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k-Type Hyperbolic Slant Helices in $\mathbb{H}^3$

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Abstract.
In the present paper, we give the notion of k-type hyperbolic slant helices in $\mathbb{H}^3$, where $k \in \{0, 1, 2, 3\}$. We give the necessary and sufficient conditions for hyperbolic curves to be k-type slant helices in terms of their hyperbolic curvature functions.

1. Introduction

The notion of a slant helix was due to Izumiya and Takeuchi ([6]). A curve γ with non-zero curvature is called a slant helix in Euclidean 3-space $\mathbb{R}^3$ if the principal normal line of γ makes a constant angle with a fixed vector in $\mathbb{R}^3$. Also some characterizations of such curves were presented in [1, 7, 8, 14]. Slant helices are the successor curves of the general helices. In particular, they are geodesics of the helix surfaces.

Further, k-type slant helices emerged and attracted attention of researchers. Ergüt et al ([5]) studied k-slant helices in Minkowski 3-space, $\mathbb{R}^3_1$. Also curves of such a type were studied in Minkowski space-time by some researchers such as [2, 10]. Lastly, in [12, 13], the authors studied k-slant helices for null curves in lightlike cone in Minkowski space-time and k-type spacelike slant helices lying on ligthlike surfaces.

On the other hand, in [9], the author considered hyperbolic curves in 3-dimensional hyperbolic space, and construct the hyperbolic frame of the hyperbolic space curves. Also, the author studied the associated curve of a hyperbolic curve in $\mathbb{H}^3$. Hyperbolic curves in $\mathbb{H}^3$ according to their Frenet frame, are characterized in [4].

In this paper, we introduce the notion of k-type hyperbolic slant helices in $\mathbb{H}^3$, where $k \in \{0, 1, 2, 3\}$. We give the necessary and sufficient conditions for hyperbolic curves to be k-type slant helices in terms of their hyperbolic curvature functions. Finally, we give the related examples.

2. Priliminaries

The Minkowski space-time $\mathbb{E}^4_1$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$\langle\ ,\ \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

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where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_1^4$. Recall that a vector $v \in \mathbb{E}_1^4 \backslash \{0\}$ can be spacelike if $\langle v, v\rangle > 0$, timelike if $\langle v, v\rangle < 0$ and null (lightlike) if $\langle v, v\rangle = 0$. In particular, the vector $v = 0$ is said to be a spacelike. The norm of a vector v is given by $\|v\| = \sqrt{|\langle v, v\rangle|}$. Two vectors v and w are said to be orthogonal, if $\langle v, w\rangle = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$, can locally be spacelike, timelike or null (lightlike), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null [11].

A null curve α is parameterized by pseudo-arc s if $\langle \alpha''(s), \alpha''(s)\rangle = 1$ [3]. On the other hand, a non-null curve α is parametrized by the arc-length parameter s if $\langle \alpha'(s), \alpha'(s)\rangle = \pm 1$.

Let m be a fixed point and $r > 0$ be a constant. The pseudo-Riemannian hyperbolic space is defined by

$$\mathbb{H}^3(m, r) = \{u \in \mathbb{E}_1^4 : \langle u - m, u - m\rangle = -r^2\}.$$

When $m = 0$ and $r = 1$, we denote $\mathbb{H}^3(0, 1)$ by $\mathbb{H}^3$.

For the regular curve $x(s) \subset \mathbb{H}^3 \subset \mathbb{E}_1^4$ with hyperbolic Frenet frame $\{x(s), \alpha(s), \beta(s), y(s)\}$ and hyperbolic curvature functions $\kappa(s)$, $\tau(s)$, the Frenet formulas of hyperbolic space curve $x(s)$ in $\mathbb{H}^3$ can be written as

$$\begin{cases} x'(s) &= \alpha(s), \\ \alpha'(s) &= x(s) + \kappa(s)\, y(s), \\ \beta'(s) &= \tau(s)\, y(s), \\ y'(s) &= -\kappa(s)\,\alpha(s) - \tau(s)\,\beta(s), \end{cases} \tag{1}$$

where for all s,

$$\langle x(s), x(s)\rangle = -1, \quad \langle \alpha(s), \alpha(s)\rangle = \langle \beta(s), \beta(s)\rangle = \langle y(s), y(s)\rangle = 1,$$

$$\langle x(s), \alpha(s)\rangle = \langle x(s), \beta(s)\rangle = \langle x(s), y(s)\rangle = 0,$$

$$\langle \alpha(s), \beta(s)\rangle = \langle \alpha(s), y(s)\rangle = \langle \beta(s), y(s)\rangle = 0.$$

If $\langle x''(s), x''(s)\rangle = -1$, together with $\langle x(s), x(s)\rangle = \langle x(s), x''(s)\rangle = -1$ we know that $x''(s) = x(s)$. So we assume that $\langle x''(s), x''(s)\rangle > -1$ and call the curve regular ([9]).

3. k-type hyperbolic slant helices in 3-dimensional hyperbolic space $\mathbb{H}^3$

In this section, we study k-type hyperbolic slant helices in hyperbolic space $\mathbb{H}^3$. Let us set that

$$V_0 = x, \quad V_1 = \alpha, \quad V_2 = \beta, \quad V_3 = y.$$

In the following definition, we introduce the k-type slant helices lying in pseudohyperbolic space $\mathbb{H}^3$.

Definition 3.1. *A hyperbolic space curve $x(s)$ parametrized by arc-length s with hyperbolic Frenet frame $\{V_0, V_1, V_2, V_3\}$ in pseudohyperbolic space $\mathbb{H}^3$ is called a k-type hyperbolic slant helix for $k \in \{0, 1, 2, 3\}$ if there exists a non-zero fixed vector $U \in \mathbb{E}_1^4$ such that the following holds*

$$\langle V_k, U\rangle = \text{constant}.$$

Firstly, we consider 0-type hyperbolic slant helices in $\mathbb{H}^3$.

Theorem 3.2. *Let $x(s)$ be a hyperbolic space curve in $\mathbb{H}^3$ parametrized by arc-length s with non-zero curvatures κ, τ. Then $x(s)$ is a 0-type hyperbolic slant helix if and only if*

$$\left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'' + \frac{\tau}{\kappa} = 0. \tag{2}$$

Proof. Assume that $x(s)$ is a 0-type hyperbolic slant helix in $\mathbb{H}^3$ parametrized by arc-length s with non-zero curvatures κ, τ. Then there exists a non-zero fixed vector $U \in \mathbb{E}_1^4$ such that

$$\langle x, U\rangle = c, \quad c \in \mathbb{R}. \tag{3}$$

Taking derivative of the equation (3) with respect to s and using Frenet equations (1), we get

$$\langle \alpha, U\rangle = 0, \quad \langle y, U\rangle = -\frac{c}{\kappa}. \tag{4}$$

By using (4), we can write U with respect to the frame $\{x, \alpha, \beta, y\}$ as follows

$$U = -cx + \lambda\beta - \frac{c}{\kappa}y, \tag{5}$$

where λ is some differentiable function of s and $c \in \mathbb{R}\setminus\{0\}$. Taking derivative of the equation (5) with respect to s and using Frenet equations (1), we have

$$\left(\lambda' + c\frac{\tau}{\kappa}\right)\beta + \left(\lambda\tau - c\left(\frac{1}{\kappa}\right)'\right)y = 0$$

which implies that

$$\left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'' + \frac{\tau}{\kappa} = 0.$$

Conversely, assume that (2) holds. Choosing the vector U as

$$U = -c\left[x - \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\beta + \frac{1}{\kappa}y\right], \tag{6}$$

we get $U' = 0$ and $\langle x, U\rangle = c$ (constant). Thus $x(s)$ is a 0-type hyperbolic slant helix. □

Example 3.3. *The hyperbolic curvature functions*

$$\kappa = \frac{\sqrt{s^4 + 6s^2 + 10}}{s^2 + 2} \qquad \text{and} \qquad \tau = \frac{2s^2}{s^4 + 6s^2 + 10}$$

satisfy (2)*. The hyperbolic curve* $x(s)$ *with the hyperbolic curvature functions* κ *and* τ *can be written as*

$$x(s) = \left(\sqrt{s^2 + 2}, s\cos A, 1, s\sin A\right)$$

with

$$\begin{aligned}
\alpha(s) &= \left(\frac{s}{\sqrt{s^2+2}}, \cos A - \frac{s\sin A}{\sqrt{s^2+2}}, 0, \sin A + \frac{s\cos A}{\sqrt{s^2+2}}\right), \\
y(s) &= \left(\frac{-s^4 - 4s^2 - 2}{\sqrt{s^2+2}\sqrt{s^4+6s^2+10}}, \frac{-s\sqrt{s^2+2}\left(3+s^2\right)\cos A - \left(4+s^2\right)\sin A}{\sqrt{s^2+2}\sqrt{s^4+6s^2+10}},\right. \\
&\qquad \left.\frac{-s^2-2}{\sqrt{s^4+6s^2+10}}, \frac{\left(4+s^2\right)\cos A - s\sqrt{s^2+2}\left(3+s^2\right)\sin A}{\sqrt{s^2+2}\sqrt{s^4+6s^2+10}}\right), \\
\beta(s) &= \left(\frac{2\sqrt{s^2+2}}{\sqrt{s^4+6s^2+10}}, \frac{s\sqrt{s^2+2}\cos A - \left(s^2+2\right)\sin A}{\sqrt{s^2+2}\sqrt{s^4+6s^2+10}},\right. \\
&\qquad \left.\frac{4+s^2}{\sqrt{s^4+6s^2+10}}, \frac{s\sqrt{s^2+2}\sin A + \left(s^2+2\right)\cos A}{\sqrt{s^2+2}\sqrt{s^4+6s^2+10}}\right),
\end{aligned}$$

where $A = \operatorname{arcsinh}\frac{s}{\sqrt{2}}$. *So we get*

$$U = -c\left[x - \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\beta + \frac{1}{\kappa}y\right] = (0, 0, c, 0)$$

and $\langle x, U\rangle = c$ *(constant). Thus* $x(s)$ *is a* 0*-type hyperbolic slant helix.*

Example 3.4. *The following hyperbolic curvature functions satisfy* (2).
(i) $\kappa = 1/\cos s$, $\tau = 1$ $\quad$ (ii) $\kappa = 1/\cos(\ln s)$, $\tau = 1/s$

Corollary 3.5. *The axis of a* 0*-type hyperbolic slant helix is given by*

$$U = -c\left[x - \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\beta + \frac{1}{\kappa}y\right] \tag{7}$$

where $c \in \mathbb{R}\setminus\{0\}$.

Corollary 3.6. *Let* $x(s)$ *be a hyperbolic space curve in* $\mathbb{H}^3$ *parametrized by arc-length* s *with non-zero curvatures* κ, τ. *Then* $x(s)$ *is a* 0*-type hyperbolic slant helix if and only if*

$$\frac{1}{\tau^2}\left(\left(\frac{1}{\kappa}\right)'\right)^2 + \frac{1}{\kappa^2} = constant. \tag{8}$$

Proof. Assume that $x(s)$ is a 0-type hyperbolic slant helix in $\mathbb{H}^3$ parametrized by arc-length s with non-zero curvatures κ, τ. From (7), we have

$$\frac{1}{\tau^2}\left(\left(\frac{1}{\kappa}\right)'\right)^2 + \frac{1}{\kappa^2} = \text{constant}.$$

Conversely, assume that the relation (8) holds. Then taking derivative of the equation (8) with respect to s, we get

$$\left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'' + \frac{\tau}{\kappa} = 0$$

which means that $x(s)$ is a 0-type hyperbolic slant helix. □

Secondly, we consider 1-type hyperbolic slant helices in $\mathbb{H}^3$.

Theorem 3.7. *Let* $x(s)$ *be a hyperbolic space curve in* $\mathbb{H}^3$ *parametrized by arc-length* s *with non-zero curvatures* κ, τ. *Then* $x(s)$ *is a* 1*-type hyperbolic slant helix if and only if*

$$c_1\left(\tfrac{1}{\tau}\right)'\left(\tfrac{1}{\kappa} - \kappa\right) - \left(\tfrac{1}{\tau}\right)'\left(\tfrac{1}{\kappa}\right)'(-c_1 s + c_2) + c_1\tfrac{1}{\tau}\left(2\left(\tfrac{1}{\kappa}\right)' - \kappa'\right) - \tfrac{1}{\tau}\left(\tfrac{1}{\kappa}\right)''(-c_1 s + c_2) - \tfrac{\tau}{\kappa}(-c_1 s + c_2) = 0, \tag{9}$$

where $c_1, c_2 \in \mathbb{R}$ *and* $(c_1, c_2) \neq (0, 0)$.

Proof. Assume that $x(s)$ is a 1-type hyperbolic slant helix in $\mathbb{H}^3$ parametrized by arc-length s with non-zero curvatures κ, τ. Then there exists a non-zero fixed vector $U \in \mathbb{E}_1^4$ such that

$$\langle \alpha, U\rangle = c_1, \quad c_1 \in \mathbb{R}. \tag{10}$$

Then we can write U with respect to the frame $\{x, \alpha, \beta, y\}$ as follows

$$U = \lambda_1 x + c_1\alpha + \lambda_3\beta + \lambda_4 y \tag{11}$$

where λ_1, λ_3 and λ_4 are some differentiable functions of s. Differentiating the equation (11) with respect to s and using Frenet equations (1), we get

$$0 = \left(\lambda_1' + c_1\right) x + (\lambda_1 - \kappa\lambda_4)\,\alpha + \left(\lambda_3' - \tau\lambda_4\right)\beta + \left(c_1\kappa + \lambda_3\tau + \lambda_4'\right) y$$

which implies that

$$\begin{cases} \lambda_1' + c_1 = 0, \\ \lambda_1 - \kappa\lambda_4 = 0, \\ \lambda_3' - \tau\lambda_4 = 0, \\ c_1\kappa + \lambda_3\tau + \lambda_4' = 0. \end{cases} \tag{12}$$

Solving (12), we get

$$c_1\left(\tfrac{1}{\tau}\right)'\left(\tfrac{1}{\kappa} - \kappa\right) - \left(\tfrac{1}{\tau}\right)'\left(\tfrac{1}{\kappa}\right)'(-c_1 s + c_2) + c_1\tfrac{1}{\tau}\left(2\left(\tfrac{1}{\kappa}\right)' - \kappa'\right) - \tfrac{1}{\tau}\left(\tfrac{1}{\kappa}\right)''(-c_1 s + c_2) - \tfrac{\tau}{\kappa}(-c_1 s + c_2) = 0,$$

where $c_1, c_2 \in \mathbb{R}$ and $(c_1, c_2) \neq (0, 0)$.

Conversely, assume that the relation (9) holds. Then choosing the vector U as follows

$$U = (-c_1 s + c_2)\,x + c_1\alpha + \frac{1}{\tau}\left[c_1\left(\frac{1}{\kappa} - \kappa\right) - \left(\frac{1}{\kappa}\right)'(-c_1 s + c_2)\right]\beta + \frac{1}{\kappa}(-c_1 s + c_2)\,y,$$

we get $U' = 0$ and $\langle \alpha, U\rangle = c_1$ (constant). Thus $x(s)$ is a 1-type hyperbolic slant helix. □

Example 3.8. *The following hyperbolic curvature functions satisfy* (9).
(i) $c_1 = 0$, $c_2 = 1$, $\kappa = 1/\sin s$, $\tau = 1$.

Corollary 3.9. *The axis of a* 1*-type hyperbolic slant helix is given by*

$$U = (-c_1 s + c_2)\,x + c_1\alpha + \frac{1}{\tau}\left[c_1\left(\frac{1}{\kappa} - \kappa\right) - \left(\frac{1}{\kappa}\right)'(-c_1 s + c_2)\right]\beta + \frac{1}{\kappa}(-c_1 s + c_2)\,y,$$

where $c_1, c_2 \in \mathbb{R}$ *and* $(c_1, c_2) \neq (0, 0)$.

Assume that $c_1 = 0$ in (9), Then we have $c_2 \neq 0$ and

$$\left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'' + \frac{\tau}{\kappa} = 0.$$

Then $x(s)$ is a 0-type hyperbolic slant helix. Thus we give the following corollary.

Corollary 3.10. *Let $x(s)$ be a hyperbolic space curve in $\mathbb{H}^3$ parametrized by arc-length s with non-zero curvatures κ, τ. Then $x(s)$ is a* 0*-type hyperbolic slant helix if and only if $x(s)$ is a* 1*-type hyperbolic slant helix whose axis U satisfies $\langle \alpha, U\rangle = 0$.*

Thirdly, we consider 2-type hyperbolic slant helices in $\mathbb{H}^3$.

Theorem 3.11. *Let $x(s)$ be a hyperbolic space curve in $\mathbb{H}^3$ parametrized by arc-length s with non-zero curvatures κ, τ. Then $x(s)$ is a* 2*-type hyperbolic slant helix if and only if*

$$\left(\frac{\tau}{\kappa}\right)'' - \frac{\tau}{\kappa} = 0, \tag{13}$$

or equivalently

$$\frac{\tau}{\kappa} = c_1 e^{s} + c_2 e^{-s}.$$

Proof. Assume that $x(s)$ is a 2-type hyperbolic slant helix in $\mathbb{H}^3$ parametrized by arc-length s with non-zero curvatures κ, τ. Then there exists a non-zero fixed vector $U \in \mathbb{E}^4_1$ such that

$$\langle \beta, U \rangle = c, \quad c \in \mathbb{R}. \tag{14}$$

Assume that $c = 0$. Then $U = 0$ which is a contradiction. So $c \neq 0$.

Taking derivative of the equation (14) with respect to s and using Frenet equations (1), we get

$$\langle \alpha, U \rangle = -\frac{\tau}{\kappa} c, \quad \langle y, U \rangle = 0. \tag{15}$$

By using (15), we can write U with respect to the frame $\{x, \alpha, \beta, y\}$ as follows

$$U = \lambda x - \frac{\tau}{\kappa} c\alpha + c\beta \tag{16}$$

where λ is some differentiable function of s. Differentiating the equation (16) with respect to s and using Frenet equations (1), we get

$$0 = \left(\lambda' - \frac{\tau}{\kappa} c\right) x + \left(\lambda - c\left(\frac{\tau}{\kappa}\right)'\right)\alpha$$

which implies that

$$\left(\frac{\tau}{\kappa}\right)'' - \frac{\tau}{\kappa} = 0,$$

or equivalently

$$\frac{\tau}{\kappa} = c_1 e^s + c_2 e^{-s}.$$

Conversely, assume that the relation (13) holds. Then choosing the vector U as follows

$$U = c\left(\frac{\tau}{\kappa}\right)' x - \frac{\tau}{\kappa} c\alpha + c\beta,$$

where $c \in \mathbb{R} \setminus \{0\}$, we get $U' = 0$ and $\langle \beta, U \rangle = c$ (constant). Thus $x(s)$ is a 2-type hyperbolic slant helix. □

Example 3.12. *The following hyperbolic curvature functions satisfy* (13).
(*i*) $\kappa = 1, \tau = e^s$ (*ii*) $\kappa = e^s, \tau = 1$

Corollary 3.13. *The axis of a* 2*-type hyperbolic slant helix is given by*

$$U = c\left(c_1 e^s - c_2 e^{-s}\right) x - \left(c_1 e^s + c_2 e^{-s}\right)\alpha + c\beta,$$

where $c \in \mathbb{R} \setminus \{0\}$.

Lastly, we consider 3-type hyperbolic slant helices in $\mathbb{H}^3$.

Theorem 3.14. *Let* $x(s)$ *be a hyperbolic space curve in* $\mathbb{H}^3$ *parametrized by arc-length* s *with non-zero curvatures* κ, τ. *Then* $x(s)$ *is a* 3*-type hyperbolic slant helix if and only if*

$$\int \left(\frac{\tau}{\kappa} \int \tau ds\right) ds - \left(\frac{\tau}{\kappa}\right)' \int \tau ds = \frac{\kappa^2 + \tau^2}{\kappa}. \tag{17}$$

Proof. Assume that $x(s)$ is a 3-type hyperbolic slant helix in $\mathbb{H}^3$ parametrized by arc-length s with non-zero curvatures κ, τ. Then there exists a non-zero fixed vector $U \in \mathbb{E}_1^4$ such that

$$\langle y, U\rangle = c, \quad c \in \mathbb{R}\setminus\{0\}. \tag{18}$$

Then we can write U with respect to the frame $\{x, \alpha, \beta, y\}$ as follows

$$U = \lambda_1 x + \lambda_2\alpha + \lambda_3\beta + cy \tag{19}$$

where λ_1, λ_2 and λ_3 are some differentiable functions of s. Differentiating the equation (19) with respect to s and using Frenet equations (1), we get

$$0 = \left(\lambda_1' + \lambda_2\right)x + \left(\lambda_1 + \lambda_2' - c\kappa\right)\alpha + \left(\lambda_3' - c\kappa\right)\beta + \left(\lambda_2\kappa + \lambda_3\tau\right)y,$$

which implies that

$$\begin{cases} \lambda_1' + \lambda_2 = 0, \\ \lambda_1 + \lambda_2' - c\kappa = 0, \\ \lambda_3' - c\kappa = 0, \\ \lambda_2\kappa + \lambda_3\tau = 0. \end{cases} \tag{20}$$

Solving (20), we get

$$\int\left(\frac{\tau}{\kappa}\int\tau ds\right)ds - \left(\frac{\tau}{\kappa}\right)'\int\tau ds = \frac{\kappa^2+\tau^2}{\kappa}.$$

Conversely, assume that the relation (13) holds. Then choosing the vector U as follows

$$U = \left(\int\left(\frac{\tau}{\kappa}\int\tau ds\right)ds\right)x - \left(\frac{\tau}{\kappa}\int\tau ds\right)\alpha + \int\tau ds\beta + y,$$

we get $U' = 0$ and $\langle y, U\rangle = 1$ (constant). Thus $x(s)$ is a 3-type hyperbolic slant helix. □

Example 3.15. *The following hyperbolic curvature functions satisfy* (17).
(*i*) $\kappa = s, \tau = 1$ (*ii*) $\kappa = -s, \tau = 1$

Corollary 3.16. *The axis of a* 3*-type hyperbolic slant helix is given by*

$$U = c\left(\int\left(\frac{\tau}{\kappa}\int\tau ds\right)ds\right)x - c\left(\frac{\tau}{\kappa}\int\tau ds\right)\alpha + c\int\tau ds\beta + cy,$$

where $c \in \mathbb{R}\setminus\{0\}$.

Assume that $c = 0$ in (20), then we have

$$\begin{cases} \lambda_1' + \lambda_2 = 0, \quad \lambda_1 + \lambda_2' = 0, \\ \lambda_3' = 0, \quad \lambda_2\kappa + \lambda_3\tau = 0. \end{cases}$$

which implies that

$$\frac{\tau}{\kappa} = c_1e^s + c_2e^{-s}.$$

Then $x(s)$ is a 2-type hyperbolic slant helix. Thus we give the following corollary.

Corollary 3.17. *Let $x(s)$ be a hyperbolic space curve in $\mathbb{H}^3$ parametrized by arc-length s with non-zero curvatures κ, τ. Then $x(s)$ is a* 2*-type hyperbolic slant helix if and only if $x(s)$ is a* 3*-type hyperbolic slant helix whose axis U satisfies $\langle y, U\rangle = 0$.*

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