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Value Distribution of Some Differential Monomials

Bikash Chakraborty[a], Sudip Saha[a], Amit Kumar Pal[b], Jayanta Kamila[a]

[a]*Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, West Bengal 700 118, India.*
[b]*Department of Mathematics, University of Kalyani, Kalyani, West Bengal 741 235, India.*

Abstract. Let f be a transcendental meromorphic function defined in the complex plane $\mathbb{C}$ and $k \in \mathbb{N}$. We consider the value distribution of the differential polynomial $f^{q_0}(f^{(k)})^{q_k}$, where $q_0(\geq 2), q_k(\geq 1)$ are integers. We obtain a quantitative estimation of the characteristic function $T(r, f)$ in terms of $\overline{N}\left(r, \frac{1}{f^{q_0}(f^{(k)})^{q_k}-1}\right)$.

Our result generalizes the results obtained by Xu et al. (Math. Inequal. Appl., Vol. 14, PP. 93-100, 2011); Karmakar and Sahoo (Results Math., Vol. 73, 2018) for a particular class of transcendental meromorphic functions.

1. Introduction

Throughout this paper, we assume that the readers are familiar with the standard notations of Nevanlinna theory ([3]). Also, we assume that f is a transcendental meromorphic function defined in the complex plane $\mathbb{C}$. It will be convenient to let that E denote any set of positive real numbers of finite linear (Lebesgue) measure, not necessarily same at each occurrence. For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \text{ as } r \to \infty,\ r \notin E.$$

Definition 1.1. *Let f be a non-constant meromorphic function. A meromorphic function $a(z)(\not\equiv 0, \infty)$ is called a "small function" with respect to f if $T(r, a(z)) = S(r, f)$.*

Definition 1.2. *Let f be non-constant meromorphic function defined in the complex plane $\mathbb{C}$, and k be a positive integer. We say*

$$M[f] = (f)^{q_0}(f')^{q_1}\ldots(f^{(k)})^{q_k}$$

is a differential monomial generated by f, where $q_0, q_1, \ldots, q_k$ are non-negative integers.

In this context, the terms $\mu := q_0 + q_1 + \ldots + q_k$ and $\mu_* := q_1 + 2q_2 + \ldots + kq_k$ are known as the degree and weight of the differential monomial respectively.

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Email addresses: bikashchakraborty.math@yahoo.com (Bikash Chakraborty), sudipsaha814@gmail.com (Sudip Saha), mail4amitpal@gmail.com (Amit Kumar Pal), kamilajayanta@gmail.com (Jayanta Kamila)

Definition 1.3. *([14]) Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer k, we denote*

i) by $N_{k)}(r, a; f)$ the counting function of a-points of f with multiplicity $\leq k$,
ii) by $N_{(k}(r, a; f)$ the counting function of a-points of f with multiplicity $\geq k$,

Similarly, the reduced counting functions $\overline{N}_{k)}(r, a; f)$ and $\overline{N}_{(k}(r, a; f)$ are defined.

Definition 1.4. *([8]) For a positive integer k, we denote $N_k(r, 0; f)$ the counting function of zeros of f, where a zero of f with multiplicity q is counted q times if $q \leq k$, and is counted k times if $q > k$.*

In 1959, Hayman proved the following theorem:

Theorem 1.1. *([4]) If f is a transcendental meromorphic function and $n \geq 3$, then $f^n f'$ assumes all finite values except possibly zero infinitely often.*

Moreover, Hayman ([4]) conjectured that the Theorem 1.1 remains valid for the cases $n = 1, 2$. In 1979, Mues ([10]) confirmed the Hayman's Conjecture for $n = 2$ and Chen and Fang ([2]) ensured the conjecture for $n = 1$ in 1995.

In 1992, Q. Zhang ([15]) gave the quantitative version of Mues's result as follows:

Theorem 1.2. *([15]) For a transcendental meromorphic function f, the following inequality holds :*

$$T(r, f) \leq 6N\left(r, \frac{1}{f^2 f' - 1}\right) + S(r, f).$$

In ([13]), Theorem 1.2 was improved by Xu and Yi as

Theorem 1.3. *([13]) Let f be a transcendental meromorphic function and $\phi(z)(\not\equiv 0)$ be a small function, then*

$$T(r, f) \leq 6\overline{N}\left(r, \frac{1}{\phi f^2 f' - 1}\right) + S(r, f).$$

Also, Huang and Gu ([5]) extended Theorem 1.2 by replacing f' by $f^{(k)}$, where $k(\geq 1)$ is an integer.

Theorem 1.4. *([5]) Let f be a transcendental meromorphic function and k be a positive integer. Then*

$$T(r, f) \leq 6N\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) + S(r, f).$$

A natural question was raised whether the above inequality still holds if the counting function in Theorem 1.4 is replaced by the corresponding reduced counting function. In this direction, in 2009, Xu, Yi and Zhang ([11]) proved the following theorem:

Theorem 1.5. *([11]) Let f be a transcendental meromorphic function, and $k(\geq 1)$ be a positive integer. If $N_1(r, 0; f) = S(r, f)$, then*

$$T(r, f) \leq 2\overline{N}\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) + S(r, f).$$

Later, in 2011, removing the restrictions on zeros of f, Xu, Yi and Zhang ([12]) proved the following theorem:

Theorem 1.6. *Let f be a transcendental meromorphic function, and $k(\geq 1)$ be a positive integer. Then*

$$T(r, f) \leq M\overline{N}\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) + S(r, f),$$

where M is 6 if $k = 1$, or $k \geq 3$ and $M = 10$ if $k = 2$.

Recently, Karmakar and Sahoo([7]) further improved the Theorem 1.6 and obtained the following result:

Theorem 1.7. *([7]) Let f be a transcendental meromorphic function, and $n(\geq 2)$, $k(\geq 1)$ be any integers, then*

$$T(r,f) \leq \frac{6}{2n-3}\overline{N}\left(r, \frac{1}{f^n f^{(k)} - 1}\right) + S(r,f).$$

From the above discussions the following question is obvious:

Question 1.1. *Is it possible to replace $f^n f^{(k)}$, where $n(\geq 2)$, $k(\geq 1)$ be any integers, in the above theorem by $(f)^{q_0}(f^{(k)})^{q_k}$, where $q_0(\geq 2), q_k(\geq 1)$ are integers?*

The aim of this paper is to answer above question by giving some restriction on the poles of f.

2. Main Results

Theorem 2.1. *Let f be a transcendental meromorphic function such that it has no simple pole. Also, let $q_0(\geq 2), q_k(\geq 1)$ are $(k \in \mathbb{N})$ integers. Then*

$$T(r,f) \quad \leq \quad \frac{6}{2q_0-3}\overline{N}\left(r, \frac{1}{(f)^{q_0}(f^{(k)})^{q_k} - 1}\right) + S(r,f).$$

Corollary 2.1. *Clearly, Theorem 2.1 generalise Theorem 1.7 for transcendental meromorphic function which has no simple pole.*

Remark 2.1. *Is it possible to remove the condition that "f has no simple pole" when $q_k \geq 2$?*

3. Lemmas

For a transcendental meromorphic function f, we define

$$M[f] := (f)^{q_0}(f')^{q_1} \dots (f^{(k)})^{q_k}.$$

In this paper, we assume that $q_0(\geq 1)$ and $q_k(\geq 1)$.

Lemma 3.1. *For a non constant meromorphic function g, we obtain*

$$N\left(r, \frac{g'}{g}\right) - N\left(r, \frac{g}{g'}\right) = \overline{N}(r,g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right).$$

Proof. The proof is same as the formula (12) of ([6]). □

Lemma 3.2. *Let f be a transcendental meromorphic function and $M[f]$ be a differential monomial in f, then*

$$T\Big(r, M[f]\Big) = O(T(r,f)) \;\; and \;\; S\Big(r, M[f]\Big) = S(r,f).$$

Proof. The proof is similar to the proof of the Lemma 2.4 of ([9]). □

Lemma 3.3. *([14]) Let f be a transcendental meromorphic function defined in the complex plane $\mathbb{C}$. Then*

$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = \infty.$$

Lemma 3.4. *Let $M[f]$ be differential monomial generated by a transcendental meromorphic function f. Then $M[f]$ is not identically constant.*

Proof. Here

$$\left(\frac{1}{f}\right)^{\mu}=\left(\frac{f'}{f}\right)^{q_1}\left(\frac{f''}{f}\right)^{q_2}\dots\left(\frac{f^{(k)}}{f}\right)^{q_k}\frac{1}{M[f]}.$$

Thus by the first fundamental theorem and lemma of logarithmic derivative, we have

$$\begin{aligned}
&\mu T(r,f)\\
\leq\ & \sum_{i=1}^{k} q_i N\left(r,\left(\frac{f^{(i)}}{f}\right)\right)+T(r,M[f])+S(r,f)\\
\leq\ & \sum_{i=1}^{k} iq_i\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}+T(r,M[f])+S(r,f)\\
\leq\ & \sum_{i=1}^{k} iq_i\left\{N(r,0;M[f])+N(r,\infty;M[f])\right\}+T(r,M[f])+S(r,f)\\
\leq\ & (2\mu_*+1)T(r,M[f])+S(r,f), \qquad (3.1)
\end{aligned}$$

Since f is a transcendental meromorphic function, so by Lemma 3.3 and inequality (3.1), $M[f]$ must be not identically constant. □

Lemma 3.5. *Let f be a transcendental meromorphic function and $M[f]$ be a differential monomial, given by $M[f]=(f)^{q_0}(f')^{q_1}\cdots(f^{(k)})^{q_k}$, where $q_0(\geq 2), q_1, q_2, \cdots, q_k$ (≥ 1) are $k(\geq 1)$ non negative integers. Let $g(z):=M[f]-1$, $h(z):=\frac{M'[f]}{f^{q_0-1}}$, and*

$$F(z):=2\left(\frac{g'(z)}{g(z)}\right)^2+3\left(\frac{g'(z)}{g(z)}\right)'-2\left(\frac{h'(z)}{h(z)}\right)'+\left(\frac{h'(z)}{h(z)}\right)^2-4\left(\frac{g'(z)h'(z)}{g(z)h(z)}\right). \qquad (3.2)$$

Then $F\not\equiv 0$.

Proof. On contrary, let us assume that $F\equiv 0$. Now,

$$M'[f]=g'=f^{q_0-1}h. \qquad (3.3)$$

Thus

$$N(r,0;f)\leq N(r,0;g'). \qquad (3.4)$$

Claim 1: First we claim that $g(z)\neq 0$.
Proof of Claim 1:
If z_1 is a zero of g of multiplicity l ($l\geq 1$), then $g(z_1)=M[f](z_1)-1=0$. Thus $f(z_1)\neq 0,\infty$. Now, we consider two cases :
Case -1.1 $l\geq 2$.

In this case, z_1 is a zero of h of order $l-1$. Using Laurent series expansion of F about z_1, one can see that z_1 is a pole of F of order 2 if the coefficient of $(z-z_1)^{-2}$ in F is non zero, i.e., if $(2l^2-3l+(l-1)^2+2(l-1)-4l(l-1))\neq 0$ for all l, i.e., the polynomial $-l^2+l-1$ has no real solution, which is true by the given condition. Thus z_1 is a pole of F, which contradicts the fact that $F\equiv 0$. Thus on our assumption $F\equiv 0$, $g(z)\neq 0$.
Case -1.2 $l=1$.

The equation (3.3) yields that $h(z_1)\neq 0$. In this case, the coefficient of $(z-z_1)^{-2}$ in F is (-1). Thus $z=z_1$ is a pole of F of order 2, which contradicts the fact that $F\equiv 0$.
Hence the claim is true, i.e., g has no zero.

Claim 2: Next we claim that $h(z)\neq 0$.

Proof of Claim 2:
Let z_2 be a zero of h of order m. Thus from equation (3.3), $M'[f](z_2) = 0$, i.e., $g'(z_2) = 0$. Hence $g(z_2) \neq \infty$. Also, by **Claim 1**, $g(z_2) \neq 0$. Now, we consider two cases :
Case -2.1 $m \geq 2$.

If $m \geq 2$, then z_2 is the zero of $h'(z)$ of order $(m-1)$. So by Laurent series expansion, one can see that the coefficient of $(z-z_2)^{-2}$ in F is (m^2+2m), which is non zero. Thus $z = z_2$ is a pole of F of order 2, which contradicts the fact that $F \equiv 0$.
Case -2.2 $m = 1$.

If $m = 1$, then the coefficient of $(z-z_2)^{-2}$ in F is 3, which again contradicts the fact that $F \equiv 0$.
Hence Claim 2 is true.

Claim 3: All zeros of $f(z)$ are simple.
Proof of Claim 3:
If z_3 be a zero of f of order ≥ 2, then by definition of h, $h(z_3) = 0$, which contradicts the **Claim 2**. Thus the Claim 3 is true.

Now, we define another function as $\phi(z) = \dfrac{h(z)}{g(z)}$. Then

$$\frac{g'}{g} = \phi f^{q_0-1}, \tag{3.5}$$

and

$$\frac{h'}{h} = \phi f^{q_0-1} + \frac{\phi'}{\phi}. \tag{3.6}$$

Clearly $\phi \not\equiv 0$, otherwise $\dfrac{g'}{g} \equiv 0$, which contradicts Lemma 3.4.

Claim 4: $\phi(z)$ is an entire function.
Proof of Claim 4:
As g and h has no zero, so poles of ϕ comes from the poles of h. Thus poles of ϕ comes from the poles of f. Again, zeros of ϕ comes from the poles of g, i.e., from poles of f.

Let z_4 be a pole of f of order t. Then z_4 is a pole of g of order $t\mu+\mu_*$ and pole of h of order $t\mu+\mu_*+1-t(q_0-1)$. Thus z_4 is a pole of ϕ of order $1-t(q_0-1)$ if $1-t(q_0-1) > 0$ and z_4 is a zero of ϕ of order $t(q_0-1)-1$ if $t(q_0-1)-1 > 0$.

As $q_0 \geq 2$, so ϕ is an entire function. Also, if $q_0 = 2$, then zeros of ϕ occur only at multiple poles of f and if $q_0 > 2$, then zeros of ϕ occur at only poles of f.

Next, in view of Lemma 3.4, we can write

$$\frac{1}{f^{\mu}} = \frac{g+1}{f^{\mu}} - \frac{g'}{f^{\mu}}\frac{g}{g'}. \tag{3.7}$$

and

$$\begin{aligned} \frac{\mu}{q_0-1}T(r,\phi) = \frac{\mu}{q_0-1}m(r,\phi) &= \frac{\mu}{q_0-1}m\left(r,\frac{g'}{g}\frac{1}{f^{q_0-1}}\right) \\ &\leq m\left(r,\frac{1}{f^{\mu}}\right) + S(r,f). \end{aligned} \tag{3.8}$$

Thus using Lemma 3.1, equation (3.7) and inequality (3.8), (3.4), we have

$$\begin{aligned}\frac{\mu}{q_0-1}T(r,\phi) &= \frac{\mu}{q_0-1}m(r,\phi)\\ &\le m\left(r,\frac{g}{g'}\right)+S(r,f)\\ &\le N\left(r,\frac{g'}{g}\right)-N\left(r,\frac{g}{g'}\right)+S(r,f)\\ &\le \overline{N}(r,g)+N\left(r,\frac{1}{g}\right)-N\left(r,\frac{1}{g'}\right)+S(r,f)\\ &\le \overline{N}(r,f)-N\left(r,\frac{1}{f}\right)+S(r,f)\end{aligned}$$

Again, using (3.7), we have

$$\mu\cdot m\left(r,\frac{1}{f}\right)\le \overline{N}(r,f)-N\left(r,\frac{1}{f}\right)+S(r,f)$$

i.e., $$\mu T(r,f)\le N(r,f)+(\mu-1)N\left(r,\frac{1}{f}\right)+S(r,f)$$

i.e., $$m(r,f)+(\mu-1)m\left(r,\frac{1}{f}\right)\le S(r,f).$$

Hence,

$$\begin{aligned}m(r,f) &= m\left(r,\frac{1}{f}\right)=S(r,f), &(3.9)\\ T(r,\phi) &= S(r,f). &(3.10)\end{aligned}$$

Next, we consider two cases :
Case-1 Assume $q_0>2$.
If z_4 is a pole of f of order t, then z_4 is a zero of ϕ of order $t(q_0-1)-1$. As $t(q_0-1)-1\ge 2t-1\ge t$, so

$$N(r,f)\le N(r,0,\phi) \tag{3.11}$$

Combining (3.9),(3.10) and (3.11), we get

$$T(r,f)=S(r,f), \tag{3.12}$$

which is absurd as f is a non constant transcendental meromorphic function. Thus our assumption is wrong. Hence $F\not\equiv 0$.
Case-2 Next, we assume that $q_0=2$.
Substituting (3.5) and (3.6) in (3.2) and using the fact that $F\equiv 0$, we obtain

$$f^2\phi^2+\left[2\left(\frac{\phi'}{\phi}\right)'-\left(\frac{\phi'}{\phi}\right)^2\right]+f\phi'-f'\phi\equiv 0. \tag{3.13}$$

From Lemma (3.4), it is clear that $\phi\not\equiv 0$. If z_5 is the zero of f, then $\phi(z_5)\neq 0,\infty$. Thus proceeding similarly as in Lemma 3 of ([5]), we can write

$$f^{(i)}(z_5)=\frac{l_{i1}(z_5)}{\phi(z_5)}, \tag{3.14}$$

where $l_{i1}(z)$ are the differential monomials in $\dfrac{\phi'}{\phi}$ for $i=1,2,\cdots,k$.

Since $g(z_5) = -1$ and $h(z_5) = q_0\Big((f')^{q_1+1}(f'')^{q_2}\cdots(f^{(k)})^{q_k}\Big)(z_5)$, so,

$$\phi(z_5) = -q_0\Big((f')^{q_1+1}(f'')^{q_2}\cdots(f^{(k)})^{q_k}\Big)(z_5). \tag{3.15}$$

Thus using (3.15) and (3.14), we have

$$\phi(z_5) = -q_0\Big(\frac{(l_{11})^{q_1+1}(l_{21})^{q_2}\cdots(l_{k1})^{q_k}}{(\phi)^{q_1+q_2+\cdots+q_k+1}}\Big)(z_5). \tag{3.16}$$

Next we define

$$G := \phi^{q_1+q_2+\cdots+q_k+2} + q_0\ (l_{11})^{q_1+1}(l_{21})^{q_2}\cdots(l_{k1})^{q_k}.$$

If $G \not\equiv 0$, then

$$N(r,0;f) = \overline{N}(r,0;f) \leq N(r,0;G) \leq O(T(r,\phi)) + O(1) = S(r,f).$$

Thus $T(r,f) = T(r,\frac{1}{f}) + O(1) = S(r,f)$, a contradiction as f is non constant transcendental meromorphic function.
If $G \equiv 0$, then

$$\phi^{q_1+q_2+\cdots+q_k+2} = -q_0\ (l_{11})^{q_1+1}(l_{21})^{q_2}\cdots(l_{k1})^{q_k}.$$

Thus by lemma of logarithmic derivative, $T(r,\phi) = m(r,\phi) = S(r,\phi)$, i.e., ϕ is a polynomial or a constant (as ϕ is an entire function).

Now, proceeding similarly as in Lemma 3 of ([5]), one can show that f is rational, which is impossible. Hence the proof. □

Lemma 3.6. *([8]) Let f be a transcendental meromorphic function and $\alpha = \alpha(z)(\not\equiv 0,\infty)$ be a small function of f. If $\psi = \alpha(f)^n(f^{(k)})^p$, where $n(\geq 0)$, $p(\geq 1)$, $k(\geq 1)$ are integers, then for any small function $a = a(z)(\not\equiv 0,\infty)$ of ψ,*

$$(p+n)T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + pN_k(r,0;f) + \overline{N}(r,a;\psi) + S(r,f).$$

4. Proof of the Theorem

Proof. **[Proof of Theorem 2.1]** We define

$$g(z) := (f)^{q_0}(f^{(k)})^{q_k} - 1,$$

where $q_0(\geq 2), q_k\ (\geq 1)\ \ (k \in \mathbb{N})$ are non negative integers, and $h(z) := \dfrac{g'}{f^{q_0-1}}$. Also,

$$F(z) = 2\left(\frac{g'(z)}{g(z)}\right)^2 + 3\left(\frac{g'(z)}{g(z)}\right)' - 2\left(\frac{h'(z)}{h(z)}\right)' + \left(\frac{h'(z)}{h(z)}\right)^2 - 4\Big(\frac{g'(z)h'(z)}{g(z)h(z)}\Big), \tag{4.1}$$

Clearly, $F(z) \not\equiv 0$, and f has no simple pole. Next we define another function as

$$\beta \ := \ q_0(f')(f^{(k)})^{q_k} + q_k f(f^{(k)})^{q_k-1}f^{(k+1)} - f(f^{(k)})^{q_k}\frac{g'}{g}.$$

Then

$$f^{q_0-1}\beta = -\frac{g'}{g}, \tag{4.2}$$

and

$$h = -\beta g, \tag{4.3}$$

and

$$\beta^2 F = \beta^2\{(\frac{g'}{g})' - (\frac{g'}{g})^2\} - 2\beta\beta'(\frac{g'}{g}) + (\beta')^2 - 2(\beta\beta'' - (\beta')^2) \tag{4.4}$$

We note that

i) Equation (4.3) gives that the zeros of h come from the zeros of β or, the zeros of g.

ii) Equation (4.2) gives that the multiple poles of f with multiplicity $p(\geq 2)$ are the zeros of β with multiplicity $(q_0 - 1)p - 1$.

iii) If z_0 is a zero of g, then it can not be a pole of f. Thus from equation (4.2), it is clear that z_0 is a simple pole of β.

iv) From (iii) and equation (4.4) gives that the poles of $\beta^2 F$ only come from the zeros of g. Moreover, poles of $\beta^2 F$ have multiplicity atmost 4. Thus

$$N(r, \beta^2 F) \leq 4\overline{N}(r, \frac{1}{g}). \tag{4.5}$$

Since $m(r, F) = S(r, f)$ and $m(r, \beta) = S(r, f)$, therefore $m(r, \beta^2 F) = S(r, f)$. Thus

$$T(r, \beta^2 F) \leq 4\overline{N}(r, \frac{1}{g}) + S(r, f). \tag{4.6}$$

Let z_0 be a zero of f of multiplicity $q(\geq k+1)$. Then equation (4.2) gives that it is a zero of β of order atleast $q_k(q-k)+(q-1)$. Therefore it is a zero of $\beta^2 F$ of order at least $2(q_k(q-k)+(q-1))-2 = (2q-2)+2q_k(q-k)-2 \geq (2q-2)$. Thus

$$2N(r, \frac{1}{f}) - 2\overline{N}(r, \frac{1}{f}) \leq N(r, \frac{1}{\beta^2 F}) \leq T(r, \beta^2 F) + O(1) \leq 4\overline{N}(r, \frac{1}{g}) + S(r, f). \tag{4.7}$$

Again, from Lemma 3.6, we have

$$(q_0 + q_k)T(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + q_k N_k(r, 0; f) + \overline{N}(r, \frac{1}{g}) + S(r, f). \tag{4.8}$$

Combining twice of (4.8) and (4.7), we obtain

$$\begin{aligned} & 2(q_0 + q_k)T(r, f) + 2N(r, \frac{1}{f}) - 2\overline{N}(r, \frac{1}{f}) \\ \leq\ & 2\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f) + 2q_k N_k(r, 0; f) + 6\overline{N}(r, \frac{1}{g}) + S(r, f). \end{aligned}$$

Since f has no simple pole, so we have

$$2(q_0 + q_k)T(r, f) \leq 2\overline{N}_{(2}(r, \infty; f) + 2\overline{N}(r, 0; f) + 2q_k N_k(r, 0; f) + 6\overline{N}(r, \frac{1}{g}) + S(r, f). \tag{4.9}$$

i.e.,

$$\begin{aligned} & (2q_0 - 3)T(r, f) + m(r, f) + N(r, f) + (2 + 2q_k)m(r, \frac{1}{f}) + (2 + 2q_k)N(r, \frac{1}{f}) \\ \leq\ & 2\overline{N}_{(2}(r, \infty; f) + 2\overline{N}(r, 0; f) + 2q_k N_k(r, 0; f) + 6\overline{N}(r, \frac{1}{g}) + S(r, f). \end{aligned} \tag{4.10}$$

i.e.,

$$\begin{aligned} & (2q_0 - 3)T(r, f) + N(r, f) + (2 + 2q_k)N(r, \frac{1}{f}) \\ \leq\ & 2\overline{N}_{(2}(r, \infty; f) + 2\overline{N}(r, 0; f) + 2q_k N_k(r, 0; f) + 6\overline{N}(r, \frac{1}{g}) + S(r, f). \end{aligned} \tag{4.11}$$

i.e.,

$$(2q_0 - 3)T(r, f) + N(r, f) \leq 2\overline{N}_{(2}(r, \infty; f) + 6\overline{N}(r, \frac{1}{g}) + S(r, f). \tag{4.12}$$

Thus

$$(2q_0 - 3)T(r, f) \leq 6\overline{N}(r, \frac{1}{g}) + S(r, f). \tag{4.13}$$

This completes the proof. □

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