and sufficiently large μ, ν, we have

$$\tau^{>,10}_{\mu\nu}(u) - \sigma^{10}_{\mu\nu}(u) = \frac{\lambda_\mu + 1}{\lambda_\mu - \mu}\left(\sigma^{10}_{\lambda_\mu,\nu} - \sigma^{10}_{\mu\nu}\right).$$

(ii) For $0 < \lambda < 1$ and sufficiently large μ, ν, we have

$$\tau^{<,10}_{\mu\nu}(u) - \sigma^{10}_{\mu\nu}(u) = \frac{\lambda_\mu + 1}{\mu - \lambda_\mu}\left(\sigma^{10}_{\mu\nu} - \sigma^{10}_{\lambda_\mu,\nu}\right).$$

In [5], Çanak proved that a generator sequence is convergent under some suitable conditions. Inspiring this theorem given for the single sequences, we indicate under which conditions a double generator sequence of $(u_{\mu\nu})$ in sense $(1, 0)$ is P-convergent.

Lemma 4.4. *For a double sequence $u = (u_{\mu\nu})$ of real numbers, let the assumptions*

$$\lim_{\lambda\to1^+}\liminf_{\mu,\nu\to\infty}\left(\sigma^{10}_{\lambda_\mu,\nu} - \tau^{>,10}_{\mu\nu}\right) \geq 0 \quad \text{and} \quad \lim_{\lambda\to1^-}\limsup_{\mu,\nu\to\infty}\left(\sigma^{10}_{\lambda_\mu,\nu} - \tau^{<,10}_{\mu\nu}\right) \leq 0 \tag{39}$$

hold. If the conditions

$$\lim_{\lambda\to1^+}\liminf_{\mu,\nu\to\infty}\left(\tau^{>,10}_{\mu\nu} - u_{\mu\nu}\right) \geq 0 \tag{40}$$

and

$$\lim_{\lambda\to1^-}\liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu} - \tau^{<,10}_{\mu\nu}\right) \geq 0 \tag{41}$$

are satisfied, then the generator sequence $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is P-convergent to 0.

Proof. Assume that conditions (39)-(41) are satisfied. In order to prove that $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is P-convergent to 0, we examine difference between the general terms of sequences $(u_{\mu\nu})$ and $(\sigma^{10}_{\mu\nu}(u))$ in two cases $\lambda > 1$ and $0 < \lambda < 1$. We firstly consider the case $\lambda > 1$. If we take the lim sup of both sides of equality (35) as $\mu, \nu \to \infty$, then we obtain that for each $\lambda > 1$

$$\begin{aligned}\limsup_{\mu,\nu\to\infty}\left(u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u)\right) &\leq \limsup_{\mu,\nu\to\infty}\frac{\lambda_\mu + 1}{\mu + 1}\limsup_{\mu,\nu\to\infty}\left(-\left(\sigma^{10}_{\lambda_\mu,\nu} - \tau^{>,10}_{\mu\nu}\right)\right) + \limsup_{\mu,\nu\to\infty}\left(-\left(\tau^{>,10}_{\mu\nu} - u_{\mu\nu}\right)\right)\\ &= -\lambda\liminf_{\mu,\nu\to\infty}\left(\sigma^{10}_{\lambda_\mu,\nu} - \tau^{>,10}_{\mu\nu}\right) - \liminf_{\mu,\nu\to\infty}\left(\tau^{>,10}_{\mu\nu} - u_{\mu\nu}\right)\end{aligned}$$

because of that

$$\lim_{\mu\to\infty} \frac{\lambda_\mu + 1}{\mu + 1} = \lambda. \tag{42}$$

If we take the limit of both sides of last inequality as $\lambda \to 1^+$, we get

$$\limsup_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u)\right) \leq -\lim_{\lambda\to 1^+} \liminf_{\mu,\nu\to\infty} \left(\sigma^{10}_{\lambda_\mu,\nu} - \tau^{>,10}_{\mu\nu}\right) - \lim_{\lambda\to 1^+} \liminf_{\mu,\nu\to\infty} \left(\tau^{>,10}_{\mu\nu} - u_{\mu\nu}\right).$$

From assumptions in (39) and (40), it follows that

$$\limsup_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u)\right) \leq 0. \tag{43}$$

On the other hand, we consider the case $0 < \lambda < 1$. If we take the lim inf of both sides of equality (36) as $\mu, \nu \to \infty$, then we obtain that for each $0 < \lambda < 1$

$$\begin{aligned}
\liminf_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u)\right) &\geq \liminf_{\mu,\nu\to\infty} \frac{\lambda_\mu + 1}{\mu + 1} \liminf_{\mu,\nu\to\infty} \left(-\left(\sigma^{10}_{\lambda_\mu,\nu} - \tau^{<,10}_{\mu\nu}\right)\right) + \liminf_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \tau^{<,10}_{\mu\nu}\right) \\
&= -\lambda \limsup_{\mu,\nu\to\infty} \left(\sigma^{10}_{\lambda_\mu,\nu} - \tau^{<,10}_{\mu\nu}\right) + \liminf_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \tau^{<,10}_{\mu\nu}\right)
\end{aligned}$$

because of limit in (42). If we take the limit of both sides of last inequality as $\lambda \to 1^-$, we get

$$\liminf_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u)\right) \geq -\lim_{\lambda\to 1^-} \limsup_{\mu,\nu\to\infty} \left(\sigma^{10}_{\lambda_\mu,\nu} - \tau^{<,10}_{\mu\nu}\right) + \lim_{\lambda\to 1^-} \liminf_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \tau^{<,10}_{\mu\nu}\right).$$

From assumptions in (39) and (41), it follows that

$$\liminf_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u)\right) \geq 0. \tag{44}$$

If we combine inequalities (43) with (44), we conclude

$$\lim_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \sigma^{10}_{\mu\nu}(u)\right) = 0$$

which means by the double Kronecker identity that $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is P-convergent to 0. □

4.2. Main Results

In this subsection, we prove a Tauberian theorem for double sequences that P-convergence follows from the $(C, 1, 0)$ summability under the conditions of slow decrease of the generator sequence $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ in sense $(1, 0)$ and we present some corollaries related to this theorem. In the sequel, we end this part by giving some Tauberian conditions for the $(C, 1, 0)$ summability method.

Theorem 4.5. *Let a double sequence $(u_{\mu\nu})$ be $(C, 1, 0)$ summable to a number ℓ. If the generator sequence $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is slowly decreasing in sense $(1, 0)$, then $(u_{\mu\nu})$ is P-convergent to ℓ.*

Proof. Suppose that a double sequence $(u_{\mu\nu})$ is $(C, 1, 0)$ summable to ℓ and $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is slowly decreasing in sense $(1, 0)$. In order to prove that $(u_{\mu\nu})$ is P- convergent to same number, we firstly indicate that $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is P-convergent to 0. Since $(\sigma^{10}_{\mu\nu}(u))$ is P-convergent to ℓ and the $(C, 1, 0)$ summability method is regular, we obtain that $(\sigma^{10^{(2)}}_{\mu\nu}(u))$ is also convergent to same number. It follows from the double Kronecker identity that $(V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u))$ is P-convergent to 0. For $\lambda > 1$, if we replace $u_{\mu\nu}$ by $V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u)$ in Lemma 4.2 (*i*), we obtain

$$\begin{aligned} V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u) - V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u) &= \frac{\lambda_\mu + 1}{\lambda_\mu - \mu}\left(V^{10^{(1)}}_{\lambda_\mu,\nu} - V^{10^{(1)}}_{\mu\nu}\right) - \frac{1}{\lambda_\mu - \mu}\sum_{i=\mu+1}^{\lambda_\mu}\left(V^{10^{(0)}}_{i\nu} - V^{10^{(0)}}_{\mu\nu}\right) \\ &\leq \frac{\lambda_\mu + 1}{\lambda_\mu - \mu}\left(V^{10^{(1)}}_{\lambda_\mu,\nu} - V^{10^{(1)}}_{\mu\nu}\right) - \min_{\mu<i\leq\lambda_\mu}\left(V^{10^{(0)}}_{i\nu} - V^{10^{(0)}}_{\mu\nu}\right). \end{aligned} \tag{45}$$

for sufficiently large μ, ν. Considering that $(V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u))$ is P-convergent to 0 and

$$\lim_{\mu\to\infty}\frac{\lambda_\mu + 1}{\lambda_\mu - \mu} = \frac{\lambda}{\lambda - 1}$$

if we take the lim sup of both sides of inequality (45) as $\mu, \nu \to \infty$, then we reach

$$\limsup_{\mu,\nu\to\infty}\left(V^{10^{(0)}}_{\mu\nu}(\Delta_0 u) - V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u)\right) \leq -\liminf_{\mu,\nu\to\infty}\min_{\mu<i\leq\lambda_\mu}\left(V^{10^{(0)}}_{i\nu} - V^{10^{(0)}}_{\mu\nu}\right).$$

If we take the limit of both sides of last inequality as $\lambda \to 1^+$, then we find

$$\limsup_{\mu,\nu\to\infty}\left(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u) - V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u)\right) \leq -\lim_{\lambda\to1^+}\liminf_{\mu,\nu\to\infty}\min_{\mu<i\leq\lambda_\mu}\left(V^{10^{(0)}}_{i\nu} - V^{10^{(0)}}_{\mu\nu}\right) \leq 0 \tag{46}$$

due to the fact that $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is slowly decreasing in sense $(1, 0)$. Following a similar procedure to above for $0 < \lambda < 1$, if we replace $u_{\mu\nu}$ by $V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u)$ in Lemma 4.2 (*ii*), we obtain

$$\begin{aligned} V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u) - V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u) &= \frac{\lambda_\mu + 1}{\mu - \lambda_\mu}\left(V^{10^{(1)}}_{\mu\nu} - V^{10^{(1)}}_{\lambda_\mu,\nu}\right) + \frac{1}{\mu - \lambda_\mu}\sum_{i=\lambda_\mu+1}^{\mu}\left(V^{10^{(0)}}_{\mu\nu} - V^{10^{(0)}}_{i\nu}\right) \\ &\geq \frac{\lambda_\mu + 1}{\mu - \lambda_\mu}\left(V^{10^{(1)}}_{\mu\nu} - V^{10^{(1)}}_{\lambda_\mu,\nu}\right) + \min_{\lambda_\mu<i\leq\mu}\left(V^{10^{(0)}}_{\mu\nu} - V^{10^{(0)}}_{i\nu}\right). \end{aligned} \tag{47}$$

for sufficiently large μ, ν. Considering that $(V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u))$ is P-convergent to 0 and

$$\lim_{\mu\to\infty}\frac{\lambda_\mu + 1}{\mu - \lambda_\mu} = \frac{\lambda}{1 - \lambda}$$

if we take the lim inf of both sides of inequality (47) as $\mu, \nu \to \infty$, then we reach

$$\liminf_{\mu,\nu\to\infty}\left(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u) - V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u)\right) \geq \liminf_{\mu,\nu\to\infty}\min_{\mu<i\leq\lambda_\mu}\left(V^{10^{(0)}}_{\mu\nu} - V^{10^{(0)}}_{i\nu}\right).$$

If we take the limit of both sides of last inequality as $\lambda \to 1^-$, then we find

$$\liminf_{\mu,\nu\to\infty}\left(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u) - V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u)\right) \geq \lim_{\lambda\to1^-}\liminf_{\mu,\nu\to\infty}\min_{\mu<i\leq\lambda_\mu}\left(V^{10^{(0)}}_{\mu\nu} - V^{10^{(0)}}_{i\nu}\right) \geq 0 \tag{48}$$

due to the fact that $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is slowly decreasing in sense $(1, 0)$. If we combine inequalities (46) with (48), we conclude

$$\lim_{\mu,\nu\to\infty}V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u) = \lim_{\mu,\nu\to\infty}V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u).$$

Because $(V^{10^{(1)}}_{\mu\nu}(\Delta_{10}u))$ is P- convergent to 0, $(V^{10^{(0)}}_{\mu\nu}(\Delta_{10}u))$ is also P-convergent to 0, which means from the double Kronecker identity that $(u_{\mu\nu})$ is P-convergent to ℓ. $\square$

In consideration of Theorem 4.5, we can state the following corollary.

Corollary 4.6. *Let a double sequence $(u_{\mu\nu})$ be $(C, 1, 0)$ summable to a number ℓ. If condition*

$$\mu\Delta_{10}V_{\mu\nu}^{10^{(0)}}(\Delta_{10}u) \geq -C$$

is satisfied for some $C \geq 0$, then $(u_{\mu\nu})$ is P-convergent to ℓ.

Analogous results for double sequences of complex numbers can be formulated as follow.

Theorem 4.7. *Let a double sequence $(u_{\mu\nu})$ be $(C, 1, 0)$ summable to a number ℓ. If its generator sequence $(V_{\mu\nu}^{10^{(0)}}(\Delta_{10}u))$ is slowly oscillating in sense $(1, 0)$, then $(u_{\mu\nu})$ is P-convergent to ℓ.*

In consideration of Theorem 4.7, we can state the following corollary.

Corollary 4.8. *Let a double sequence $(u_{\mu\nu})$ be $(C, 1, 0)$ summable to a number ℓ. If condition*

$$|\mu\Delta_{10}V_{\mu\nu}^{10^{(0)}}(\Delta_{10}u)| \leq M$$

is satisfied for some $M \geq 0$, then $(u_{\mu\nu})$ is P-convergent to ℓ.

Before finishing this subsection, we examine some conditions needed for the $(C, 1, 0)$ summable double sequences to be convergent.

Theorem 4.9. *Let a double sequence $(u_{\mu\nu})$ be $(C, 1, 0)$ summable to a number ℓ. If conditions*

$$\lim_{\lambda\to 1^+} \liminf_{\mu,\nu\to\infty} \left(\tau_{\mu\nu}^{>,10} - u_{\mu\nu}\right) \geq 0 \tag{49}$$

and

$$\lim_{\lambda\to 1^-} \liminf_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \tau_{\mu\nu}^{<,10}\right) \geq 0 \tag{50}$$

are satisfied, then $(u_{\mu\nu})$ is P-convergent to ℓ.

Proof. Suppose that a double sequence $(u_{\mu\nu})$ is $(C, 1, 0)$ summable to ℓ and conditions (49) and (50) are satisfied. In order to prove that $(u_{\mu\nu})$ is P-convergent to same number, it is enough to indicate that conditions in (39) are verified. For $\lambda > 1$, we have from Lemma 3.3 (i)

$$\sigma_{\lambda_\mu,\nu}^{10} - \tau_{\mu\nu}^{>,10} = \frac{\mu+1}{\lambda_\mu - \mu}\left(\sigma_{\mu\nu}^{10} - \sigma_{\lambda_\mu,\nu}^{10}\right) \tag{51}$$

for sufficiently large μ, ν. Considering that $(\sigma_{\mu\nu}^{10}(u))$ is P-convergent to ℓ,

$$\lim_{\mu\to\infty} \frac{\lambda_\mu + 1}{\lambda_\mu - \mu} = \frac{\lambda}{\lambda - 1}, \tag{52}$$

if we take the lim inf of both sides of equality (51) as $\mu, \nu \to \infty$, then we reach

$$\liminf_{\mu,\nu\to\infty} \left(\sigma_{\lambda_\mu,\nu}^{10} - \tau_{\mu\nu}^{>,10}\right) \geq \frac{1}{\lambda - 1} \liminf_{\mu,\nu\to\infty} \left(\sigma_{\mu\nu}^{10} - \sigma_{\lambda_\mu,\nu}^{10}\right).$$

If we take the limit of both sides of last inequality as $\lambda \to 1^+$, then we find

$$\lim_{\lambda\to 1^+} \liminf_{\mu,\nu\to\infty} \left(\sigma_{\lambda_\mu,\nu}^{10} - \tau_{\mu\nu}^{>,10}\right) \geq 0.$$

Thus, we can declare that the first of conditions in (39) is verified. Following a similar procedure to above for $0 < \lambda < 1$, we can observe that second one is also verified. In that case, we obtain from Lemma 4.4 that $(V_{\mu\nu}^{10^{(0)}}(\Delta_{10}u))$ is P-convergent to 0. Therefore, we conclude from the double Kronecker identity that $(u_{\mu\nu})$ is P-convergent to ℓ. $\square$

5. Some Results for the $(C, 0, 1)$ Summable Double Sequences

This section essentially consists of two parts. In the first part, we present some lemmas to be benefited in the proofs of main results of this section for double sequences. In the second part, we discuss various Tauberian conditions which pave the way for a Tauberian conclusion from the $(C, 0, 1)$ summability to convergence for double sequences. In the sequel, we end this section by some corollaries.

5.1. Lemmas

In this subsection, we express and prove the following assertions to be utilized in the proofs of main results of this section for double sequences. The following lemma presents two representations of difference between the general terms of double sequences $(u_{\mu\nu})$ and $(\sigma^{01}_{\mu\nu}(u))$ by the aid of the de la Vallée Poussin means of the sequence $(u_{\mu\nu})$ in sense $(0, 1)$.

Lemma 5.1. *Let $u = (u_{\mu\nu})$ be a double sequence.*

(i) For $\lambda > 1$ and sufficiently large μ, ν, we have

$$u_{\mu\nu} - \sigma^{01}_{\mu\nu}(u) = -\frac{\lambda_\nu + 1}{\nu + 1}\left(\sigma^{01}_{\mu,\lambda_\nu} - \tau^{>,01}_{\mu\nu}\right) - \left(\tau^{>,01}_{\mu\nu} - u_{\mu\nu}\right). \tag{53}$$

(ii) For $0 < \lambda < 1$ and sufficiently large μ, ν, we have

$$u_{\mu\nu} - \sigma^{01}_{\mu\nu}(u) = -\frac{\lambda_\nu + 1}{\nu + 1}\left(\sigma^{01}_{\mu,\lambda_\nu} - \tau^{<,01}_{\mu\nu}\right) + \left(u_{\mu\nu} - \tau^{<,01}_{\mu\nu}\right). \tag{54}$$

Proof. This is similar to the proof of parts (*i*) and (*ii*) of Lemma 4.1. □

In the next lemma being a consequence of Lemma 3.2, the difference between the general terms of double sequences $(u_{\mu\nu})$ and $(\sigma^{01}_{\mu\nu}(u))$ interprets differently from the statement given in Lemma 5.1.

Lemma 5.2. *([19], Lemma 5) Let $u = (u_{\mu\nu})$ be a double sequence.*

(i) For $\lambda > 1$ and sufficiently large μ, ν, we have

$$u_{\mu\nu} - \sigma^{01}_{\mu\nu}(u) = \frac{\lambda_\nu + 1}{\lambda_\nu - \nu}\left(\sigma^{01}_{\mu,\lambda_\nu} - \sigma^{01}_{\mu\nu}\right) - \frac{1}{\lambda_\nu - \nu}\sum_{j=\nu+1}^{\lambda_\nu}\left(u_{\mu j} - u_{\mu\nu}\right).$$

(ii) For $0 < \lambda < 1$ and sufficiently large μ, ν, we have

$$u_{\mu\nu} - \sigma^{01}_{\mu\nu}(u) = \frac{\lambda_\nu + 1}{\nu - \lambda_\nu}\left(\sigma^{01}_{\mu\nu} - \sigma^{01}_{\mu,\lambda_\nu}\right) + \frac{1}{\nu - \lambda_\nu}\sum_{j=\lambda_\nu+1}^{\nu}\left(u_{\mu\nu} - u_{\mu j}\right).$$

Lemma 5.3. *([19], Lemma 4) Let $u = (u_{\mu\nu})$ be a double sequence.*

(i) For $\lambda > 1$ and sufficiently large μ, ν, we have

$$\tau^{>,01}_{\mu\nu} - \sigma^{01}_{\mu\nu}(u) = \frac{\lambda_\nu + 1}{\lambda_\nu - \nu}\left(\sigma^{01}_{\mu,\lambda_\nu} - \sigma^{01}_{\mu\nu}\right).$$

(ii) For $0 < \lambda < 1$ and sufficiently large μ, ν, we have

$$\tau^{<,01}_{\mu\nu} - \sigma^{01}_{\mu\nu}(u) = \frac{\lambda_\nu + 1}{\nu - \lambda_\nu}\left(\sigma^{01}_{\mu\nu} - \sigma^{01}_{\mu,\lambda_\nu}\right).$$

In [5], Çanak proved that a generator sequence is convergent under some suitable conditions. Inspiring this theorem given for the single sequences, we indicate under which conditions a double generator sequence of $(u_{\mu\nu})$ in sense $(0,1)$ is P-convergent.

Lemma 5.4. *For a double sequence $u = (u_{\mu\nu})$ of real numbers, let the assumptions*

$$\lim_{\lambda\to 1^+}\liminf_{\mu,\nu\to\infty}\left(\sigma^{01}_{\mu,\lambda_\nu} - \tau^{>,01}_{\mu\nu}\right) \geq 0 \quad \text{and} \quad \lim_{\lambda\to 1^-}\limsup_{\mu,\nu\to\infty}\left(\sigma^{01}_{\mu,\lambda_\nu} - \tau^{<,01}_{\mu\nu}\right) \leq 0 \tag{55}$$

hold. If the conditions

$$\lim_{\lambda\to 1^+}\liminf_{\mu,\nu\to\infty}\left(\tau^{>,01}_{\mu\nu} - u_{\mu\nu}\right) \geq 0 \tag{56}$$

and

$$\lim_{\lambda\to 1^-}\liminf_{\mu,\nu\to\infty}\left(u_{\mu\nu} - \tau^{<,01}_{\mu\nu}\right) \geq 0 \tag{57}$$

are satisfied, then the generator sequence $(V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u))$ is P-convergent to 0.

Proof. This is similar to the proof of Lemma 4.4. □

5.2. Main Results

In this subsection, we prove a Tauberian theorem for double sequences that P-convergence follows from the $(C,0,1)$ summability under the conditions of slow decrease of the generator sequence $(V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u))$ in sense $(0,1)$ and we present some corollaries related to this theorem. In the sequel, we end this part by giving some Tauberian conditions for the $(C,0,1)$ summability method.

Theorem 5.5. *Let a double sequence $(u_{\mu\nu})$ be $(C,0,1)$ summable to a number ℓ. If its generator sequence $(V^{01^0}_{\mu\nu}(\Delta_{01}u))$ is slowly decreasing in sense $(0,1)$, then $(u_{\mu\nu})$ is P-convergent to ℓ.*

Proof. This is similar to the proof of Theorem 4.5. □

In consideration of Theorem 5.5, we can state the following corollary.

Corollary 5.6. *Let a double sequence $(u_{\mu\nu})$ be $(C,0,1)$ summable to a number ℓ. If condition*

$$\mu\Delta_{01}V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u) \geq -C$$

is satisfied for some $C \geq 0$, then $(u_{\mu\nu})$ is P-convergent to ℓ.

Analogous results for double sequences of complex numbers can be formulated as follow.

Theorem 5.7. *Let a bounded double sequence $(u_{\mu\nu})$ be $(C,0,1)$ summable to a number ℓ. If its generator sequence $(V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u))$ is slowly oscillating in sense $(0,1)$, then $(u_{\mu\nu})$ is P-convergent to ℓ.*

In consideration of Theorem 5.7, we can state the following corollary.

Corollary 5.8. *Let a double sequence $(u_{\mu\nu})$ be $(C,0,1)$ summable to a number ℓ. If condition*

$$|\mu\Delta_{01}V^{01^{(0)}}_{\mu\nu}(\Delta_{01}u)| \leq M$$

is satisfied for some $M \geq 0$, then $(u_{\mu\nu})$ is P-convergent to ℓ.

Before finishing this subsection, we examine some conditions needed for the $(C,0,1)$ summable double sequences to be convergent.

Theorem 5.9. *Let a double sequence $(u_{\mu\nu})$ be $(C, 0, 1)$ summable to a number ℓ. If conditions*

$$\lim_{\lambda\to 1^+} \liminf_{\mu,\nu\to\infty} \left(\tau^{>,01}_{\mu\nu} - u_{\mu\nu}\right) \geq 0 \tag{58}$$

and

$$\lim_{\lambda\to 1^-} \liminf_{\mu,\nu\to\infty} \left(u_{\mu\nu} - \tau^{<,01}_{\mu\nu}\right) \geq 0 \tag{59}$$

are satisfied, then $(u_{\mu\nu})$ is P-convergent to ℓ.

Proof. This is similar to the proof of Theorem 4.9. $\square$

6. Conclusion

In this paper, we have obtained several Tauberian conditions in terms of the generator sequences in certain senses under which P-convergence of a double sequences follows from its $(C, 1, 1)$ summability. Similar results have been given for $(C, 1, 0)$ and $(C, 0, 1)$ summability methods. In a forthcoming work, we plan to obtain analogous results for the weighted mean of double sequences.

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References

[1] R. P. Agnew, On summability of multiple sequences, Amer. J. Math. 1 (4) (1934) 62–68.
[2] C. Belen, Some Tauberian theorems for weighted means of bounded double sequences, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.). 63 (1) (2017) 115–122.
[3] T. J. I'A. Bromwich, An introduction to the theory of infinite series, Chelsea Publishing, Rhode Island, 1991.
[4] H. Çakalli, R. F. Patterson, Functions preserving slowly oscillating double sequences, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.). 62 (2) (2016) 531–536.
[5] İ. Çanak, A theorem for convergence of generator sequences, Comput. Math. Appl. 61 (2) (2011) 408–411.
[6] C. -P. Chen and J. -M. Hsu, Tauberian theorems for weighted means of double sequences, Anal. Math. 26 (4) (2000) 243–262.
[7] O. H. H. Edely, M. Mursaleen, Tauberian theorems for statistically convergent double sequences, Inform. Sci. 176 (7) (2006) 875–886.
[8] G. Fındık, İ. Çanak, Ü. Totur, An alternative proof of the generalized Littlewood Tauberian theorem for Cesàro summable double sequences, Tbilisi Math. J. 12 (2019) 131–148.
[9] G. H. Hardy, On the convergence of certain multiple series, Proc. London Math. Soc.(2). 1 (1904) 124–128.
[10] K. Knopp, Limitierungs-Umkehrsätze für Doppelfolgen, Math. Z. 45 (1939) 573–589.
[11] F. Móricz, Tauberian theorems for Cesàro summable double sequences, Studia Math. 110 (1) (1994) 83–96.
[12] M. Mursaleen, O. H. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288 (1) (2003) 223–231.
[13] Z. Önder, İ. Çanak, Tauberian theorems for statistically $(C, 1, 1)$ summable double sequences, Positivity. 23 (2019) 891–919.
[14] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, Math. Ann. 53 (3) (1900) 289–321.
[15] G. M. Robison, Divergent double sequences and series, Trans. Amer. Math. Soc. 28 (1) (1926) 50–73.
[16] E. Savaş, On certain double A-summability methods, Filomat. 30 (2) (2016) 395–401.
[17] R. Schmidt, Über divergente Folgen und lineare Mittelbildungen, Math. Z. 22 (1) (1925) 89–152.
[18] A. Tauber, Ein Satz aus der Theorie der unendlichen Reihen, Monatsh. Math. Phys. 8 (1) (1897) 273–277.
[19] Ü. Totur, Classical Tauberian theorems for the $(C, 1, 1)$ summability method, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.). 61 (2) (2015) 401–414.
[20] Ü. Totur, İ. Çanak, Tauberian theorems for the statistical convergence and the statistical $(C, 1, 1)$ summability, Filomat. 32 (1) (2018) 101–116.