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On Jleli-Samet-Ćirić-Presić Type Contractive Mappings

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Abstract. Ćirić and Presić [Acta Math. Univ. Comenian. LXXVI (2) (2007), 143-147] extended the notion of Presić contraction to kth-order Ćirić type contractive mappings on a metric space. In this paper, we extend the concept of Ćirić-Presić to Jleli-Samet-Ćirić-Presić contractive mappings and obtain some related fixed point theorems. Our results generalize some known ones in the literature. A real concrete example and an illustrating application are given in support of our main result.

1. Introduction

One of the powerful results in fixed point theory is the Banach contraction principle (BCP) [7]. It has variant applications in the resolution of linear, nonlinear, differential, integral, and fractional analysis. One can see some applications and recent results in fixed point theory in the following works [1–6, 8, 12–15, 18, 19].

Theorem 1.1. *[7] Let (X, d) be a complete metric space and $f : X \to X$ so that*

$$d(fx, fy) \leq \gamma d(x, y) \text{ for all } x, y \in X.$$

where $\gamma \in [0, 1)$. Then, there is a unique σ in X such that $\sigma = f\sigma$. Also, for each $x_0 \in X$, the iterative sequence $x_{n+1} = fx_n$ converges to σ.

The BCP has been extended and generalized in many directions. Namely, Presić [16] gave the following result.

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Theorem 1.2. *[16] Let (X,d) be a complete metric space and $f : X^k \to X$ (k is a positive integer). Suppose that*

$$d(f(x_1,...,x_k), f(x_2,...,x_{k+1})) \le \sum_{i=1}^{k} q_i d(x_i, x_{i+1}) \tag{1}$$

for all $x_1,...,x_{k+1}$ in X, where $q_i \ge 0$ and $\sum_{i=1}^{k} q_i \in [0,1)$. Then f has a unique fixed point x^* (that is, $f(x^*,...,x^*) = x^*$). Moreover, for all arbitrary points $x_1,...,x_{k+1}$ in X, sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1},...,x_{n+k-1})$, converges to x^*.

It is easy to show that for $k = 1$, Theorem 1.2 reduces to the Banach contraction principle.

Ćirić and Presić [10] generalized above theorem as follows.

Theorem 1.3. *[10] Let (X,d) be a complete metric space and $f : X^k \to X$ (k is a positive integer). Suppose that*

$$d(f(x_1,...,x_k), f(x_2,...,x_{k+1})) \le \lambda \max\{d(x_i, x_{i+1}) : 1 \le i \le k\}, \tag{2}$$

for all $x_1,...,x_{k+1}$ in X, where $\lambda \in [0,1)$. Then f has a fixed point $x^* \in X$. Also, for all points $x_1,...,x_{k+1} \in X$, the sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1},...,x_{n+k-1})$, converges to x^*. If

$$d(f(\rho,...,\rho), f(\varrho,...,\varrho)) < d(\rho, \varrho),$$

for all $\rho, \varrho \in X$ with $\rho \ne \varrho$, then x^* is the unique fixed point of f.

Obviously, any Presić contraction is a Ćirić-Presić type contraction. For other related works on Presić type contractions, see [9, 16, 17].

Consistent with [11] we denote by Θ_0 the set of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

θ_1. θ is increasing;

θ_2. for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

θ_3. there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Recall the following result.

Theorem 1.4. *[11, Corollary 2.1] Let (X,d) be a complete metric space and let $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta_0$ and $k \in (0,1)$ such that*

$$x, y \in X, \quad d(Tx,Ty) \ne 0 \ \Rightarrow\ \theta(d(Tx,Ty)) \le \theta(d(x,y))^k.$$

Then T has a unique fixed point.

Note that the Banach contraction principle is a special case of the above Theorem.

In this paper, we establish some fixed point results for self maps satisfying Jleli-Samet-Ćirić-Presić type contractions defined on a metric space. An illustrated example is presented. At the end, applying one of our main results, we ensure the existence of a solution for an integral type equation.

2. Main results

First, like in [11], we denote by Θ the set of all functions $\theta : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

(θ1) θ is a strictly increasing function and continuous from right at 0;

(θ2) for each sequence $\{l_n\} \subseteq [0,\infty)$, $\lim_{n\to\infty} \theta(l_n) = 1$ iff $\lim_{n\to\infty} l_n = 0$;

(θ3) $\theta^{-1}[(\theta(l))^{\alpha}] \leq \sqrt{\alpha l e^{l}}$ for all $l \geq 0$ and $0 \leq \alpha < 1$.

Example 2.1. *The functions $\theta_i : [0,\infty) \to [1,\infty)$ defined by $\theta_1(t) = e^t$, $\theta_2(t) = e^{te^t}$ and $\theta_3(t) = e^{\sqrt{te^t}}$, belong to Θ.*

Theorem 2.2. *Let (X,d) be a complete metric space and $f : X^k \to X$ (k is a positive integer). Assume that*

$$\theta(d(f(x_1,...,x_k), f(x_2,...,x_{k+1}))) \leq [\theta(\max\{d(x_i,x_{i+1}) : i = 1,\cdots,k\})]^{\lambda} \tag{3}$$

for all $x_1,...,x_{k+1}$ in X, where $0 \leq \lambda < 1$. Then, for all arbitrary points $x_1,...,x_k$ in X, the sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1},...,x_{n+k-1})$, converges to a fixed point of f. Moreover, if for all $\rho, \varrho \in X$ with $\rho \neq \varrho$,

$$\theta(d(f(\rho,...,\rho), f(\varrho,...,\varrho))) \leq [\theta(d(\rho,\varrho)]^{\lambda}, \tag{4}$$

then the fixed point of f is unique.

Proof. Consider the arbitrary points $x_1,...,x_k$ in X and define a sequence $\{x_n\}$ by $x_{n+k} = f(x_n, x_{n+1},...,x_{n+k-1})$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}&\theta(d(x_{n+k}, x_{n+k+1})) = \theta(d(f(x_n,...,x_{n+k-1}), f(x_{n+1},...,x_{n+k})))\\ &\leq [\theta(\max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}),...,d(x_{n+k-1},x_{n+k+1})\})]^{\lambda}.\end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned}\theta(d(x_{k+1},x_{k+2})) &= \theta(d(f(x_1,...,x_k), f(x_2,...,x_{k+1})))\\ &\leq [\theta(\max\{d(x_1,x_2), d(x_2,x_3),...,d(x_k,x_{k+1})\})]^{\lambda} = [\theta(M)]^{\lambda},\end{aligned}$$

where $M = \max\{d(x_1,x_2), d(x_2,x_3),...,d(x_k,x_{k+1})\}$. Now,

$$\begin{aligned}\theta(d(x_{k+2},x_{k+3})) &= \theta(d(f(x_2,...,x_{k+1}), f(x_3,...,x_{k+2})))\\ &\leq [\theta(\max\{d(x_2,x_3), d(x_3,x_4),...,d(x_{k+1},x_{k+2})\})]^{\lambda}\\ &\leq [\max\{\theta(M), [\theta(M)]^{\lambda}\}]^{\lambda} = [\theta(M)]^{\lambda},\end{aligned}$$

...

$$\begin{aligned}\theta(d(x_{2k},x_{2k+1})) &= \theta(d(f(x_k,...,x_{2k-1}), f(x_{k+1},...,x_{2k})))\\ &\leq [\theta(\max\{d(x_k,x_{k+1}), d(x_{k+1},x_{k+2}),...,d(x_{2k-1},x_{2k})\})]^{\lambda}\\ &\leq [\max\{\theta(M), [\theta(M)]^{\lambda}\}]^{\lambda} = [\theta(M)]^{\lambda},\end{aligned}$$

...

$$\begin{aligned}\theta(d(x_{2k+1},x_{2k+2})) &= \theta(d(f(x_{k+1},...,x_{2k}), f(x_{k+2},...,x_{2k+1})))\\ &\leq [\theta(\max\{d(x_{k+1},x_{k+2}), d(x_{k+2},x_{k+3}),...,d(x_{2k},x_{2k+1})\})]^{\lambda}\\ &\leq [[\theta(M)]^{\lambda}]^{\lambda} = [\theta(M)]^{\lambda^2},\end{aligned}$$

...

$$\begin{aligned}\theta(d(x_{3k},x_{3k+1})) &= \theta(d(f(x_{2k},...,x_{3k-1}), f(x_{2k+1},...,x_{3k})))\\ &\leq [\theta(\max\{d(x_{2k},x_{2k+1}), d(x_{2k+1},x_{2k+2}),...,d(x_{3k-1},x_{3k})\})]^{\lambda}\\ &\leq [\max\{[\theta(M)]^{\lambda}, [\theta(M)]^{\lambda^2}\}]^{\lambda} = [\theta(M)]^{\lambda^2}.\end{aligned}$$

Continuing this process, we get

$$\theta(d(x_{pk+i}, x_{pk+i+1})) \leq [\theta(M)]^{\lambda^p}, \; for\,all \; p \in \mathbb{N}\,and \; i \in \{1,2,...,k\}.$$

We conclude that $\lim_{p\to\infty} \theta(d(x_{pk+i}, x_{pk+i+1})) = 1$. From ($\theta 2$), we obtain

$$\lim_{p\to\infty} d(x_{pk+i}, x_{pk+i+1}) = 0, \quad for\, all\ i \in \{1, 2, ..., k\}.$$

Thus,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{6}$$

We claim that $\{x_n\}$ is Cauchy. Consider two elements $m, n \in \mathbb{N}$ so that $n < m$. Then, there are $p, q \in \mathbb{N}$ and $i, j \in \{1, 2, ..., k\}$ such that $p \leq q$, $n = pk + i$ and $m = qk + j$. Now, we have

$$\begin{aligned} d(x_n, x_m) &= d(x_{pk+i}, x_{qk+j}) \leq \Sigma_{r=p}^{q} \Sigma_{l=1}^{k} d(x_{rk+l}, x_{rk+l+1}) \\ &\leq \Sigma_{r=p}^{q} \Sigma_{l=1}^{k} \theta^{-1}[(\theta(M))^{\lambda^r}] \leq \Sigma_{r=p}^{q} k\sqrt{\lambda^r M e^M} \\ &= k\sqrt{Me^M}\Sigma_{r=p}^{q}[\sqrt{\lambda}]^r = k[\sqrt{\lambda}]^p \frac{1-[\sqrt{\lambda}]^{q-p+1}}{1-\sqrt{\lambda}} \\ &\leq k[\sqrt{\lambda}]^p \frac{1}{1-\sqrt{\lambda}}. \end{aligned} \tag{7}$$

As $n, m \to \infty$, we have $p, q \to \infty$. Thus, the last term in (7) converges to 0, and so $\{x_n\}$ is a Cauchy sequence. Completeness of (X, d) yields that there is $v \in X$ so that

$$\lim_{n\to\infty} d(x_n, v) = 0. \tag{8}$$

Now, we shall prove that v is a fixed point of f. To see this, we have

$$\begin{aligned} d(x_{n+k}, f(v, ..., v)) &= d(f(x_n, x_{n+1} ..., x_{n+k-1}), f(v, ..., v)) \\ &\leq d(f(x_n, x_{n+1}, ..., x_{n+k-1}), f(x_{n+1}, x_{n+2}, ..., x_{n+k-1}, v)) \\ &+ d(f(x_{n+1}, x_{n+2}, ..., x_{n+k-1}, v), f(x_{n+2}, x_{n+3}, ..., x_{n+k-1}, v, v)) \\ &+ ... + d(f(x_{n+k-1}, v, ..., v), f(v, v, ..., v)) \\ &\leq \theta^{-1}[\Big(\theta(\max\{d(x_n, x_{n+1}), ..., d(x_{n+k-2}, x_{n+k-1}), d(x_{n+k-1}, v)\})\Big)^{\lambda}] \\ &+ \theta^{-1}[\Big(\theta(\max\{d(x_n + 1, x_{n+2}), ..., d(x_{n+k-2}, x_{n+k-1}), d(x_{n+k-1}, v)\})\Big)^{\lambda}] \\ &+ ... + \theta^{-1}[\Big(\theta(d(x_{n+k-1}, v))\Big)^{\lambda}]. \end{aligned} \tag{9}$$

Using ($\theta 1$) and ($\theta 2$) and letting $n \to \infty$, we get taking in account (6) and (8), the right-hand side of (9) goes to 0. Hence, $\lim_{n\to\infty} d(x_{n+k}, f(v, ..., v)) = 0$. The continuity of the metric d yields that

$$d(v, f(v, ..., v)) = \lim_{n\to\infty} d(x_{n+k}, f(v, ..., v)) = 0.$$

Therefore, $v = f(v, ..., v)$. Suppose u, v are two distinct fixed points of f. By hypothesis,

$$\theta(d(u, v)) = \theta(d(f(u, ..., u), f(v, ..., v))) \leq [\theta(d(u, v)]^{\lambda} < \theta(d(u, v),$$

which is a contradiction. Thus, the fixed point of f is unique. □

Note that by taking $\theta(t) = e^t$, the above theorem reduces to Theorem 1.3. The following is a straightforward result of Theorem 2.2.

Theorem 2.3. *Let (X, d) be a complete metric space and $f : X^k \to X$ (k is a positive integer). Suppose that*

$$\theta(d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1}))) \leq \prod_{i=1}^{k} [\theta(d(x_i, x_{i+1}))]^{q_i} \tag{10}$$

for all $x_1, ..., x_{k+1}$ in X, where $0 \leq \sum_{i=1}^{k} q_i < 1$. Then for all points $x_1, ..., x_k$ in X, the sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1} ..., x_{n+k-1})$, converges to a fixed point of f. Also, if for all $\rho, \varrho \in X$ with $d(f(\rho, ..., \rho), f(\varrho, ..., \varrho)) > 0$,

$$\theta(d(f(\rho, ..., \rho), f(\varrho, ..., \varrho))) \leq [\theta(d(\rho, \varrho)]^{\sum_{i=1}^{k} q_i}, \tag{11}$$

then the fixed point of f is unique.

Taking $\theta(t) = e^{te^t}$ in Theorem 2.2, we obtain the following.

Corollary 2.4. *Let (X, d) be a complete metric space and $f : X^k \to X$ (k is a positive integer). Suppose that*

$$\frac{d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1}))e^{d(f(x_1,...,x_k),f(x_2,...,x_{k+1}))-\max\{d(x_i,x_{i+1}):i=1,\cdots,k\}}}{\max\{d(x_i, x_{i+1}) : i = 1, \cdots, k\}} \leq \lambda \tag{12}$$

for all $x_1, ..., x_{k+1}$ in X with $d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) > 0$, where $0 \leq \lambda < 1$. Then, for all points $x_1, ..., x_k$ in X, the sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1}, ..., x_{n+k-1})$, converges to a fixed point of f. Also, if for all $\rho, \varrho \in X$ with $d(f(\rho, ..., \rho), f(\varrho, ..., \varrho))) > 0$,

$$d(f(\rho, ..., \rho), f(\varrho, ..., \varrho))) < d(\rho, \varrho),$$

then the fixed point of f is unique.

We present an example in support of our main result.

Example 2.5. *Let $X = \{\tau_n = \frac{n(n+1)}{2} : n = 1, 2, \cdots\}$, $d(\rho, \varrho) = |\rho - \varrho|$ and define $f : X \to X$ by*

$$f(\tau_n, \tau_m) = \begin{cases} \min\{\tau_{n-1}, \tau_{m-1}\}, & n, m > 1, \\ \tau_1, & n = 1 \, or \, m = 1. \end{cases}$$

Firstly, note that for all $m, n \in \mathbb{N}$ with $m, n > 1$, one writes

$$\begin{aligned} &\frac{d(\tau_{n-1}, \tau_{m-1})e^{d(\tau_{n-1},\tau_{m-1})-d(\tau_n,\tau_m)}}{d(\tau_n, \tau_m)} \\ &= \frac{(\frac{m(m-1)}{2} - \frac{n(n-1)}{2})e^{\frac{m(m-1)}{2} - \frac{n(n-1)}{2} - (\frac{m(m+1)}{2} - \frac{n(n+1)}{2})}}{\frac{m(m+1)}{2} - \frac{n(n+1)}{2}} \\ &= \frac{(m + n - 1)e^{-(m-n)}}{m + n + 1} \leq e^{-1} = \lambda. \end{aligned}$$

Also, for $m = 1$ and $n > 1$, we have

$$\begin{aligned} &\frac{d(\tau_1, \tau_{n-1})e^{d(\tau_1,\tau_{n-1})-d(\tau_1,\tau_n)}}{d(\tau_1, \tau_n)} \\ &= \frac{(\frac{n(n-1)}{2} - 1)e^{\frac{n(n-1)}{2} - 1 - (\frac{n(n+1)}{2} - 1)}}{\frac{n(n+1)}{2} - 1} \leq e^{-n} \leq e^{-1} = \lambda. \end{aligned}$$

Now, Let $\rho = \tau_n$, $\varrho = \tau_m$ and $\varsigma = \tau_p$. If $m \leq \min\{n, p\}$, then $d(f(\rho, \varrho), f(\varrho, \varsigma)) = 0$. So, we may assume that either $n < m$ or $p < m$. We treat the following:

***(Case 1)**: $n < m \leq p$. Here, if $n = 1$, then*

$$\begin{aligned} &\frac{d(f(\rho, \varrho), f(\varrho, \varsigma))e^{d(f(\rho,\varrho),f(\varrho,\varsigma))-\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}}}{\max\{d(\rho, \varrho), d(\varrho, \varsigma)\}} \\ &\leq \frac{d(\tau_1, \tau_{m-1})e^{d(\tau_1,\tau_{m-1})-d(\tau_1,\tau_m)}}{d(\tau_1, \tau_m)} \leq e^{-m} \leq e^{-1} = \lambda, \end{aligned}$$

and if $n > 1$, then

$$\frac{d(f(\rho,\varrho),f(\varrho,\varsigma))e^{d(f(\rho,\varrho),f(\varrho,\varsigma))-\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}}}{\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}} \leq \frac{d(\tau_{n-1},\tau_{m-1})e^{d(\tau_{n-1},\tau_{m-1})-d(\tau_n,\tau_m)}}{d(\tau_n,\tau_m)} \leq e^{-1} = \lambda.$$

***(Case 2)**: $p < m \leq n$. Here, if $p = 1$, then*

$$\frac{d(f(\rho,\varrho),f(\varrho,\varsigma))e^{d(f(\rho,\varrho),f(\varrho,\varsigma))-\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}}}{\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}} \leq \frac{d(\tau_1,\tau_{m-1})e^{d(\tau_1,\tau_{m-1})-d(\tau_1,\tau_m)}}{d(\tau_1,\tau_m)} \leq e^{-m} \leq e^{-1} = \lambda,$$

and if $p > 1$, then

$$\frac{d(f(\rho,\varrho),f(\varrho,\varsigma))e^{d(f(\rho,\varrho),f(\varrho,\varsigma))-\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}}}{\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}} \leq \frac{d(\tau_{m-1},\tau_{p-1})e^{d(\tau_{m-1},\tau_{p-1})-d(\tau_m,\tau_p)}}{d(\tau_m,\tau_p)} \leq e^{-1} = \lambda.$$

***(Case 3)**: $n < p < m$. In this case, if $n = 1$, then*

$$\frac{d(f(\rho,\varrho),f(\varrho,\varsigma))e^{d(f(\rho,\varrho),f(\varrho,\varsigma))-\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}}}{\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}} \leq \frac{d(\tau_1,\tau_{p-1})e^{d(\tau_1,\tau_{p-1})-d(\tau_1,\tau_p)}}{d(\tau_1,\tau_p)} \leq e^{-p} \leq e^{-1} = \lambda,$$

and if $n > 1$, then

$$\frac{d(f(\rho,\varrho),f(\varrho,\varsigma))e^{d(f(\rho,\varrho),f(\varrho,\varsigma))-\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}}}{\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}} \leq \frac{d(\tau_{n-1},\tau_{p-1})e^{d(\tau_{n-1},\tau_{p-1})-d(\tau_n,\tau_p)}}{d(\tau_n,\tau_p)} \leq e^{-1} = \lambda.$$

***(Case 4)**: $p < n < m$. Here, if $p = 1$, then*

$$\frac{d(f(\rho,\varrho),f(\varrho,\varsigma))e^{d(f(\rho,\varrho),f(\varrho,\varsigma))-\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}}}{\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}} \leq \frac{d(\tau_1,\tau_{n-1})e^{d(\tau_1,\tau_{n-1})-d(\tau_1,\tau_n)}}{d(\tau_1,\tau_n)} \leq e^{-n} \leq e^{-1} = \lambda,$$

and if $p > 1$, then

$$\frac{d(f(\rho,\varrho),f(\varrho,\varsigma))e^{d(f(\rho,\varrho),f(\varrho,\varsigma))-\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}}}{\max\{d(\rho,\varrho),d(\varrho,\varsigma)\}} \leq \frac{d(\tau_{n-1},\tau_{p-1})e^{d(\tau_{n-1},\tau_{p-1})-d(\tau_n,\tau_p)}}{d(\tau_n,\tau_p)} \leq e^{-1} = \lambda.$$

Also, for $u, v \in X$ with $d(f(u,u), f(v,v)) > 0$, let $u = \tau_n, v = \tau_m$ and $n < m$. If $n = 1$, then

$$d(f(u,u),f(v,v)) = d(f(\tau_1,\tau_1),f(\tau_m,\tau_m))$$

and

$$d(\tau_1,\tau_{m-1}) = \frac{m(m-1)}{2} - 1 < \frac{m(m+1)}{2} - 1 = d(\tau_1,\tau_m) = d(u,v),$$

and if $n > 1$, then

$$\begin{aligned} d(f(u,u), f(v,v)) &= d(f(\tau_n,\tau_n), f(\tau_m,\tau_m)) \\ &= d(\tau_{n-1},\tau_{m-1}) = \frac{m(m-1)}{2} - \frac{n(n-1)}{2} \\ &= \frac{(m-n)(m+n-1)}{2} < \frac{(m-n)(m+n+1)}{2} = \\ &\frac{m(m+1)}{2} - \frac{n(n+1)}{2} = d(\tau_n,\tau_m) = d(u,v). \end{aligned}$$

We see that all of conditions of Corollary 2.4 are satisfied. Thus, f has a unique fixed point. Here, $f(\tau_1,\tau_1) = \tau_1$ and τ_1 is the unique fixed point. Note that Theorem 1.3 is not applicable. In fact,

$$\sup_{n>1} \frac{d(f(\tau_1,\tau_n), f(\tau_n,\tau_n))}{\max\{d(\tau_1,\tau_n), d(\tau_n,\tau_n)\}} = \sup_{n>1} \frac{d(\tau_{n-1},\tau_1)}{d(\tau_1,\tau_n)} = \sup_{n>1} \frac{\frac{n(n-1)}{2} - 1}{\frac{n(n+1)}{2} - 1} = 1.$$

Thus, Theorem 2.2 is a real generalization of Cirić-Presić's result (Theorem 1.3).

3. Application

In this section, we study the existence of solutions for the following integral equation:

$$x(t) = f\left(t, \int_0^{\varrho(t)} g(t, y, \underbrace{x(\rho(y)), \cdots, x(\rho(y))}_{n \text{ times}})dy\right). \tag{13}$$

where $t \in [0,\infty)$.

We will ensure such an existence by applying Theorem 2.2.

Let $BC[0,\infty)$ be the space of all real, bounded and continuous functions on the interval $[0,\infty)$. We endow it with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in [0,\infty)\}.$$

Recall that the associated metric on $BC[0,\infty)$ is defined by

$$d(x,y) = \sup\{|x(t) - y(t)| : t \in [0,\infty)\}.$$

Theorem 3.1. *Suppose that the following assumptions are satisfied:*

(i) $\rho, \varrho : [0,\infty) \longrightarrow [0,\infty)$ *are continuous functions so that*

$$\Lambda = \sup\{|\varrho(t)| : t \in [0,\infty)\} < 1,$$

(ii) *The function $f : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous so that*

$$\left|f\big(t,x\big) - f\big(t,\nu\big)\right| \le \left|x - \nu\right|, \tag{14}$$

for all $t \in [0,\infty)$ and $x, \nu \in \mathbb{R}$,

(iii)

$$\theta\Big(\Big|g(t,y,\underbrace{x_1(\rho(y)),\cdots,x_k(\rho(y))}) - g(t,y,\underbrace{x_2(\rho(y)),\cdots,x_{k+1}(\rho(y))})\Big|\Big) \le [\theta(\max_{i=1,\cdots,k}\{d(x_i,x_{i+1})\})]^{\lambda} \tag{15}$$

where $g:[0,\infty)^2\times\mathbb{R}^k \longrightarrow \mathbb{R}$ *is continuous and* $\theta(\lambda t)\le[\theta(t)]^{\lambda}$ *for all* $\lambda\in[0,1)$,

(iv) $M=\max\{f(t,0,0,0): t\in[0,\infty)\}<\infty$ *and*
$G=\sup\Big\{\Big|g(t,y,0,\cdots,0)\Big| : t\in[0,\infty)\Big\}<\infty.$

Then the integral equation (13) *has at least one solution in the space* $BC[0,\infty)$.

Proof. Let us consider the operator $\Upsilon : BC[0,\infty)^k \longrightarrow BC[0,\infty)$ defined by

$$\Upsilon(x_1,x_2,\cdots,x_k)(t) \tag{16}$$
$$= f\Big(t,\int_0^{\varrho(t)} g(t,y,x_1(\rho(y)),x_2(\rho(y)),\cdots,x_k(\rho(y)))dy\Big). \tag{17}$$

In view of given assumptions, we infer that the function $\Upsilon(x_1,x_2,\cdots,x_n)$ is continuous for arbitrarily $x_1,x_2,\cdots,x_k\in BC[0,\infty)$. Now, we show that $\Upsilon(x_1,x_2,\cdots,x_k)$ is bounded in $BC[0,\infty)$. As

$$\begin{aligned}&|\Upsilon(x_1,x_2\cdots,x_k)(t)|\\ &=\Big|f\Big(t,\int_0^{\varrho(t)} g(t,y,x_1(\rho(y)),x_2(\rho(y))\cdots,x_k(\rho(t)))dy\Big)\Big|\\ &\le\Big|f\Big(t,\int_0^{\varrho(t)} g(t,y,x_1(\rho(y)),x_2(\rho(y))\cdots,x_k(\rho(t)))dy\Big)-f(t,0)\Big|+\Big|f(t,0)\Big|,\end{aligned}$$

we have

$$\begin{aligned}&\Big|f\Big(t,\int_0^{\varrho(t)} g(t,y,x_1(\rho(y)),x_2(\rho(y))\cdots,x_k(\rho(t)))dy\Big)-f(t,0)\Big|\\ &\le\int_0^{\varrho(t)} g(t,y,x_1(\rho(y)),x_2(\rho(y)),\cdots x_n(\rho(y)))dy\Big|\\ &\le\Lambda\max\{\|x_1\|,\|x_2\|,\cdots,\|x_k\|\}+\Lambda G.\end{aligned}$$

Thus,

$$\begin{aligned}&\Big|f\Big(t,\int_0^{\varrho(t)} g(t,y,x_1(\rho(y)),x_2(\rho(y)),\cdots x_n(\rho(y)))dy\Big)-f(t,0)\Big|\\ &\le\Lambda\max\{\|x_1\|,\|x_2\|,\cdots,\|x_n\|\}+\Lambda G.\end{aligned}$$

From the above calculations, we have

$$\|\Upsilon(x_1,x_2,\cdots,x_k)(t)\|\le\Lambda\max\{\|x_1\|,\|x_2\|,\cdots,\|x_k\|\}+\Lambda G+M. \tag{18}$$

Due to the above inequality, the function Υ is bounded.

Now, we show that Υ satisfies all the conditions of Theorem 2.2. Let $x_1,x_2,\cdots x_k,x_{k+1}$ be some elements of $BC[0,\infty)$. Then we have

$$\begin{aligned}
&\theta\Big(\Big|\Upsilon(x_1,x_2,\cdots x_k)(t)-\Upsilon(x_2,x_3,\cdots x_{k+1})(t)\Big|\Big)\\
&\leq \theta\Big(\Big|f\Big(t,\int_0^{\varrho(t)} g(t,y,x_1(\rho(y)),x_2(\rho(y)),\cdots x_k(\rho(y))dy)\\
&\quad -f\Big(t,\int_0^{\varrho(t)} g(t,y,x_2(\rho(t)),x_3(\rho(t)),\cdots x_{k+1}(\rho(t)))dy\Big)\Big|\Big)\\
&\leq \theta\Big(\Big|\int_0^{\varrho(t)} g(t,y,x_1(\rho(y)),x_2(\rho(y)),\cdots x_k(\rho(y))dy\\
&-\int_0^{\varrho(t)} g(t,y,x_2(\rho(t)),x_3(\rho(t)),\cdots x_{k+1}(\rho(t)))dy\Big|\Big)\\
&\leq \theta\Big(\varrho(t)(\max\{d(x_i,x_{i+1}):i=1,\cdots,k\})\Big)\\
&\leq \theta\Big(\Lambda(\max\{d(x_i,x_{i+1}):i=1,\cdots,k\})\Big)
\end{aligned} \tag{19}$$

Thus, we obtain that

$$\theta(d(\Upsilon(x_1,...,x_k),\Upsilon(x_2,...,x_{k+1}))) \leq [\theta(\max\{d(x_i,x_{i+1}):i=1,\cdots,k\})]^{\Lambda}. \tag{20}$$

Using Theorem 2.2, we obtain that the operator Υ has a fixed point. Thus, the functional integral equation (13) has at least one solution in $BC[0,\infty)$. ■ □

4. Example

Example 4.1. *Consider the following integral equation*

$$x(t) = \tfrac{1}{2}e^{-t^2} + \tfrac{1}{8}\arctan\Big(\int_0^{\tanh t} \tfrac{s(|\sinh x(s)|+\arctan x(s))}{2e^t}ds\Big) \tag{21}$$

We observe that the integral equation (21) *is a special case of* (13) *with* $\rho(t)=t$ *and* $\varrho(t)=\tanh t$, *where* $t\in[0,\infty)$. *Also,*

$$f(t,x) = \frac{1}{2}e^{-t^2} + \frac{\arctan(x)}{8},$$

and

$$g(t,s,x_1,x_2) = \frac{s(|\sinh x_1| + \arctan x_2)}{2e^t}.$$

To show the existence of a solution for this equation, we need to verify the conditions (i)-(iv) *of Theorem 2.2. Condition (i) is clearly evident. Take* $\theta(t)=\cosh t$. *Now,*

$$\begin{aligned}
\Big|f(t,x)-f(t,u)\Big| &\leq \frac{|\arctan x - \arctan u|}{8}\\
&\leq \frac{\arctan|x-u|}{8}\\
&\leq |x-u|.
\end{aligned} \tag{22}$$

So, we find that f *satisfies condition (ii) of Theorem 3.1. Also,*

$$M = \sup\{|f\big(t,0\big)| : t\in[0,\infty)\} = \sup\{\frac{1}{2}e^{-t^2} : t\in[0,\infty)\} = 0.5$$

Obviously, condition (iii) of Theorem 3.1 is valid, that is, g is continuous on $[0,\infty)\times[0,\infty)\times\mathbb{R}^2$*, and*

$$G = \sup\left\{\Big|\int_0^{\tanh t}\frac{s}{2e^t}ds\Big| : t\in[0,\infty)\right\} \tag{23}$$
$$= \sup_{t\in[0,\infty)}\left(\frac{\tanh t}{2e^t}\right)\simeq 0.1501.$$

On the other hand,

$$\theta\Big(\Big|g(t,s,\underbrace{x_1(\rho(s)),x_2(\rho(s))}) - g(t,s,\underbrace{x_2(\rho(s)),x_3(\rho(s))})\Big|\Big) \tag{24}$$
$$= \cosh\Big(\frac{s(|\sinh x_1(s)| + \arctan x_2(s))}{2e^t} - \frac{s(|\sinh x_2(s)| + \arctan x_3(s))}{2e^t}\Big) \tag{25}$$
$$\le \cosh\Big(\frac{s(|\sinh x_1(s) - \sinh x_2(s)| + \arctan x_2(s) - \arctan x_3(s))}{2e^t}\Big) \tag{26}$$
$$\le \sqrt{\theta(\max_{i=1,\cdots,2}\{d(x_i,x_{i+1})\})} \tag{27}$$

Consequently, all the conditions of Theorem 2.2 are satisfied. Hence, the integral equation (21) *has at least one solution, which belongs to the space* $BC[0,\infty)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the manuscript.

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