



Codazzi and Statistical Connections on Almost Product Manifolds

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Abstract. Considering an almost product manifold, we get the necessary and sufficient conditions for Codazzi connections on it. Also, we show that a Codazzi adapted connection on an almost product manifold, gives two type of Codazzi connections on it's distributions, and moreover we study the conditions of holding the converse of this. Finally, we study the Codazzi (and statistical) structures for Schouten-Van Kampen and Vranceanu connections as two important special cases of adapted connections, and then we present some important examples of them.

1. Introduction

The notion of non-holonomic manifold as a need for a geometric interpretation of non-holonomic mechanical systems have introduced independently by Vranceanu [19] and Horak [13]. Then Bejancu-Farran in [6] have presented a modern approach to the geometry of non-holonomic manifolds as manifolds endowed with non-integrable distributions, and extend this study to almost product manifolds. Their approach is mainly based on adapted linear connections, stressing the important role of Schouten-Van Kampen and Vranceanu connections for understanding the geometry of distributions, in general, and the geometry of non-holonomic manifolds, in particular. When a semi-Riemannian metric is considered on the manifold, they have compared the intrinsic and induced connections on a semi-Riemannian manifold, and have got the local structure of the manifold when these connections coincide.

The mathematical scope of information geometry arose in 1945 by C. R. Rao from the idea that using Fisher information, it is possible to define a Riemannian metric in spaces of probability distributions ([16]). This powerful branch of mathematics implements the methods of differential geometry to the extent of probability theory. Information geometry leads us to a geometrical interpretation of probability theory and statistics and enables us to survey the invariant properties of statistical manifolds. It was realized by the works of S. Amari that the differential geometric structure of a statistical manifold can be obtained from divergence functions, giving a Riemannian metric and a pair of affine connections ([1, 2]). As the affine connections play important roles in information geometry, recently many researchers worked on these connections (see [4, 10, 14, 18]). Information geometry has many applications in various fields of research. These applications can be found for example in image processing, physics, computer science and machine learning (see for instance [7, 17]).

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A statistical manifold is a Riemannian manifold such that each of its points is a probability distribution. Let Θ be an open subset of \mathbb{R}^n . If S is a set of probability density functions on a sample space Ω with parameter $\theta = (\theta^1, \dots, \theta^n)$ such that

$$S = \left\{ p(x; \theta) : \int_{\Omega} p(x; \theta) = 1, p(x; \theta) > 0, \theta \in \Theta \subseteq \mathbb{R}^n \right\},$$

then S is called a statistical model. The semi-definite Fisher information matrix $g(\theta) = (g_{ij}(\theta))$ is defined on a statistical model S by

$$g_{ij}(\theta) := \int_{\Omega} \partial_i l_{\theta} \partial_j l_{\theta} p(x; \theta) dx = E_p \left[\partial_i l_{\theta} \partial_j l_{\theta} \right],$$

where $l_{\theta} := \log p(x; \theta)$, $\partial_i = \frac{\partial}{\partial \theta^i}$ and $E_p[f]$ is the expectation of $f(x)$ with respect to $p(x; \theta)$. Equipping S to this metric, S is called an info-manifold or a statistical manifold.

In Section 2, we recall the known concepts on lift objects on tangent bundle of a Riemannian manifold. Also, we present the known concepts on statistical manifolds. In Section 3, considering adapted linear connections on an almost product manifold, we obtain the necessary and sufficient conditions such that these connections reduce to the Codazzi connections. Also, we present an example of these connections on an almost product manifold. Then we show that from a Codazzi connection on an almost product structure $(M, \mathcal{D}, \mathcal{D}')$, we can construct two Codazzi connections on \mathcal{D} and \mathcal{D}' . Also, we show that this result is holds for statistical connections, when \mathcal{D} and \mathcal{D}' are involutive distributions. Moreover, we study the conditions that the converse of this result is hold. Then end of this section is depended to the intrinsic connections on the distributions of an almost product structure $(M, \mathcal{D}, \mathcal{D}')$ and constructing a unique statistical connection on M by using them. Section 4 is contain two important adapted connections on an almost product structure, namely, Schouten-Van Kampen and Vrăncăanu connections. Then we study the Codazzi (and statistical) conditions for these connections. Finally, using the Sasaki and horizontal lift metrics on the tangent bundle of a Riemannian manifold, we introduce some Codazzi connections on it and we study the Codazzi and statistical conditions for the Schouten-Van Kampen and Vrăncăanu connections induced by them.

2. Preliminaries

Let (M, g) be a Riemannian manifold with the unique Levi-Civita connection $\overset{g}{\nabla}$. Considering the tangent bundle (TM, π, M) , we put $\mathcal{V}_{(x,y)} = \text{Ker}(d\pi_{(x,y)})$ as the vertical subspace of $T_{(x,y)}TM$ at the point (x, y) . Indeed, $\mathcal{V}_{(x,y)} = T_{(x,y)}T_xM$. A horizontal subspace is any choice of \mathcal{H} such that

$$T_{(x,y)}TM = \mathcal{H}_{(x,y)} \oplus \mathcal{V}_{(x,y)}, \tag{1}$$

holds. As $\mathcal{V} = \text{Ker}(d\pi)$, thus $d\pi : \mathcal{H}_{(x,y)} \rightarrow T_xM$ is a vector space isomorphism. The fiber $\mathcal{H}_{(x,y)}$ is called the horizontal subspace to TM at (x, y) . If the splitting (1) is hold, then the horizontal lift of a tangent vector $X_x \in T_xM$, is the unique vector $X_x^h \in \mathcal{H}_{(x,y)}$ such that $d\pi(X_x^h) = X_x$ and its vertical lift is the unique one $X_x^v \in \mathcal{V}_{(x,y)}$ such that $X_x^v(df) = X_x(f)$ for all $f \in C^\infty(M)$.

Let (x, \mathcal{U}) be a local chart on M by $x = (x^1, \dots, x^n)$ where x^i 's belong to $C^\infty(M)$. If we consider $x^i \circ \pi$ and denote it again by x^i , defining

$$y^i(X) = X(x^i) = dx^i(X), \quad i \in \{1, \dots, n\}, \quad X \in \mathfrak{X}(M),$$

one can make a local chart $(x^1, \dots, x^n, y^1, \dots, y^n) : \pi^{-1}(\mathcal{U}) \rightarrow \mathbb{R}^{2n}$ on TM . Moreover, it can be verified that if $X = X^i \frac{\partial}{\partial x^i}$, then

$$X^v = X^i \frac{\partial}{\partial y^i}, \quad X^h = X^i \frac{\partial}{\partial x^i} - X^j y^k \overset{g}{\Gamma}{}^i_{jk} \frac{\partial}{\partial y^i},$$

where Γ^g_{jk} 's are Christoffel symbols of the Levi-Civita connection ∇^g . If $\overset{g}{R}$ denotes the Riemann curvature tensor of ∇^g , then

$$[X^v, Y^v] = 0, \quad [X^h, Y^v] = (\nabla_X^g Y)^v, \quad [X^h, Y^h] = [X, Y]^h - (\overset{g}{R}(X, Y)y)^v, \tag{2}$$

for any $X, Y, Z \in \mathfrak{X}(M)$ and any point $(x, y) \in TM$. The Sasaki metric g^S on the tangent bundle TM is a natural lift of the metric g given by

$$g^S(X^h, Y^h)_{(x,y)} = g_x(X, Y), \quad g^S(X^v, Y^h)_{(x,y)} = 0, \quad g^S(X^v, Y^v)_{(x,y)} = g_x(X, Y). \tag{3}$$

Also, the horizontal lift metric g^h on the tangent bundle TM is a natural lift of the metric g given by

$$g^h(X^h, Y^h)_{(x,y)} = g^h(X^v, Y^v)_{(x,y)} = 0, \quad g^h(X^v, Y^h)_{(x,y)} = g_x(X, Y). \tag{4}$$

If $\{\frac{\partial}{\partial x^i}|_{(x,y)}, \frac{\partial}{\partial y^i}|_{(x,y)}\}_{i=1}^n$ is the natural basis of $T_{(x,y)}TM$, then $\mathcal{H}_{(x,y)}$ could be spanned by $\{\frac{\delta}{\delta x^i}|_{(x,y)}\}_{i=1}^n$, where

$$\frac{\delta}{\delta x^i}|_{(x,y)} = \frac{\partial}{\partial x^i}|_{(x,y)} - y^k \Gamma_{ki}^j(x) \frac{\partial}{\partial y^j}|_{(x,y)}.$$

So, its dual basis is $\{dx^i, \delta y^i\}_{i=1}^n$, where $\delta y^i = dy^i + y^k \Gamma_{kj}^i(x) dx^j$. In continue, to simplify notation, we write ∂_i, δ_i and ∂_j instead of $\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^i}$ and $\frac{\partial}{\partial y^i}$, respectively.

2.1. Statistical connections on Riemannian manifolds

A linear connection ∇ on Riemannian manifold (M, g) is called *Codazzi connection* if the cubic tensor field $C = \nabla g$ is totally symmetric, namely the Codazzi equations hold:

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X) \quad (= (\nabla_Z g)(X, Y)), \forall X, Y, Z \in \mathfrak{X}(M).$$

An *statistical manifold* is a data (M, g, ∇) where g is a Riemannian metric on manifold M and ∇ is a symmetric Codazzi connection.

In the local coordinate, C has the following form

$$C(\partial_i, \partial_j, \partial_k) = \partial_i g(\partial_j, \partial_k) - g(\nabla_{\partial_i} \partial_j, \partial_k) - g(\partial_j, \nabla_{\partial_i} \partial_k),$$

and so

$$C_{ijk} = \partial_i(g_{jk}) - \Gamma_{ij}^h g_{kh} - \Gamma_{ik}^h g_{jh}, \quad C_{ijk} = C_{jki} = C_{kij},$$

where Γ_{ik}^h 's are the Christoffel symbols of ∇ . Thus for every statistical manifold (M, g, ∇) , there exists a naturally associated totally symmetric covariant tensor field C of degree 3. Conversely, let (M, g, C) be a Riemannian manifold with a totally symmetric covariant tensor field C of degree 3. If we define the tensor field A by

$$g(A(X)Y, Z) = C(X, Y, Z), \tag{5}$$

and a linear connection ∇ by $\nabla = \overset{g}{\nabla} - \frac{A}{2}$, then the triplet (M, g, ∇) becomes a statistical manifold. Thus to equip a statistical structure (g, ∇) is equivalent to equip a pair of structure (g, C) consisting of a semi-Riemannian g and a totally symmetric trilinear C .

For a statistical structure (∇, g) we define the difference tensor field $K := K^{(\nabla, g)} \in \Gamma(TM^{(1,2)})$ as

$$K(X, Y) = \nabla_X Y - \overset{g}{\nabla}_X Y.$$

It is easy to check that K is symmetric and moreover,

$$g(K(X, Y), Z) = g(Y, K(X, Z)). \tag{6}$$

Conversely, if there exists a symmetric tensor field $K \in \Gamma(TM^{(1,2)})$ on a Riemannian manifold (M, g) such that satisfies in the above equation, then $(\nabla = \overset{g}{\nabla} + K, g)$ becomes a statistical structure on M (see [11, 12] for more details). It is remarkable that considering $K = -\frac{A}{2}$, two above directions are the same.

Let ∇ be a linear connection on (M, g) . The linear connection $\overset{*}{\nabla}$ given by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \overset{*}{\nabla}_X Z), \quad \forall X, Y, Z \in \Gamma(TM), \tag{7}$$

is called the *dual connection of ∇ with respect to g* . It is known that if ∇ is a Codazzi (statistical) connection on M , then $\overset{*}{\nabla}$ is a Codazzi (statistical) connection on M , too (see [3], for more details).

2.2. Adapted connections on almost product manifolds

We consider on M two complementary distributions \mathcal{D} and \mathcal{D}' , that is, TM has the decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}'. \tag{8}$$

Denote by Q and Q' the projection morphisms of TM on \mathcal{D} and \mathcal{D}' , respectively. Then we have

$$Q^2 = Q, \quad Q'^2 = Q', \quad QQ' = Q'Q = 0, \quad Q + Q' = Id_M.$$

Also, defining the $(1, 1)$ -tensor field $F = Q - Q'$, it is easy to see that $F^2 = Id_M$, i.e., F is an *almost product structure* on M . For this reason the triple $(M, \mathcal{D}, \mathcal{D}')$ is called an *almost product manifold*. It is known that [6]

$$\Gamma(\mathcal{D}) = \{X \in TTM | FX = X\}, \quad \Gamma(\mathcal{D}') = \{X \in TTM | FX = -X\}.$$

Let \mathcal{D} be an n -distribution on an $(n + p)$ - dimensional manifold M . A linear connection ∇ on M is said to be *adapted to \mathcal{D}* if

$$\nabla_X U \in \Gamma(\mathcal{D}), \quad \forall X \in \Gamma(TM), \quad U \in \Gamma(\mathcal{D}).$$

Now, if \mathcal{D}' is a p -distribution on M complementary to \mathcal{D} , then $(M, \mathcal{D}, \mathcal{D}')$ is an almost product manifold as we have seen in the above. \mathcal{D} and \mathcal{D}' are called the *structural* and *transversal* distributions, respectively.

A linear connection ∇ on an almost product manifold $(M, \mathcal{D}, \mathcal{D}')$ is said to be an *adapted connection* if it is adapted to both distributions \mathcal{D} and \mathcal{D}' . Thus ∇ is adapted if and only if $\nabla_X QY \in \Gamma(\mathcal{D})$ and $\nabla_X Q'Y \in \Gamma(\mathcal{D}')$, for all $X, Y \in \Gamma(TM)$.

Next, we suppose that \mathcal{D} and \mathcal{D}' are locally represented on a coordinate neighbourhood $\mathcal{U} \subset M$ by vector fields $\{E_i\}$ and $\{E_\alpha\}$, respectively. Then we call $\{E_A\} = \{E_i, E_\alpha\}$, $A \in \{1, \dots, n + p\}$, a *non-holonomic frame field* on \mathcal{U} . An adapted connection ∇ on M is locally given by

$$\begin{cases} \nabla_{E_j} E_i = \Gamma_{ij}^k E_k, & \nabla_{E_\alpha} E_i = \Gamma_{i\alpha}^k E_k, \\ \nabla_{E_j} E_\alpha = \Gamma_{\alpha j}^\beta E_\beta, & \nabla_{E_\gamma} E_\alpha = \Gamma_{\alpha\gamma}^\beta E_\beta, \end{cases} \tag{9}$$

with respect to $\{E_i, E_\alpha\}$. Thus we can denote by $(\Gamma_{iA}^k, \Gamma_{\alpha A}^\beta)$ an adapted connection ∇ on $(M, \mathcal{D}, \mathcal{D}')$. We put

$$\begin{cases} Q[E_j, E_i] = V_{ij}^k E_k, & Q[E_\beta, E_\alpha] = V_{\alpha\beta}^k E_k, \\ Q[E_\alpha, E_i] = -Q[E_i, E_\alpha] = V_{i\alpha}^k E_k = -V_{\alpha i}^k E_k, \end{cases} \tag{10}$$

and

$$\begin{cases} Q'[E_j, E_i] = V_{ij}^\beta E_\beta, & Q'[E_\gamma, E_\alpha] = V_{\alpha\gamma}^\beta E_\beta, \\ Q'[E_i, E_\alpha] = -Q'[E_\alpha, E_i] = V_{\alpha i}^\beta E_\beta = -V_{i\alpha}^\beta E_\beta. \end{cases} \quad (11)$$

We recall that the torsion tensor field T of the linear connection ∇ is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Using the decomposition (8) and the non-holonomic frame field $\{E_A\}$ we have

$$\begin{aligned} T(E_j, E_i) &= T_{ij}^k E_k + T_{ij}^\alpha E_\alpha, \\ T(E_\alpha, E_i) &= -T(E_i, E_\alpha) = T_{i\alpha}^k E_k + T_{i\alpha}^\beta E_\beta = -T_{\alpha i}^k E_k - T_{\alpha i}^\beta E_\beta, \\ T(E_\gamma, E_\alpha) &= T_{\alpha\gamma}^k E_k + T_{\alpha\gamma}^\beta E_\beta, \end{aligned}$$

where

$$\begin{cases} T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k - V_{ij}^k, & T_{ij}^\alpha = -V_{ij}^\alpha \\ T_{i\alpha}^k = -T_{\alpha i}^k = \Gamma_{i\alpha}^k - V_{i\alpha}^k, & T_{\alpha i}^\beta = -T_{i\alpha}^\beta = \Gamma_{\alpha i}^\beta - V_{i\alpha}^\beta \\ T_{\alpha\beta}^k = V_{\alpha\beta}^k, & T_{\alpha\gamma}^\beta = \Gamma_{\alpha\gamma}^\beta - \Gamma_{\gamma\alpha}^\beta - V_{\alpha\gamma}^\beta. \end{cases} \quad (12)$$

From (12) we conclude the following result, easily:

Proposition 2.1. [6] *If an adapted connection ∇ on an almost product manifold $(M, \mathcal{D}, \mathcal{D}')$ is torsion free, then distributions \mathcal{D} and \mathcal{D}' are involutive.*

Here, we consider an $(n + p)$ -dimensional Riemannian manifold (M, g) and suppose that (\mathcal{D}, g) is a Riemannian n -distribution on M . Considering the vector bundle

$$D^\perp = \cup_{x \in M} D_x^\perp,$$

where D_x^\perp is the complementary orthogonal subspace to D_x in $(T_x M, g_x)$, then (\mathcal{D}^\perp, g) is a Riemannian p -distribution on M . Here we denoted by the same symbol g the Riemannian metrics induced by g on \mathcal{D} and \mathcal{D}^\perp . Thus we have

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp. \quad (13)$$

In what follows we keep the same notations Q and Q' for the projection morphisms of TM on \mathcal{D} and \mathcal{D}^\perp , respectively.

Defining

$$D_X QY = \nabla_X QY, \quad D'_X Q'Y = \nabla_X Q'Y, \quad \forall X, Y \in \Gamma(TM),$$

easily we can see that D and D' are linear connections on \mathcal{D} and \mathcal{D}' , respectively. Conversely, if D and D' are two linear connections on \mathcal{D} and \mathcal{D}' , respectively, then we construct an adapted connection ∇ on $(M, \mathcal{D}, \mathcal{D}')$, by the formula

$$\nabla_X Y = D_X QY + D'_X Q'Y, \quad \forall X, Y \in \Gamma(TM). \quad (14)$$

Moreover, the restrictions of ∇_X to $\Gamma(\mathcal{D})$ and $\Gamma(\mathcal{D}')$ are exactly D_X and D'_X respectively (see [6], for more details).

Here, we consider the almost product manifold $(M, \mathcal{D}, \mathcal{D}')$ and we let D and D' are linear connections on \mathcal{D} and \mathcal{D}' , respectively. We consider the tensor field

$${}_D T(X, QY) = D_X QY - D_{QY} QX - Q[X, QY], \quad \forall X, Y \in \Gamma(TM), \quad (15)$$

and we call it the \mathcal{D}' -torsion tensor field of D . Similarly, the \mathcal{D} -torsion tensor field of D' is defined as follows:

$${}_{D'}T(X, Q'Y) = D'_X Q'Y - D_{Q'Y} Q'X - Q'[X, Q'Y], \quad \forall X, Y \in \Gamma(TM).$$

It is known that ${}_D T$ and ${}_{D'} T$ have the following locally expression with respect to $\{E_A\}$:

$$\begin{aligned} {}_D T(E_j, E_i) &= {}_D T_{ij}^k E_k, \quad {}_D T(E_\alpha, E_i) = {}_D T_{i\alpha}^k E_k, \\ {}_{D'} T(E_\gamma, E_\alpha) &= {}_{D'} T_{\alpha\gamma}^\beta E_\beta, \quad {}_{D'} T(E_i, E_\alpha) = {}_{D'} T_{\alpha i}^\beta E_\beta, \end{aligned}$$

where

$$\begin{cases} {}_D T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k - V_{ij}^k, & {}_D T_{i\alpha}^k = \Gamma_{i\alpha}^k - V_{i\alpha}^k, \\ {}_{D'} T_{\alpha\gamma}^\beta = \Gamma_{\alpha\gamma}^\beta - \Gamma_{\gamma\alpha}^\beta - V_{\alpha\gamma}^\beta, & {}_{D'} T_{\alpha i}^\beta = \Gamma_{\alpha i}^\beta - V_{\alpha i}^\beta. \end{cases} \quad (16)$$

From (12) and (16) we obtain the following

Proposition 2.2. [6] *Let $(M, \mathcal{D}, \mathcal{D}')$ be an almost product manifold such that \mathcal{D} and \mathcal{D}' are involutive. Then an adapted connection ∇ on M is torsion free if and only if D and D' are \mathcal{D}' -torsion free and \mathcal{D} -torsion free, respectively.*

Let $(M, \mathcal{D}, \mathcal{D}')$ be an almost product manifold and ∇ be a linear connection on M . Defining

$$\overset{SV}{\nabla}_X Y = Q\nabla_X QY + Q'\nabla_X Q'Y, \quad (17)$$

$$\overset{V}{\nabla}_X Y = Q\nabla_{QX} QY + Q'\nabla_{Q'X} Q'Y + Q[Q'X, QY] + Q'[QX, Q'Y], \quad (18)$$

for all $X, Y \in \Gamma(TM)$, it is easy to see that these connections are adapted connections on M which are called the Schouten-Van Kampen connection and the Vrănceanu connection, respectively. If ∇ is locally given by

$$\nabla_{E_B} E_A = F_{AB}^C E_C,$$

then we have $\overset{SV}{\nabla} = (\overset{SV}{\Gamma}_{iA}^k, \overset{SV}{\Gamma}_{\alpha A}^\beta)$ and $\overset{V}{\nabla} = (\overset{V}{\Gamma}_{iA}^k, \overset{V}{\Gamma}_{\alpha A}^\beta)$, where

$$\overset{SV}{\Gamma}_{iA}^k = F_{iA}^k, \quad \overset{SV}{\Gamma}_{\alpha A}^\beta = F_{\alpha A}^\beta,$$

and

$$\overset{V}{\Gamma}_{ij}^k = F_{ij}^k, \quad \overset{V}{\Gamma}_{i\alpha}^k = V_{i\alpha}^k, \quad \overset{V}{\Gamma}_{\alpha i}^\beta = V_{\alpha i}^\beta, \quad \overset{V}{\Gamma}_{\alpha\gamma}^\beta = F_{\alpha\gamma}^\beta.$$

3. Statistical connections on almost product manifolds

Some questions are raised about linear connections on almost product manifolds in the mind. One of these questions is that if ∇ is an adapted connection on an almost product manifold, is the dual connection of it adapted? In the below we study this question.

Proposition 3.1. *Let (M, g) be a $(n+p)$ -dimensional Riemannian manifold and (\mathcal{D}, g) be a Riemannian n -distribution on M . If ∇ is an adapted statistical connection on $(M, \mathcal{D}, \mathcal{D}^\perp)$, then the dual connection of ∇ with respect to g , given by (7) is an adapted statistical connection on M .*

Proof. From (7) we have

$$Xg(Q'Y, QZ) = g(\nabla_X Q'Y, QZ) + g(Q'Y, \overset{*}{\nabla}_X QZ), \quad \forall X, Y, Z \in \Gamma(TM).$$

Since $Q'Y$ and QZ are belong to $\Gamma(\mathcal{D}^\perp)$ and $\Gamma(\mathcal{D})$, respectively, then we have $g(Q'Y, QZ) = 0$. Also, since $\nabla_X Q'Y \in \Gamma(\mathcal{D}^\perp)$, then we have $g(\nabla_X Q'Y, QZ) = 0$. Thus the above equation reduces to $g(Q'Y, \overset{*}{\nabla}_X QZ) = 0$. This shows that $\overset{*}{\nabla}_X QZ \in \Gamma(\mathcal{D})$. Similarly we can deduce that $\overset{*}{\nabla}_X QZ \in \Gamma(\mathcal{D}^\perp)$. So $\overset{*}{\nabla}$ is an adapted connection on M . \square

The other question is, under what conditions an adapted connection reduces to a Codazzi connection. Using locally expression is one of the best ways to study these conditions. In below, we present these conditions and we construct an interesting example in this case.

Proposition 3.2. *Let $(M, \mathcal{D}, \mathcal{D}')$ be an almost product manifold and g be a Riemannian metric on it. Then an adapted connection $\nabla = (\Gamma_{iA}^k, \Gamma_{\alpha A}^\beta)$ is Codazzi if and only if*

$$E_k g_{ij} - \Gamma_{ik}^r g_{rj} - \Gamma_{jk}^r g_{ir} = E_i g_{jk} - \Gamma_{ji}^r g_{rk} - \Gamma_{ki}^r g_{jr} = E_j g_{ki} - \Gamma_{kj}^r g_{ri} - \Gamma_{ij}^r g_{kr}, \tag{19}$$

$$E_\gamma g_{\alpha\beta} - \Gamma_{\alpha\gamma}^\mu g_{\mu\beta} - \Gamma_{\beta\gamma}^\mu g_{\alpha\mu} = E_\alpha g_{\beta\gamma} - \Gamma_{\beta\alpha}^\mu g_{\mu\gamma} - \Gamma_{\gamma\alpha}^\mu g_{\beta\mu} = E_\beta g_{\gamma\alpha} - \Gamma_{\gamma\beta}^\mu g_{\mu\alpha} - \Gamma_{\alpha\beta}^\mu g_{\gamma\mu}, \tag{20}$$

$$E_\alpha g_{ij} - \Gamma_{i\alpha}^r g_{rj} - \Gamma_{j\alpha}^r g_{ir} = E_i g_{j\alpha} - \Gamma_{ji}^r g_{r\alpha} - \Gamma_{\alpha i}^\mu g_{j\mu} = E_j g_{\alpha i} - \Gamma_{\alpha j}^\mu g_{\mu i} - \Gamma_{ij}^r g_{\alpha r}, \tag{21}$$

$$E_\alpha g_{\beta i} - \Gamma_{\beta\alpha}^\mu g_{\mu i} - \Gamma_{i\alpha}^r g_{\beta r} = E_\beta g_{i\alpha} - \Gamma_{i\beta}^r g_{r\alpha} - \Gamma_{\alpha\beta}^\mu g_{i\mu} = E_i g_{\alpha\beta} - \Gamma_{\alpha i}^\mu g_{\mu\beta} - \Gamma_{\beta i}^\mu g_{\alpha\mu}. \tag{22}$$

Proof. Using the non-holonomic frame field $\{E_A\}$ in

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X) = (\nabla_Z g)(X, Y),$$

we can conclude the proof. \square

Corollary 3.3. *Let $(M, \mathcal{D}, \mathcal{D}^\perp)$ be an almost product manifold and g be a Riemannian metric on it. Then an adapted connection $\nabla = (\Gamma_{iA}^k, \Gamma_{\alpha A}^\beta)$ is Codazzi if and only if*

$$E_k g_{ij} - \Gamma_{ik}^r g_{rj} - \Gamma_{jk}^r g_{ir} = E_i g_{jk} - \Gamma_{ji}^r g_{rk} - \Gamma_{ki}^r g_{jr} = E_j g_{ki} - \Gamma_{kj}^r g_{ri} - \Gamma_{ij}^r g_{kr}, \tag{23}$$

$$E_\gamma g_{\alpha\beta} - \Gamma_{\alpha\gamma}^\mu g_{\mu\beta} - \Gamma_{\beta\gamma}^\mu g_{\alpha\mu} = E_\alpha g_{\beta\gamma} - \Gamma_{\beta\alpha}^\mu g_{\mu\gamma} - \Gamma_{\gamma\alpha}^\mu g_{\beta\mu} = E_\beta g_{\gamma\alpha} - \Gamma_{\gamma\beta}^\mu g_{\mu\alpha} - \Gamma_{\alpha\beta}^\mu g_{\gamma\mu}, \tag{24}$$

$$E_\alpha g_{ij} - \Gamma_{i\alpha}^r g_{rj} - \Gamma_{j\alpha}^r g_{ir} = 0, \tag{25}$$

$$E_i g_{\alpha\beta} - \Gamma_{\alpha i}^\mu g_{\mu\beta} - \Gamma_{\beta i}^\mu g_{\alpha\mu} = 0. \tag{26}$$

Example 3.4. *Let (\mathbb{R}^4, g) be the 4-dimensional Euclidean space with g given by $g(x, y) = \sum_{i=1}^4 x_i y_i$. We define the open submanifold M of \mathbb{R}^4 by*

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | 2x_3 - (x_1)^2 > 0\},$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system on \mathbb{R}^4 . Then on the Riemannian manifold (M, g) we consider the distributions \mathcal{D} and \mathcal{D}^\perp spanned by

$$\{X_1 = \frac{\partial}{\partial x_1} + L \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_2} - L \frac{\partial}{\partial x_3}\},$$

$$\{X_{\bar{1}} = \frac{\partial}{\partial x_2} - L \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_4}, X_{\bar{2}} = \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_1} + L \frac{\partial}{\partial x_4}\},$$

respectively, where $L = \sqrt{2x_3 - (x_1)^2}$. It is easy to see that \mathcal{D} and \mathcal{D}^\perp are complementary orthogonal Riemannian distributions on (M, g) . Moreover, none of them is involutive, so they are not integrable. Also, easily results that $\{X_1, X_2\}$ is an orthogonal basis in $\Gamma(\mathcal{D})$ and $\{X_{\bar{1}}, X_{\bar{2}}\}$ is an orthogonal basis in $\Gamma(\mathcal{D}^\perp)$. Moreover we have

$$g(X_1, X_1) = g(X_2, X_2) = g(X_{\bar{1}}, X_{\bar{1}}) = g(X_{\bar{2}}, X_{\bar{2}}) = H,$$

where $H = 1 + 2x_3$, which is non-zero on M (see [6]). Let $\nabla = (\Gamma_{iA}^k, \Gamma_{\bar{i}A}^{\bar{k}})$, $A \in \{\bar{j}, \bar{j}\}$, $i, j, k = 1, 2$, be an adapted connection on M . Using Corollary 3.3 we deduce that ∇ is a Codazzi linear connection on (M, g) if and only if

$$\begin{cases} H(\Gamma_{11}^2 + \Gamma_{21}^1) = 2L + 2H\Gamma_{12}^1, & \Gamma_{12}^1 = H^{-1}, \Gamma_{11}^1 = 0, \Gamma_{11}^2 = -\Gamma_{21}^1, \Gamma_{12}^2 = -\Gamma_{22}^1 \\ H(\Gamma_{\bar{1}\bar{1}}^2 + \Gamma_{\bar{2}\bar{1}}^1) = 2 - 2H\Gamma_{\bar{1}\bar{2}}^1, & \Gamma_{\bar{1}\bar{2}}^1 = -LH^{-1}, \Gamma_{\bar{1}\bar{1}}^1 = x_1 H^{-1}, \Gamma_{\bar{1}\bar{1}}^2 = -\Gamma_{\bar{2}\bar{1}}^1, \Gamma_{\bar{1}\bar{2}}^2 = \Gamma_{\bar{2}\bar{2}}^1 \end{cases} \tag{27}$$

So we can obtain several Codazzi adapted connections on (M, g) . For instance, considering $(\Gamma_{iA}^k, \bar{\Gamma}_{iA}^k)$, where all $\Gamma_{iA}^k, \bar{\Gamma}_{iA}^k$ are zero expect

$$\Gamma_{11}^2 = 2LH^{-1}, \Gamma_{12}^1 = H^{-1}, \Gamma_{11}^2 = 2H^{-1}, \Gamma_{12}^1 = -LH^{-1}, \Gamma_{11}^1 = x_1H^{-1},$$

we have a Codazzi adapted connection (M, g) , which is not compatible with g .

Here, we focus on Riemannian distributions on M and we like to introduce Codazzi (statistical) connections on these distributions. We present the following definition:

Definition 3.5. Let (\mathcal{D}, g) be a Riemannian distribution on M , \mathcal{D}' be a complementary distribution to \mathcal{D} in TM and D be a linear connection on \mathcal{D} . We say that g is \mathcal{D} -Codazzi with respect to D (or D is \mathcal{D} -Codazzi connection) if

$$(D_{QX}g)(QY, QZ) = (D_{QY}g)(QZ, QX) = (D_{QZ}g)(QX, QY), \quad \forall X, Y, Z \in \Gamma(TM).$$

Also, g is called \mathcal{D}' -parallel with respect to D (or D is \mathcal{D}' -compatible with respect to g) if $(D_{Q'X}g)(QY, QZ) = 0$. Moreover, D is called \mathcal{D} -statistical connection, if D is \mathcal{D} -Codazzi and \mathcal{D}' -torsion free connection.

Let (\mathcal{D}, g) be a Riemannian distribution on M and D be a \mathcal{D} -Codazzi connection on M . We define the linear connection $\overset{*}{D}$ on M as follows

$$Xg(QY, QZ) = g(D_XQY, QZ) + g(QY, \overset{*}{D}_XQZ), \quad \forall X, Y, Z \in \Gamma(TM), \tag{28}$$

and we call it the dual connection of D with respect to g . Now, we study the Codazzi (statistical) conditions for dual connection of D .

Equation (28) implies

$$(D_{QX}g)(QY, QZ) = g(\overset{*}{D}_{QX}QY, QZ) - g(D_{QX}QY, QZ).$$

Similarly we get

$$(D_{QY}g)(QX, QZ) = g(\overset{*}{D}_{QY}QX, QZ) - g(D_{QY}QX, QZ).$$

Since D is a \mathcal{D} -Codazzi connection, then two above equations give us

$$g(\overset{*}{D}_{QX}QY - \overset{*}{D}_{QY}QX - D_{QX}QY + D_{QY}QX, QZ) = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

Applying (15) in the above equation we obtain

$$g(T_{\overset{*}{D}}(QX, QY) - T_D(QX, QY), QZ) = 0, \quad \forall X, Y, Z \in \Gamma(TM),$$

which gives us

$$T_{\overset{*}{D}}(QX, QY) = T_D(QX, QY), \quad \forall X, Y \in \Gamma(TM). \tag{29}$$

Here we study the \mathcal{D} -Codazzi condition for $\overset{*}{D}$. (28) gives us

$$(\overset{*}{D}_{QX}g)(QY, QZ) = g(D_{QX}QY, QZ) - g(\overset{*}{D}_{QX}QY, QZ), \tag{30}$$

and

$$(\overset{*}{D}_{QY}g)(QX, QZ) = g(D_{QY}QX, QZ) - g(\overset{*}{D}_{QY}QX, QZ). \tag{31}$$

Applying (29) we conclude that (30) is equal to (31), and so $\overset{*}{D}$ is a \mathcal{D} -Codazzi condition on M . Thus we have the following

Theorem 3.6. Let (\mathcal{D}, g) be a Riemannian distribution on M and \mathcal{D}' be a complementary distribution to \mathcal{D} in TM . If D is a \mathcal{D} -Codazzi connection on M , then the dual connection of D with respect to g is a \mathcal{D} -Codazzi connection on M . Moreover the \mathcal{D}' -torsion tensor fields of D and $\overset{*}{D}$ over \mathcal{D} are the same (i.e., $T_{\overset{*}{D}}(QX, QY) = T_D(QX, QY)$).

From the above theorem we can conclude the following

Corollary 3.7. Let (\mathcal{D}, g) be a Riemannian distribution on M and \mathcal{D}' be a complementary distribution to \mathcal{D} in TM . If D is a \mathcal{D} -statistical connection on M , then the dual connection of D with respect to g is a \mathcal{D} -statistical connection on M .

Here, we study the existence of statistical connections on the distributions of an almost product manifolds.

Theorem 3.8. Let (\mathcal{D}, g) be a Riemannian distribution on M and \mathcal{D}' be a complementary distribution to \mathcal{D} in TM . If C is a totally symmetric $(3, 0)$ -tensor field on \mathcal{D} , then there exists a unique linear connection D on \mathcal{D} such that D is \mathcal{D} -statistical connection on \mathcal{D} with cubic tensor field C .

Proof. We define the differential operator $D : \Gamma(TM) \times \Gamma(\mathcal{D}) \rightarrow \Gamma(\mathcal{D})$ by

$$2g(D_{QX}QY, QZ) = QX(g(QY, QZ)) + QY(g(QZ, QX)) - QZ(g(QX, QY)) + g(Q[QX, QY], QZ) - g(Q[QY, QZ], QX) + g(Q[QZ, QX], QY) + C(QX, QY)QZ, \tag{32}$$

and

$$D_{Q'X}QY = Q[Q'X, QY], \tag{33}$$

for all $X, Y, Z \in \Gamma(TM)$. It is easy to see that D given by (32) and (33) is a linear connection on \mathcal{D} . Using (32) we get

$$g(D_{QX}QY - D_{QY}QX, QZ) = g(Q[QX, QY], QZ), \quad \forall X, Y, Z \in \Gamma(TM),$$

and so $D_{QX}QY = D_{QY}QX + Q[QX, QY]$. Thus using (33) we get

$$\begin{aligned} D_XQY - D_{QY}QX &= D_{QX}QY + D_{Q'Y}QY - D_{QY}QX \\ &= D_{Q'X}QY + Q[QX, QY] \\ &= Q[Q'X, QY] + Q[QX, QY] \\ &= Q[X, QY], \end{aligned}$$

i.e., D is \mathcal{D}' -torsion free. Again, using (32) we obtain

$$QX(g(QY, QZ)) - g(D_{QX}QY, QZ) - g(QY, D_{QX}QZ) = C(QX, QY)QZ,$$

i.e., D is \mathcal{D} -Codazzi connection on \mathcal{D} with cubic tensor field C . \square

Now, the question arises as to whether a Codazzi (statistical) adapted connection can induces Codazzi (statistical) connections on its distributions. In the following we answer to this question.

Theorem 3.9. Let (M, g) be a Riemannian manifold and ∇ be a Codazzi adapted connection on $(M, \mathcal{D}, \mathcal{D}')$. Then D and D' are \mathcal{D} -Codazzi and \mathcal{D}' -Codazzi connections, respectively.

Proof. As ∇ is an Codazzi adapted connection on $(M, \mathcal{D}, \mathcal{D}')$, then we get

$$\begin{aligned} (D_{QX}g)(QY, QZ) &= (QX)g(QY, QZ) - g(D_{QX}QY, QZ) - g(QY, D_{QX}QZ) \\ &= (QX)g(QY, QZ) - g(\nabla_{QX}QY, QZ) - g(QY, \nabla_{QX}QZ) \\ &= (\nabla_{QX}g)(QY, QZ) = (\nabla_{Q'Y}g)(QZ, QX) \\ &= (D_{Q'Y}g)(QZ, QX). \end{aligned} \tag{34}$$

Thus D is a \mathcal{D} -Codazzi connection on \mathcal{D} . In the similar way we can conclude that D' is a \mathcal{D}' -Codazzi connection on \mathcal{D}' . \square

Using Proposition 2.2 and Theorem 3.9 we can conclude the following

Corollary 3.10. *Let (M, g) be a Riemannian manifold and ∇ be an statistical adapted connection on $(M, \mathcal{D}, \mathcal{D}')$ such that \mathcal{D} and \mathcal{D}' are involutive distributions. Then D and D' are statistical connection on \mathcal{D} and \mathcal{D}' , respectively.*

In the following theorem, we present the conditions that the converse of Theorem 3.9 holds.

Theorem 3.11. *Let (M, g) be a Riemannian manifold. If g is \mathcal{D} -Codazzi with respect to D , \mathcal{D}^\perp -Codazzi with respect to D^\perp , \mathcal{D}^\perp -parallel with respect to D and \mathcal{D} -parallel with respect to D^\perp , then the adapted connection ∇ on M defined by (14) is Codazzi.*

Proof. Since $g(QX, Q'Y) = 0$, for all $X, Y \in \Gamma(TM)$, then we can obtain the following

$$(\nabla_X g)(Y, Z) = (D_{QX}g)(QY, QZ) + (D_{Q^\perp X}^\perp g)(Q'Y, Q'Z) + (D_{Q'X}g)(QY, QZ) + (D_{Q'X}^\perp g)(Q'Y, Q'Z).$$

Since g is \mathcal{D}^\perp -parallel with respect to D and \mathcal{D} -parallel with respect to D^\perp , then the third and fourth sentences in the right side of the above equation are zero and so it reduces to the following

$$(\nabla_X g)(Y, Z) = (D_{QX}g)(QY, QZ) + (D_{Q^\perp X}^\perp g)(Q^\perp Y, Q^\perp Z).$$

From the above equation we deduce

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X) = (\nabla_Z g)(X, Y),$$

because g is \mathcal{D} -Codazzi with respect to D and \mathcal{D}^\perp -Codazzi with respect to D^\perp . \square

According to Theorem 3.8 there exists a unique connection D (resp. D^\perp) on \mathcal{D} (resp. \mathcal{D}^\perp) satisfying the conditions from the theorem with respect to the decomposition (13). We call D and D^\perp the *intrinsic connections* on \mathcal{D} associated to C and \mathcal{D}^\perp associated to C' , respectively.

Theorem 3.12. *If $(M, \mathcal{D}, \mathcal{D}')$ is an almost product manifold, D is an intrinsic connection on \mathcal{D} associated to C and D^\perp is an intrinsic connection on \mathcal{D}^\perp associated to C' , then the adapted connection determined by (D, D^\perp) is the Vranceanu connection $\overset{\vee}{\nabla}$ defined by the statistical connection determined with cubic tensor field $C + C'$ on M .*

Proof. We consider the linear connection ∇ defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &+ g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y) + C(QX, QY)QZ + C'(Q'X, Q'Y)Q'Z, \end{aligned} \quad (35)$$

for all $X, Y, Z \in \Gamma(TM)$. Direct calculations imply that ∇ is torsion-free. Moreover, we obtain

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X) = (\nabla_Z g)(X, Y) = C(QX, QY)QZ + C'(Q'X, Q'Y)Q'Z.$$

Thus ∇ is a Codazzi connection on M and consequently it is a statistical connection on M . Now, using (32) and (34) we get $Q\nabla_{QX}QY = D_{QX}QY$, and so (33) implies

$$D_X QY = Q\nabla_{QX}QY + Q[Q'X, QY], \quad \forall X, Y \in \Gamma(TM).$$

In the similar way we obtain

$$D_X^\perp Q'Y = Q'\nabla_{Q^\perp X}Q'Y + Q'[QX, Q'Y], \quad \forall X, Y \in \Gamma(TM).$$

The proof is complete, using two above equation with together (18). \square

4. Statistical structures on Schouten-Van Kampen and Vranceanu connections

In this section, we study the Codazzi (statistical) conditions for Schouten-Van Kampen and Vranceanu connections. Also, we present some interesting examples in these cases.

Theorem 4.1. *Let ∇ be a Codazzi connection on $(M, \mathcal{D}, \mathcal{D}')$. Then the Schouten-Van Kampen connection $\overset{SV}{\nabla}$ determined by ∇ is Codazzi if and only if*

$$F_{ik}^\gamma g_{\gamma j} + F_{jk}^\gamma g_{\gamma i} = F_{ji}^\gamma g_{\gamma k} + F_{ki}^\gamma g_{\gamma j} = F_{kj}^\gamma g_{\gamma i} + F_{ij}^\gamma g_{\gamma k}, \tag{36}$$

$$F_{\alpha\gamma}^r g_{r\beta} + F_{\beta\gamma}^r g_{r\alpha} = F_{\beta\alpha}^r g_{r\gamma} + F_{\gamma\alpha}^r g_{r\beta} = F_{\gamma\beta}^r g_{r\alpha} + F_{\alpha\beta}^r g_{r\gamma}, \tag{37}$$

$$F_{ik}^\gamma g_{\gamma\beta} + F_{\beta k}^\gamma g_{i\gamma} = F_{ki}^\gamma g_{\gamma\beta} + F_{\beta i}^\gamma g_{k\gamma} = F_{i\beta}^\gamma g_{\gamma k} + F_{k\beta}^\gamma g_{i\gamma}, \tag{38}$$

$$F_{\alpha k}^r g_{r\beta} + F_{\beta k}^r g_{\alpha r} = F_{\beta\alpha}^r g_{rk} + F_{k\alpha}^\gamma g_{\beta\gamma} = F_{\alpha\beta}^r g_{rk} + F_{k\beta}^\gamma g_{\alpha\gamma}. \tag{39}$$

Proof. It is easy to see that

$$\begin{cases} (\nabla_{E_k} g)(E_i, E_j) = (\overset{SV}{\nabla}_{E_k} g)(E_i, E_j) - F_{ik}^\gamma g_{\gamma j} - F_{jk}^\gamma g_{\gamma i}, \\ (\nabla_{E_i} g)(E_j, E_k) = (\overset{SV}{\nabla}_{E_i} g)(E_j, E_k) - F_{ji}^\gamma g_{\gamma k} - F_{ki}^\gamma g_{\gamma j}, \\ (\nabla_{E_j} g)(E_k, E_i) = (\overset{SV}{\nabla}_{E_j} g)(E_k, E_i) - F_{kj}^\gamma g_{\gamma i} - F_{ij}^\gamma g_{\gamma k}. \end{cases} \tag{40}$$

Since ∇ is a Codazzi, then ∇g is totally symmetric with respect to $\{E_i, E_j, E_k\}$. Thus using (40) we deduce that $\overset{SV}{\nabla}$ is totally symmetric with respect to $\{E_i, E_j, E_k\}$ if and only if (36) holds. Also, in the similar way we derive that $\overset{SV}{\nabla}$ is totally symmetric with respect to $\{E_\alpha, E_\beta, E_\gamma\}$ if and only if (37) holds. Moreover, it follows that the totally symmetric property of $\overset{SV}{\nabla}$ with respect $\{E_i, E_k, E_\beta\}$ is equivalent with (38) and the totally symmetric property of $\overset{SV}{\nabla}$ with respect $\{E_\alpha, E_\beta, E_k\}$ is equivalent with (39). \square

Corollary 4.2. *Let ∇ be a Codazzi connection on $(M, \mathcal{D}, \mathcal{D}^\perp)$. Then the Schouten-Van Kampen connection $\overset{SV}{\nabla}$ determined by ∇ is Codazzi if and only if*

$$F_{ik}^\gamma g_{\gamma\beta} + F_{\beta k}^r g_{i\gamma} = F_{ki}^\gamma g_{\gamma\beta} + F_{\beta i}^r g_{k\gamma} = 0, \tag{41}$$

$$F_{\beta\alpha}^r g_{rk} + F_{k\alpha}^\gamma g_{\beta\gamma} = F_{\alpha\beta}^r g_{rk} + F_{k\beta}^\gamma g_{\alpha\gamma} = 0. \tag{42}$$

Example 4.3. *Let ∇ be a statistical connection on Riemannian manifold (M, g) and C be the cubic tensor field of ∇ with the coefficients C_{ijk} . It is easy to check that $\widetilde{\nabla}$ given by*

$$\begin{cases} \widetilde{\nabla}_{\delta_j} \delta_i = \Gamma_{ji}^k \delta_k - \frac{1}{2}(y^l R_{jil}^k + g^{kr} C_{irj}) \partial_{\bar{k}}, \\ \widetilde{\nabla}_{\delta_j} \partial_{\bar{i}} = \frac{1}{2}(y^l R_{lij}^k - g^{kr} C_{irj}) \delta_k + (\Gamma_{ji}^k - \frac{1}{2} g^{kr} C_{irj}) \partial_{\bar{k}}, \\ \widetilde{\nabla}_{\partial_{\bar{j}}} \delta_i = \frac{1}{2}(y^l R_{lji}^k - g^{kr} C_{jri}) \delta_k - \frac{1}{2} g^{kr} C_{jri} \partial_{\bar{k}}, \\ \widetilde{\nabla}_{\partial_{\bar{j}}} \partial_{\bar{i}} = -\frac{1}{2} g^{kr} C_{irj} \delta_k, \end{cases} \tag{43}$$

is a Codazzi connection on (TM, g^S) (see [5], for more details). Then the Schouten-Van Kampen connection $\overset{SV}{\nabla}$

determined by $\widetilde{\nabla}$ is as follows

$$\begin{cases} \overset{SV}{\nabla}_{\delta_j} \delta_i = \overset{g}{\Gamma^k_{ji}} \delta_k, \\ \overset{SV}{\nabla}_{\delta_j} \partial_{\bar{i}} = (\overset{g}{\Gamma^k_{ji}} - \frac{1}{2} g^{kr} C_{irj}) \partial_{\bar{k}}, \\ \overset{SV}{\nabla}_{\partial_{\bar{j}}} \delta_i = \frac{1}{2} (y^l R^k_{lji} - g^{kr} C_{jri}) \delta_k, \\ \overset{SV}{\nabla}_{\partial_{\bar{j}}} \partial_{\bar{i}} = 0. \end{cases} \tag{44}$$

Now we study the Codazzi condition for $\overset{SV}{\nabla}$. Since $g^S(\delta_i, \partial_{\bar{j}}) = 0$, then we have $VTM = (HTM)^\perp$. So we must to check relations (41) and (42), only. As $g^S(\delta_i, \delta_j) = g^S(\partial_{\bar{i}}, \partial_{\bar{j}}) = g_{ij}$, then (41) and (42) reduce to the following, respectively:

$$F^r_{ij} g_{rk} + F^r_{kj} g_{ir} = F^r_{ji} g_{rk} + F^r_{ki} g_{rj} = 0, \tag{45}$$

$$F^r_{ij} g_{rk} + F^r_{kj} g_{ir} = F^r_{ji} g_{rk} + F^r_{ki} g_{jr} = 0, \tag{46}$$

where F^C_{AB} , $A, B, C \in \{i, \bar{i}\}$, are coefficients of $\widetilde{\nabla}$. Since C is totally symmetric, then it is easy to see that

$$F^r_{ij} g_{rk} + F^r_{kj} g_{ir} = F^r_{ji} g_{rk} + F^r_{ki} g_{rj} = F^r_{ij} g_{rk} + F^r_{kj} g_{ir} = F^r_{ji} g_{rk} + F^r_{ki} g_{jr} = -C_{ijk}.$$

Thus (45) and (46) are hold if and only if $C_{ijk} = 0$. Therefore, $\overset{SV}{\nabla}$ is Codazzi connection if and only if ∇ reduces to the Levi-Civita connection on M .

Example 4.4. Let ∇ be a statistical connection on Riemannian manifold (M, g) and C be the cubic tensor field of ∇ with the coefficients C_{ijk} . It is easy to check that $\widetilde{\nabla}$ given by

$$\begin{cases} \widetilde{\nabla}_{\delta_j} \delta_i = (\overset{g}{\Gamma^k_{ji}} - \frac{1}{2} K^k_{ij}) \delta_k + y^l R^k_{lji} \partial_{\bar{k}}, \\ \widetilde{\nabla}_{\delta_j} \partial_{\bar{i}} = (\overset{g}{\Gamma^k_{ji}} - \frac{1}{2} K^k_{ij}) \partial_{\bar{k}}, \\ \widetilde{\nabla}_{\partial_{\bar{j}}} \delta_i = -\frac{1}{2} K^k_{ij} \partial_{\bar{k}}, \\ \widetilde{\nabla}_{\partial_{\bar{j}}} \partial_{\bar{i}} = 0, \end{cases} \tag{47}$$

is a Codazzi connection on (TM, g^h) (see [15], for more details). Then the Schouten-Van Kampen connection $\overset{SV}{\nabla}$ determined by $\widetilde{\nabla}$ is as follows

$$\begin{cases} \overset{SV}{\nabla}_{\delta_j} \delta_i = (\overset{g}{\Gamma^k_{ji}} - \frac{1}{2} K^k_{ij}) \delta_k, \\ \overset{SV}{\nabla}_{\delta_j} \partial_{\bar{i}} = (\overset{g}{\Gamma^k_{ji}} - \frac{1}{2} K^k_{ij}) \partial_{\bar{k}}, \\ \overset{SV}{\nabla}_{\partial_{\bar{j}}} \delta_i = \overset{SV}{\nabla}_{\partial_{\bar{j}}} \partial_{\bar{i}} = 0. \end{cases} \tag{48}$$

Now we study the Codazzi condition for $\overset{SV}{\nabla}$. Using (4), equations (36)-(39) reduce to the following

$$F^r_{ik} g_{rj} + F^r_{jk} g_{ri} = F^r_{ji} g_{rk} + F^r_{ki} g_{rj} = F^r_{kj} g_{ri} + F^r_{ij} g_{rk}, \tag{49}$$

$$F^r_{ik} g_{rj} + F^r_{jk} g_{ri} = F^r_{ji} g_{rk} + F^r_{ki} g_{rj} = F^r_{kj} g_{ri} + F^r_{ij} g_{rk}, \tag{50}$$

$$F^r_{ij} g_{rk} + F^r_{kj} g_{ir} = 0, \tag{51}$$

$$F^r_{ik} g_{rj} + F^r_{jk} g_{ri} = 0. \tag{52}$$

It is easy to check that all sides of (49), (50) and (52) are zero, and so these equations are hold. But (51) is equivalent with $K_{ij}^r g_{rk} + K_{kj}^r g_{ir} = 0$. Using (6), the last equation is equivalent with $K_{ij}^r g_{rk} = 0$ or $K_{ij}^r = 0$. Thus $\overset{SV}{\nabla}$ is Codazzi connection if and only if ∇ reduces to the Levi-Civita connection on M . In this case $\widetilde{\nabla}$ reduces to the Levi-Civita connection of (TM, g^h) , i.e., $\overset{g^h}{\nabla}$.

Theorem 4.5. Let $\widetilde{\nabla}$ be a linear connection on M and $\overset{SV}{\nabla}$ be the Schouten-Van Kampen connection determined by $\widetilde{\nabla}$. If $\widetilde{\nabla}$ is torsion-free, then $\overset{SV}{\nabla}$ is torsion-free if and only if \mathcal{D} and \mathcal{D}' are involutive and $F_{ai}^k = F_{ia}^\beta = 0$.

Proof. It is easy to see that the local components of the torsion tensor fields \widetilde{T} and $\overset{SV}{T}$ of $\widetilde{\nabla}$ and Schouten–Van Kampen with respect to the non-holonomic frame field $\{E_A\}$ are as follow:

$$\widetilde{T}_{AB}^k = F_{AB}^k - F_{BA}^k - V_{AB'}^k, \quad \widetilde{T}_{AB}^\alpha = F_{AB}^\alpha - F_{BA}^\alpha - V_{AB'}^\alpha, \tag{53}$$

and

$$\begin{cases} \overset{SV}{T}_{ij}^k = F_{ij}^k - F_{ji}^k - V_{ij'}^k, & \overset{SV}{T}_{ij}^\alpha = -V_{ij'}^\alpha, \\ \overset{SV}{T}_{i\alpha}^k = -\overset{SV}{T}_{\alpha i}^k = F_{i\alpha}^k - V_{i\alpha'}^k, & \overset{SV}{T}_{\alpha i}^\beta = -\overset{SV}{T}_{i\alpha}^\beta = F_{\alpha i}^\beta - V_{\alpha i'}^\beta, \\ \overset{SV}{T}_{\alpha\beta}^k = -V_{\alpha\beta'}^k, & \overset{SV}{T}_{\alpha\beta}^\gamma = F_{\alpha\beta}^\gamma - F_{\beta\alpha}^\gamma - V_{\alpha\beta'}^\gamma. \end{cases} \tag{54}$$

Since $\widetilde{\nabla}$ is torsion-free, then we have $\widetilde{T}_{AB}^C = 0$. Thus using (53), (54) reduces to

$$\overset{SV}{T}_{ij}^k = \overset{SV}{T}_{\alpha\beta}^\gamma = 0, \quad \overset{SV}{T}_{ij}^\alpha = -V_{ij'}^\alpha, \quad \overset{SV}{T}_{i\alpha}^k = -\overset{SV}{T}_{\alpha i}^k = F_{\alpha i}^k, \quad \overset{SV}{T}_{\alpha i}^\beta = -\overset{SV}{T}_{i\alpha}^\beta = F_{i\alpha}^\beta, \quad \overset{SV}{T}_{\alpha\beta}^k = -V_{\alpha\beta'}^k. \tag{55}$$

From the above equation we conclude that $\overset{SV}{T} = 0$ if and only if $V_{ij}^\alpha = 0$ (i.e., \mathcal{D} is involutive), $V_{\alpha\beta}^k = 0$ (i.e., \mathcal{D}' is involutive) and $F_{ai}^k = F_{ia}^\beta = 0$. These complete the proof. \square

Theorems 4.1 and 4.5 imply the following

Corollary 4.6. Let $\widetilde{\nabla}$ be a linear connection on M and $\overset{SV}{\nabla}$ be the Schouten-Van Kampen connection determined by $\widetilde{\nabla}$. If $\widetilde{\nabla}$ is statistical connection, then $\overset{SV}{\nabla}$ is statistical connection if and only if $\mathcal{D}, \mathcal{D}'$ are involutive, $F_{ai}^k = F_{ia}^\beta = 0$ and moreover, (36)–(39) are hold.

Example 4.7. We consider the Codazzi connection $\widetilde{\nabla}$ introduced by Example 4.3. It is easy to check that it is torsion-free and so it is a statistical connection on (TM, g^S) . Now we study the conditions that $\overset{SV}{\nabla}$ can be statistical connection. Since $\overset{SV}{\nabla}$ is Codazzi connection if and only if ∇ reduces to the Levi-Civita connection on M , then according to Corollary 4.6, $\overset{SV}{\nabla}$ is statistical if and only if ∇ reduces to the Levi-Civita connection on M , also HTM and VTM are involutive and moreover $F_{ji}^k = F_{ij}^k = 0$. It is known that VTM is involutive, but HTM is not involutive, unless $R_{ijl}^k = 0$, for all $i, j, k, l \in \{1, \dots, n\}$. Indeed, HTM is involutive if and only if M is locally flat. Considering $R_{ijl}^k = 0$, from (43) we get $F_{ji}^k = F_{ij}^k = 0$. Thus $\overset{SV}{\nabla}$ is statistical if and only if M is locally flat manifold and ∇ reduces to the Levi-Civita connection on (M, g) . Note that in this case, $\overset{SV}{\nabla}$ and $\widetilde{\nabla}$ are the same and they reduce to the Levi-Civita connection of (TM, g^S) .

Example 4.8. Using Corollary 4.6, it is easy to see that $\overset{SV}{\nabla}$ given by Example 4.4 is statistical connection on (TM, g^h) if and only if ∇ reduces to the locally flat Levi-Civita connection on (M, g) .

Theorem 4.9. Let ∇ be a Codazzi connection on $(M, \mathcal{D}, \mathcal{D}')$. Then the Vrănceanu connection $\overset{V}{\nabla}$ determined by ∇ is Codazzi if and only if

- (i) $F_{ik}^\gamma g_{\gamma j} + F_{jk}^\gamma g_{\gamma i} = F_{ji}^\gamma g_{\gamma k} + F_{ki}^\gamma g_{\gamma j} = F_{kj}^\gamma g_{\gamma i} + F_{ij}^\gamma g_{\gamma k}$,
- (ii) $V_{\alpha k}^{\gamma\gamma} g_{i\gamma} - F_{ik}^\gamma g_{\gamma\alpha} - F_{\alpha k}^r g_{ir} - F_{\alpha k}^\gamma g_{i\gamma} = V_{\alpha i}^{\gamma\gamma} g_{k\gamma} - F_{ki}^\gamma g_{\gamma\alpha} - F_{\alpha i}^r g_{kr} - F_{\alpha i}^\gamma g_{k\gamma}$
 $= V_{k\alpha}^r g_{ri} + V_{i\alpha}^r g_{rk} - F_{k\alpha}^r g_{ri} - F_{k\alpha}^\gamma g_{\gamma i} - F_{i\alpha}^r g_{rk} - F_{i\alpha}^\gamma g_{k\gamma}$,
- (iii) $V_{\alpha k}^{\gamma\gamma} g_{\gamma\beta} + V_{\beta k}^{\gamma\gamma} g_{\alpha\gamma} - F_{\alpha k}^r g_{r\beta} - F_{\alpha k}^\gamma g_{\gamma\beta} - F_{\beta k}^r g_{\alpha r} - F_{\beta k}^\gamma g_{\alpha\gamma}$
 $= V_{k\alpha}^r g_{\beta r} - F_{\beta\alpha}^r g_{rk} - F_{k\alpha}^r g_{\beta r} - F_{k\alpha}^\gamma g_{\beta\gamma} = V_{k\beta}^r g_{\alpha r} - F_{\alpha\beta}^r g_{rk} - F_{k\beta}^r g_{\alpha r} - F_{k\beta}^\gamma g_{\alpha\gamma}$,
- (v) $F_{\alpha k}^r g_{r\beta} + F_{\beta k}^r g_{\alpha r} = F_{\beta\alpha}^r g_{rk} + F_{k\alpha}^\gamma g_{\beta\gamma} = F_{\alpha\beta}^r g_{rk} + F_{k\beta}^\gamma g_{\alpha\gamma}$.

Proof. The proof of (i) and (v) are the same of (36) and (39). It is easy to see that

$$\begin{cases} (\nabla_{E_k} g)(E_i, E_\alpha) = (\overset{V}{\nabla}_{E_k} g)(E_i, E_\alpha) + V_{\alpha k}^{\gamma\gamma} g_{i\gamma} - F_{ik}^\gamma g_{\gamma\alpha} - F_{\alpha k}^r g_{ir} - F_{\alpha k}^\gamma g_{i\gamma}, \\ (\nabla_{E_i} g)(E_\alpha, E_k) = (\overset{V}{\nabla}_{E_i} g)(E_\alpha, E_k) + V_{\alpha i}^{\gamma\gamma} g_{k\gamma} - F_{ki}^\gamma g_{\gamma\alpha} - F_{\alpha i}^r g_{kr} - F_{\alpha i}^\gamma g_{k\gamma}, \\ (\nabla_{E_\alpha} g)(E_k, E_i) = (\overset{V}{\nabla}_{E_\alpha} g)(E_k, E_i) + V_{k\alpha}^r g_{ri} + V_{i\alpha}^r g_{rk} - F_{k\alpha}^r g_{ri} - F_{k\alpha}^\gamma g_{\gamma i} - F_{i\alpha}^r g_{rk} - F_{i\alpha}^\gamma g_{k\gamma}. \end{cases} \tag{56}$$

Since ∇ is a Codazzi, then ∇g is totally symmetric with respect to $\{E_i, E_k, E_\alpha\}$. Thus using (56) we deduce that $\overset{V}{\nabla}$ is totally symmetric with respect to $\{E_i, E_k, E_\alpha\}$ if and only if (ii) holds. Also, in the similar way we derive that $\overset{V}{\nabla}$ is totally symmetric with respect to $\{E_\alpha, E_\beta, E_k\}$ if and only if (iii) holds. \square

Corollary 4.10. Let ∇ be a Codazzi connection on $(M, \mathcal{D}, \mathcal{D}^\perp)$. Then the Vrănceanu connection $\overset{V}{\nabla}$ determined by ∇ is Codazzi if and only if

- (i) $-F_{ik}^\gamma g_{\gamma\alpha} - F_{\alpha k}^r g_{ir} = -F_{ki}^\gamma g_{\gamma\alpha} - F_{\alpha i}^r g_{kr} = V_{k\alpha}^r g_{ri} + V_{i\alpha}^r g_{rk} - F_{k\alpha}^r g_{ri} - F_{i\alpha}^r g_{rk}$,
- (ii) $V_{\alpha k}^{\gamma\gamma} g_{\gamma\beta} + V_{\beta k}^{\gamma\gamma} g_{\alpha\gamma} - F_{\alpha k}^r g_{r\beta} - F_{\beta k}^r g_{\alpha r} = -F_{\beta\alpha}^r g_{rk} - F_{k\alpha}^\gamma g_{\beta\gamma} = -F_{\alpha\beta}^r g_{rk} - F_{k\beta}^\gamma g_{\alpha\gamma}$,
- (iii) $F_{\beta\alpha}^r g_{rk} + F_{k\alpha}^\gamma g_{\beta\gamma} = F_{\alpha\beta}^r g_{rk} + F_{k\beta}^\gamma g_{\alpha\gamma} = 0$.

Example 4.11. Let ∇ be a statistical connection on Riemannian manifold (M, g) , C be the cubic tensor field of ∇ with the coefficients C_{ijk} and $\widetilde{\nabla}$ be the Codazzi connection $\widetilde{\nabla}$ given by (43) on (TM, g^S) . Since $V_{ij}^k = 0$ and $V_{ji}^{\bar{k}} = \Gamma_{ji}^k$, then we have

$$\overset{V}{\Gamma}_{ij}^k = F_{ij}^k, \quad \overset{V}{\Gamma}_{i\bar{j}}^k = 0, \quad \overset{V}{\Gamma}_{i\bar{j}}^{\bar{k}} = \overset{g}{\Gamma}_{ij}^k, \quad \overset{V}{\Gamma}_{i\bar{j}}^{\bar{k}} = F_{i\bar{j}}^{\bar{k}}$$

and so the Vrănceanu connection $\overset{V}{\nabla}$ determined by $\widetilde{\nabla}$ is as follows

$$\begin{cases} \overset{V}{\nabla}_{\delta_i} \delta_i = \overset{g}{\Gamma}_{ji}^k \delta_k, & \overset{V}{\nabla}_{\delta_i} \partial_{\bar{i}} = \overset{g}{\Gamma}_{ji}^k \partial_{\bar{k}}, \\ \overset{V}{\nabla}_{\partial_{\bar{j}}} \delta_i = \overset{V}{\nabla}_{\partial_{\bar{j}}} \partial_{\bar{i}} = 0. \end{cases} \tag{57}$$

Direct calculations give $F_{\bar{j}\bar{i}}^r + F_{ki}^r g_{jr} = -C_{ijk}$. So (iii) of the above corollary is hold if and only if $C_{ijk} = 0$, i.e., ∇ reduces to the Levi-Civita connection on (M, g) . Using it, we get

$$V_{ik}^{\bar{r}\bar{r}} g_{rj} + V_{jk}^{\bar{r}\bar{r}} g_{ir} - F_{ik}^{\bar{r}} g_{rj} - F_{jk}^{\bar{r}} g_{ir} = \overset{g}{\Gamma}_{ik}^r g_{rj} + \overset{g}{\Gamma}_{jk}^r g_{ir} - \overset{g}{\Gamma}_{ik}^r g_{rj} - \overset{g}{\Gamma}_{jk}^r g_{ir} = 0.$$

Similarly the second and third sentences of (ii) of the above corollary are zero, and so (ii) is holds. We have

$$F_{ik}^r g_{rj} + F_{jk}^r g_{ir} = -\frac{1}{2} y^l (R_{kilj} - R_{ljki}) = 0.$$

Thus the first sentence of (i) of the above corollary is zero. Similarly, we deduce that the second sentence of it, is zero, too. Moreover, since $V_{kj}^r = V_{jk}^r = 0$, then the third sentence if it also is zero. Therefore (i) is hold. According to these explanations we conclude that $\overset{V}{\nabla}$ is statistical connection if and only if ∇ reduces to the Levi-Civita connection.

Example 4.12. Let ∇ be a statistical connection on Riemannian manifold (M, g) , C be the cubic tensor field of ∇ with the coefficients C_{ijk} and $\tilde{\nabla}$ be the Codazzi connection on (TM, g^h) given by Example 4.4. Then the Vrănceanu connection $\overset{V}{\nabla}$ determined by $\tilde{\nabla}$ is as follows

$$\begin{cases} \overset{V}{\nabla}_{\delta_j} \delta_i = (\overset{g}{\Gamma}{}^k_{ji} - \frac{1}{2} K_{ij}^k) \delta_k, \\ \overset{V}{\nabla}_{\delta_j} \partial_{\bar{i}} = \overset{g}{\Gamma}{}^k_{ji} \partial_{\bar{k}}, \\ \overset{V}{\nabla}_{\partial_j} \delta_i = \overset{V}{\nabla}_{\partial_j} \partial_{\bar{i}} = 0. \end{cases} \tag{58}$$

Now we check the Codazzi conditions for $\overset{V}{\nabla}$. Using (4), (i)-(v) of Theorem 4.9 reduces to the following

$$F_{ik}^r g_{rj} + F_{jk}^r g_{ri} = F_{ji}^r g_{rk} + F_{ki}^r g_{rj} = F_{kj}^r g_{ri} + F_{ij}^r g_{rk}, \tag{59}$$

$$V_{jk}^r g_{ir} - F_{jk}^r g_{ir} = V_{ji}^r g_{kr} - F_{ji}^r g_{kr} = -F_{kj}^r g_{ri} - F_{ij}^r g_{kr}, \tag{60}$$

$$F_{ik}^r g_{rj} + F_{jk}^r g_{ir} = F_{ki}^r g_{jr} = F_{kj}^r g_{ir} = 0. \tag{61}$$

We get

$$F_{ik}^r g_{rj} + F_{jk}^r g_{ri} = y^l (R_{lkij} + R_{lkji}) = 0.$$

Thus the first sentence of (59) is zero. Similarly we the second and third sentences of (59) are zero and so (59) is holds. Easily we can see that (61) is holds. We can see that the first and second sentences of (60) are equal to $\frac{1}{2} K_{jk}^r$, while the third sentence of is equal to K_{jk}^r . Thus (60) is holds if and only if $K_{jk}^r = 0$, i.e., ∇ reduces to the Levi-Civita connection on (M, g) . Therefore $\overset{V}{\nabla}$ is statistical if and only if ∇ reduces to the Levi-Civita connection on (M, g) .

Theorem 4.13. Let ∇ be a linear connection on M and $\overset{V}{\nabla}$ be the Vrănceanu connection determined by ∇ . If ∇ is torsion-free, then $\overset{V}{\nabla}$ is torsion-free if and only if \mathcal{D} and \mathcal{D}' are involutive.

Proof. It is easy to see that the local components of the torsion tensor field $\overset{V}{T}$ of Vrănceanu connection with respect to the non-holonomic frame field $\{E_A\}$ are as follow:

$$\begin{cases} \overset{V}{T}_{ij}^k = F_{ij}^k - F_{ji}^k - V_{ij}^k, & \overset{V}{T}_{ij}^\alpha = -V_{ij}^{\alpha'}, \\ \overset{V}{T}_{ia}^k = -\overset{V}{T}_{ai}^k = 0, & \overset{V}{T}_{ai}^\beta = -\overset{V}{T}_{ia}^\beta = 0, \\ \overset{V}{T}_{\alpha\beta}^k = -V_{\alpha\beta}^k, & \overset{V}{T}_{\alpha\beta}^{\gamma'} = F_{\alpha\beta}^{\gamma'} - F_{\beta\alpha}^{\gamma'} - V_{\alpha\beta}^{\gamma'}. \end{cases} \tag{62}$$

Since ∇ is torsion-free, then we have $T_{AB}^C = 0$. Thus using (53), (62) reduces to

$$T_{ij}^k = T_{\alpha\beta}^\gamma = 0, \quad T_{ij}^\alpha = -V_{ij}^{\prime\alpha}, \quad T_{i\alpha}^k = -T_{\alpha i}^k = 0, \quad T_{\alpha i}^\beta = -T_{i\alpha}^\beta = 0, \quad T_{\alpha\beta}^k = -V_{\alpha\beta}^k. \quad (63)$$

From the above equation we conclude that $T = 0$ if and only if $V_{ij}^{\prime\alpha} = 0$ (i.e., \mathcal{D} is involutive), $V_{\alpha\beta}^k = 0$ (i.e., \mathcal{D}' is involutive). \square

Corollary 4.14. Let ∇ be a linear connection on M and $\tilde{\nabla}$ be the Vrănceanu connection determined by ∇ . If ∇ is statistical connection, then $\tilde{\nabla}$ is statistical connection if and only if \mathcal{D} , \mathcal{D}' are involutive and moreover, (i)-(v) of Theorem 4.9 are hold.

Example 4.15. We consider the statistical connection $\tilde{\nabla}$ introduced by Example 4.3 on (TM, g^S) . Now we study the conditions that $\tilde{\nabla}$ can be statistical connection. Since $\tilde{\nabla}$ is Codazzi linear connection if and only if ∇ reduces to the Levi-Civita connection, then according to Corollary 4.14, $\tilde{\nabla}$ is statistical if and only if HTM and VTM are involutive and moreover ∇ reduces to the Levi-Civita connection. Thus $\tilde{\nabla}$ is statistical if and only if reduces to the flat Levi-Civita connection on (M, g) . Note that in this case, $\tilde{\nabla}$ and $\tilde{\tilde{\nabla}}$ are the same and they reduce to the Levi-Civita connection of (TM, g^S) .

Example 4.16. Similar to the above example, it is easy to see that $\tilde{\nabla}$ given by Example 4.4 is statistical connection on (TM, g^h) if and only if ∇ reduces to the locally flat Levi-Civita connection on (M, g) .

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