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On the Upper Dual Zariski Topology

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Abstract. Let R be a ring with identity and M be a left R-module. The set of all second submodules of M is called the *second spectrum* of M and denoted by $Spec^s(M)$. For each prime ideal p of R we define $Spec^s_p(M) := \{S \in Spec^s(M) : ann_R(S) = p\}$. A second submodule Q of M is called an upper second submodule if there exists a prime ideal p of R such that $Spec^s_p(M) \neq \emptyset$ and $Q = \sum_{S \in Spec^s_p(M)} S$. The set of all upper second submodules of M is called upper second spectrum of M and denoted by $u.Spec^s(M)$. In this paper, we discuss the relationships between various algebraic properties of M and the topological conditions on $u.Spec^s(M)$ with the dual Zarsiki topology. Also, we topologize $u.Spec^s(M)$ with the patch topology and the finer patch topology. We show that for every left R-module M, $u.Spec^s(M)$ with the finer patch topology is a Hausdorff, totally disconnected space and if M is Artinian then $u.Spec^s(M)$ is a compact space with the patch and finer patch topology. Finally, by applying Hochster's characterization of a spectral space, we show that if M is an Artinian left R-module, then $u.Spec^s(M)$ with the dual Zariski topology is a spectral space.

1. Introduction

Throughout this paper all rings will be associative rings with identity elements and all modules will be unital left modules. Unless otherwise stated R will denote a ring. By a proper submodule N of a left R-module M, we mean a submodule N with $N \neq M$. Given a left R-module M, we shall denote the annihilator of M (in R) by $ann_R(M)$.

A non-zero R-module M is called a *prime module* if $ann_R(M) = ann_R(K)$ for every non-zero submodule K of M. A proper submodule N of a module M is called a *prime submodule* of M if M/N is a prime module. If N is a prime submodule of a module M, then $ann_R(M/N) = p$ is a prime ideal of R and in this case N is called a *p-prime submodule* of M. The set of all prime submodules of a module M is called the *prime spectrum* of M and denoted by $Spec(M)$. Also, the set of all p-prime submodules of M is denoted by $Spec_p(M)$ for a prime ideal p of R. Several authors investigated and topologized the prime spectrum of a given module (see for example [7], [8], [18], [19], [22]).

In [23], S. Yassemi introduced second submodules of modules over commutative rings as the dual notion of prime submodules. Second modules over arbitrary rings were defined in [2] and used as a tool for the study of attached primes over noncommutative rings. A right R-module M is called a *second module* provided $M \neq (0)$ and $ann_R(M) = ann_R(M/N)$ for every proper submodule N of M. By a *second submodule* of a module, we mean a submodule which is also a second module. If N is a second submodule of a module

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M, then $ann_R(N) = p$ is a prime ideal of R and in this case N is called a *p-second submodule* of M. Recently, second submodules have attracted attention of many authors and they have been extensively studied in a number of papers (see for example [1], [3], [4], [5], [6], [10], [11], [12], [13]).

The set of all second submodules of a module M is called the *second spectrum* of M and denoted by $Spec^s(M)$. Also, the set of all p-second submodules of M is denoted by $Spec^s_p(M)$ for a prime ideal p of R. Following [1, Lemma 4.1], for any submodule N of a left R-module M we define $V^s(N) = \{S \in Spec^s(M) : ann_R(N) \subseteq ann_R(S)\}$. Let $Z^s(M) := \{V^s(N) : N \leq M\}$. Then [1, Lemma 4.1] shows that $Z^s(M)$ satisfies the axioms for closed sets in a topological space and so it induces a topology on $Spec^s(M)$. We call this topology the *dual Zariski topology* on $Spec^s(M)$. The dual Zariski topology of modules over commutative rings has been investigated in [1], [5], [10] and [14].

In [9], Behboodi and Shojaee introduced the notion of lower prime submodules and investigated a topology on the set of these submodules. A prime submodule Q of a module M is called a *lower prime* submodule if there exists a prime ideal p of R such that $Spec_p(M) \neq \emptyset$ and $Q = \bigcap_{P \in Spec_p(M)} P$. Motivated by this notion, in this paper, we define the concept of upper second submodule and investigate some topologies on the set of these submodules.

2. Upper Second Submodules and Upper Dual Zariski Topology

Note that sum of p-second submodules is also a second submodule.

Definition 2.1. *Let M be a left R-module. For a prime ideal p of R, if there exits a p-second submodule of M, then the sum of all p-second submodules of M is called upper p-second submodule or an upper second submodule for short. The set of all upper second submodules of M is called upper second spectrum of M and denoted by $u.Spec^s(M)$.*

Clearly, $u.Spec^s(M) \subseteq Spec^s(M)$ and if $S, Q \in u.Spec^s(M)$, then $S = Q$ if and only if $ann_R(S) = ann_R(Q)$.

In [15], we characterized the upper second submodules of an Artinian module. To do this we generalized the notion of p-interior of a submodule which was defined in [3] for modules over commutative rings. Let R be an arbitrary ring, p be a prime ideal of R and M be an R-module. In [15] we generalized the p-interior of a submodule N of M as follows.

$I^M_p(N) = \cap\{L : L$ is a completely irreducible submodule of M and $AN \subseteq L$ for some ideal $A \nsubseteq p\}$.

Clearly, $I^M_p(N)$ is a submodule of M and $I^M_p(N) \subseteq N$.

Theorem 2.2. *[15, Theorem 3]Let N be a submodule of a left R-module M such that $ann_R(N) = p$ is a prime ideal of R. If $M/I^M_p(N)$ is a finitely cogenerated R-module, then $I^M_p(N) = \sum_{S \in Spec^s_p(N)} S$, i.e. $I^M_p(N)$ is an upper second submodule of N.*

Corollary 2.3. *[15, Corollary 4]Let p be a prime ideal of R and M be an R-module such that $M/I^M_p((0 :_M p))$ is a finitely cogenerated R-module. If $I^M_p((0 :_M p)) \neq 0$, then $I^M_p((0 :_M p))$ is an upper second submodule of M.*

Corollary 2.4. *[15, Corollary 5]Let p be a prime ideal of R and M be an R-module such that $M/I^M_p((0 :_M p))$ is a finitely cogenerated R-module. Then the following statements are equivalent.*

(1) *$ann_R((0 :_M p)) = p$.*

(2) *$I^M_p((0 :_M p))$ is an upper second submodule of M.*

(3) *There exists a second submodule K of M such that $p = ann_R(K)$.*

(4) *$I^M_p((0 :_M p)) \neq 0$.*

Corollary 2.5. *[15, Corollary 6]Let M be an Artinian left R-module. Then*
$u.Spec^s(M) = \{I^M_p((0 :_M p)) : p$ is a prime ideal of R and $Spec^s_p(M) \neq \emptyset\}$.

Let M be a left R-module. For a submodule N of M, we define $V^u(N) := \{Q \in u.Spec^s(M) : ann_R(N) \subseteq ann_R(Q)\}$. Clearly, $V^u(N) = V^s(N) \cap u.Spec^s(M)$. Since the family $Z^s(M) = \{V^s(N) : N \leq M\}$ is the family of closed subsets of $Spec^s(M)$ with respect to dual Zariski topology, the family $Z^u(M) := \{V^u(N) : N \leq M\}$ is the family of closed subsets of $u.Spec^s(M)$ with respect to the subspace topology on $u.Spec^s(M)$ induced by the dual Zariski topology on $Spec^s(M)$. We call this topology as the *upper dual Zariski topology* of M. The following properties are easily obtained.

(i) $V^u(0) = \emptyset$ and $V^u(M) = u.Spec^s(M)$.

(ii) $\cap_{i\in I} V^u(N_i) = V^u(\cap_{i\in I}(0 :_M ann_R(N_i)))$ where $\{N_i\}_{i\in I}$ is a family of submodules of M.

(iii) $V^u(N) \cup V^u(L) = V^u(N + L)$ where $N, L \leq M$.

Also, for each submodule N of M we denote the complement of $V^u(N)$ in $u.Spec^s(M)$ by $U^u(N)$.

3. Finer Upper Patch and Upper Patch Topologies of a Module

Definition 3.1. *Let M be a left R-module.*

(1) *Let $\beta^u(M) = \{V^u(N) \cup U^u(K) : N, K \leq M, U^u(K)$ is an upper dual Zariski-quasi-compact subset$\}$. Clearly, $\beta^u(M)$ is closed under finite unions and contains $u.Spec^s(M)$ and $\emptyset$ since $\emptyset = V^u(0) \cup U^u(M)$ and $u.Spec^s(M) = V^u(M) \cup U^u(M)$. Therefore, $\beta^u(M)$ is a basis for the family of closed subsets of a topology on $u.Spec^s(M)$, and we call this topology upper patch topology of M.*

(2) *Let $\gamma^u(M) = \{V^u(N) \cap U^u(K) : N, K \leq M\}$. Clearly, $\gamma^u(M)$ is closed under finite intersections and contains $u.Spec^s(M)$ and $\emptyset$ since $u.Spec^s(M) = V^u(M) \cap U^u(0)$ and $\emptyset = V^u(0) \cap U^u(M)$. Therefore, $\gamma^u(M)$ is a basis for the family of open subsets of a topology on $u.Spec^s(M)$ and we call this topology finer upper patch topology of M.*

Note that the family $\widetilde{\beta^u}(M) = \{V^u(K) \cap U^u(N) : N, K \leq M, U^u(K)$ is an upper dual Zariski-quasi-compact subset$\}$ is a basis for the open subsets of upper patch topology, i.e., the upper patch-open subsets of $u.Spec^s(M)$ are precisely the unions of sets from $\widetilde{\beta^u}(M)$.

Lemma 3.2. *Let M be a left R-module and $S \in u.Spec^s(M)$. Then for each finer upper patch neighborhood G^u of S, there exists a submodule L of M such that $ann_R(S) \subsetneq ann_R(L)$ and $S \in V^u(S) \cap U^u(L) \subseteq G^u$.*

Proof. Since $S \in G^u$, there exists a neighborhood of the form $V^u(K)\cap U^u(N) \subseteq G^u$ such that $S \in V^u(K)\cap U^u(N)$. So, $ann_R(K) \subseteq ann_R(S)$ and $ann_R(N) \nsubseteq ann_R(S)$. Since $S \in V^u(S) \subseteq V^u(K)$, we have $S \in V^u(S) \cap U^u(N) \subseteq G^u$. Now, we claim that $V^u(S) \cap U^u(N) = V^u(S) \cap U^u((0 :_M I + p))$ where $p = ann_R(S)$ and $I = ann_R(N)$. Since $U^u((0 :_M I)) \subseteq U^u((0 :_M I+p))$, we have $V^u(S)\cap U^u(N) = V^u(S)\cap U^u((0 :_M I)) \subseteq V^u(S)\cap U^u((0 :_M I+p))$. Suppose that $Q \in V^u(S) \cap U^u((0 :_M I + p))$. Then $Q \notin U^u(S)$. On the other hand, $Q \in U^u((0 :_M I + p)) = U^u(N) \cup U^u(S)$. This implies that $Q \in U^u(N)$. Thus $Q \in V^u(S) \cap U^u(N)$ and so $V^u(S) \cap U^u(N) = V^u(S) \cap U^u((0 :_M I + p))$. Now, let $L := (0 :_M I + p)$. Then, $p \subsetneq I + p \subseteq ann_R(L)$ and $S \in V^u(S) \cap U^u(L) \subseteq G^u$ as desired. □

Theorem 3.3. *Let M be a left R-module. Then $u.Spec^s(M)$ is Hausdoff with the finer upper patch topology. Moreover, $u.Spec^s(M)$ with this topology is totally disconnected.*

Proof. Suppose that $S, Q \in u.Spec^s(M)$ are distinct points. Then $ann_R(S) \neq ann_R(Q)$. Therefore either $ann_R(S) \nsubseteq ann_R(Q)$ or $ann_R(Q) \nsubseteq ann_R(S)$. Without loss of generality we may assume that $ann_R(S) \nsubseteq ann_R(Q)$. Then $U_1 := U^u(0) \cap V^u(S)$ is a finer upper patch neighborhood of S and since $ann_R(S) \nsubseteq ann_R(Q)$, $U_2 := U^u(S) \cap V^u(Q)$ is a finer upper patch neighborhood of Q. Clearly, $U_1 \cap U_2 = \emptyset$. Thus, $u.Spec^s(M)$ is a Hausdorff space. It is well-known that if a Hausdorff space has an open base whose sets are also closed, then X is totally disconnected. For every submodule N od M, observe that $U^u(N) = V^u(M) \cap U^u(N)$ and $V^u(N) = V^u(N) \cap U^u(0)$. Therefore the sets $U^u(N)$ and $V^u(N)$ are both open and closed. Thus, the finer upper patch topology of M has a base of open sets which are also closed, and hence $u.Spec^s(M)$ is totally disconnected with this topology. □

Definition 3.4. *Let M be a left R-module. M is called quasi-secondful if for each prime ideal p of R such that $ann_R((0 :_M p)) = p$, there exists a second submodule S of M such that $ann_R(S) = p$.*

Example 3.5. (1) *Let R be a ring such that the ring R/p is right or left Goldie for every prime ideal p and M be a non-zero injective R-module. Then $(0 :_M p)$ is a p-second submodule if $ann_R((0 :_M p)) = p$. Thus M is a quasi-secondful R-module.*

(2) *Every Artinian module is quasi-secondful by Corollary 2.4.*

Theorem 3.6. *Let M be a quasi-secondful left R-module such that $R/ann_R(M)$ satisfies ascending chain condition on ideals. Then $u.Spec^s(M)$ is a compact space with the finer upper patch topology.*

Proof. Suppose that M is a quasi-secondful left R-module such that $R/ann_R(M)$ satisfies ascending chain condition on ideals. Let $\mathcal{A}$ be a family of finer upper patch-open sets covering $u.Spec^s(M)$ and suppose that no finite subfamily of $\mathcal{A}$ covers $u.Spec^s(M)$. Let $\mathcal{S} = \{L : L$ is an ideal of R such that $ann_R(M) \subseteq L$ and no finite subfamily of $\mathcal{A}$ covers $V^u((0 :_M L))\}$. Since $V^u((0 :_M ann_R(M)) = V^u(M) = u.Spec^s(M)$, we have $ann_R(M) \in \mathcal{S}$ and so $\mathcal{S} \neq \emptyset$. By the hypothesis we can choose a maximal element q of $\mathcal{S}$. Clearly, $(0 :_M q) \neq 0$. We claim that q is a prime ideal of R. For if not, suppose that I and J are two ideals of R properly containing q and $IJ \subseteq q$. Then $V^u((0 :_M I))$ and $V^u((0 :_M J))$ are covered by a finite subfamily of $\mathcal{A}$. Suppose $N \in V^u((0 :_M IJ))$. Then $IJ \subseteq ann_R(N) := p$. Since p is prime, either $I \subseteq p$ or $J \subseteq p$. This implies that either $N \in V^u((0 :_M I))$ or $N \in V^u((0 :_M J))$. Thus $V^u((0 :_M IJ)) \subseteq V^u((0 :_M I)) \cup V^u((0 :_M J))$. This shows that $V^u((0 :_M IJ))$ can be covered by a finite subfamily of $\mathcal{A}$. Since $IJ \subseteq q$, we have $V^u((0 :_M q)) \subseteq V^u((0 :_M IJ))$. Thus $V^u((0 :_M q))$ can be covered by a finite subfamily of $\mathcal{A}$, a contradiction. Hence, q is a prime ideal of R. We claim that $q = ann_R((0 :_M q))$. For if not, then there exists an ideal q_1 of R such that $q_1 = ann_R((0 :_M q))$ and $q \subsetneq q_1$. This implies that $(0 :_M q) = (0 :_M q_1)$ and so no finite subfamily of $\mathcal{A}$ covers $V^u((0 :_M q_1))$. This shows that $q_1 \in \mathcal{S}$, contrary to maximality of q. Therefore, $q = ann_R(0 :_M q))$. Since M is quasi-secondful, there exits $Q \in u.Spec^s(M)$ such that $q = ann_R(Q)$. Let $\mathcal{U} \in \mathcal{A}$ such that $Q \in \mathcal{U}$. By Lemma 3.2, there exists a submodule K of M such that $q = ann_R(Q) \subsetneq ann_R(K)$ and $Q \in V^u(Q) \cap U^u(K) \subseteq \mathcal{U}$. Put $I := ann_R(K)$. Then $U^u(K) = U^u((0 :_M I))$ and $V^u(Q) = V^u((0 :_M q))$. It follows that $Q \in U^u((0 :_M I)) \cap V^u((0 :_M q)) \subseteq \mathcal{U}$. Since $q \subsetneq I$, $V^u((0 :_M I))$ can be covered by a finite subfamily $\mathcal{A}'$ of $\mathcal{A}$. But $V^u((0 :_M q))\backslash V^u((0 :_M I)) = V^u((0 :_M q))\backslash[U^u((0 :_M I))]^c = V^u((0 :_M q)) \cap U^u((0 :_M I)) \subseteq \mathcal{U}$. So $V^u((0 :_M q))$ can be covered by $\mathcal{A}' \cup \{\mathcal{U}\}$, contrary to our choice of q. Thus there exists a finite subfamily of A which covers $u.Spec^s(M)$. Also, $u.Spec^s(M)$ is Hausdorff by Theorem 3.3. Therefore, $u.Spec^s(M)$ is compact with the finer upper patch topology of M. □

Theorem 3.7. *Let M be an Artinian left R-module. Then $u.Spec^s(M)$ is a compact space with the finer upper patch topology.*

Proof. Let M be an Artinian left R-module and let $\mathcal{A}$ be a family of finer upper patch-open sets covering $u.Spec^s(M)$. Suppose that no finite subfamily of $\mathcal{A}$ covers $u.Spec^s(M)$. Let $\mathcal{T} = \{(0 :_M L) : L$ is an ideal of R such that no finite subfamily of $\mathcal{A}$ covers $V^u((0 :_M L))\}$. Since $V^u((0 :_M 0)) = V^u(M) = u.Spec^s(M)$, $\mathcal{T} \neq \emptyset$. By the hypothesis, we can choose an ideal F of R such that $(0 :_M F)$ is a minimal element of $\mathcal{T}$. Let $ann_R((0 :_M F)) = q$. Then $V^u((0 :_M F)) = V^u((0 :_M q))$. Clearly, $(0 :_M q) \neq 0$. We claim that q is a prime ideal of R. For if not, suppose that I and J are two ideals of R properly containing q anf $IJ \subseteq q$. Then $(0 :_M I) \subsetneq (0 :_M F)$ and $(0 :_M J) \subsetneq (0 :_M F)$. By the minimality of F, $V^u((0 :_M I))$ and $V^u((0 :_M J))$ are covered by a finite subfamily of $\mathcal{A}$. As in the proof of Theorem 3.6, we have $V^u((0 :_M IJ) \subseteq V^u((0 :_M I)) \cup V^u((0 :_M J))$ and so $V^u((0 :_M IJ))$ can be covered by a finite subfamily of $\mathcal{A}$. Since $IJ \subseteq q$, $V^u((0 :_M q)) \subseteq V^u((0 :_M IJ)$. Thus $V^u((0 :_M q)) = V^u((0 :_M F))$ can be covered by a finite subfamily of $\mathcal{A}$, a contradiction.Thus q is a prime ideal of R. Now, we claim that $ann_R((0 :_M q)) = q$. For if not, there exists an ideal q_1 of R such that hence $q_1 = ann_R((0 :_M q))$ and $q \subsetneq q_1$. It follows that $(0 :_M F) \subseteq (0 :_M q) \subseteq (0 :_M q_1)$ and hence $q_1 \subseteq ann_R((0 :_M F)) = q$, a contradiction. Thus $ann_R((0 :_M q)) = q$. Since M is quasi-secondful, there exists $Q \in u.Spe0c^s(M)$ such that $ann_R(Q) = q$. Let $\mathcal{U} \in \mathcal{A}$ such that $Q \in \mathcal{U}$. By Lemma 3.2, there exists a submodule K of M such that $q = ann_R(Q) \subsetneq ann_R(K)$ and $Q \in V^u(Q) \cap U^u(K) \subseteq \mathcal{U}$. Put $I := ann_R(K)$. Then $U^u(K) = U^u((0 :_M I))$ and $V^u(Q) = V^u((0 :_M q))$. So $Q \in V^u((0 :_M q)) \cap U^u((0 :_M I)) \subseteq \mathcal{U}$. Since $F \subseteq q \subsetneq I$, we have $(0 :_M I) \subsetneq (0 :_M F)$. By the minimality of $(0 :_M F)$, $V^u((0 :_M I))$ can be covered by a finite subfamily $\mathcal{A}'$ of $\mathcal{A}$. But $V^u((0 :_M q))\backslash V^u((0 :_M I)) = V^u((0 :_M q))\backslash[U^u((0 :_M I))]^c = V^u((0 :_M q)) \cap U^u((0 :_M I)) \subseteq \mathcal{U}$. So $V^u((0 :_M q)) = V^u((0 :_M F))$ can be covered by $\mathcal{A}' \cup \{\mathcal{U}\}$, contrary to our choice of F. Thus there exists a finite subfamily of A which covers $u.Spec^s(M)$. Also, $u.Spec^s(M)$ is Hausdorff by Theorem 3.3. Therefore, $u.Spec^s(M)$ is compact with the finer upper patch topology of M. □

Let τ and τ^* be two topologies on a set X such that $\tau \subseteq \tau^*$. It is easy to observe that if X is τ^*-quasi-compact, then X is τ-quasi-conpact.

Theorem 3.8. *Let M be a left R-module. If $u.Spec^s(M)$ is compact with the finer upper patch topology, then for each submodule N of M, $U^u(N)$ is a quasi-compact subset of $u.Spec^s(M)$ with respect to upper dual Zariski topology. Consequently, $u.Spec^s(M)$ is quasi-compact with upper dual Zariski topology.*

Proof. For each submodule N of M, $V^u(N) = V^u(N) \cap U^u(0)$ is an open subset of $u.Spec^s(M)$ with the finer upper patch topology, and hence for each submodule N of M, $U^u(N)$ is a closed subset in $u.Spec^s(M)$ with the finer upper patch topology. Since every closed subset of a compact space is compact, $U^u(N)$ is compact with the finer upper patch topology. Thus $U^u(N)$ is also quasi-compact with the upper dual Zariski topology. Now, since $u.Spec^s(M) = U^u(0)$, $u.Spec^s(M)$ is quasi-compact with the upper dual Zariski topology. □

Corollary 3.9. *Let M be a left R-module. If $u.spec^s(M)$ is compact with the finer upper patch topology, then the finer upper patch topology and the upper patch topology of M coincide.*

Proof. By Theorem 3.8, for each submodule K of M, $U^u(K)$ is quasi-compact with the upper dual Zariski topology. Therefore, the bases of the finer upper patch topology and the upper patch topology are the same. Thus these two topologies coincide. □

Corollary 3.10. *Suppose that M is an Artinian left R-module or M is a quasi-secondful left R-module such that $R/ann_R(M)$ satisfies ascending chain condition on ideals. Then:*

(1) *The finer upper patch topology and the upper patch topology of M coincide.*

(2) *The upper dual Zariski topology of M is quasi-compact.*

Proof. By Theorems 3.6, 3.7, 3.8 and Corollary 3.9. □

4. Modules Whose Upper Dual Zariski Topologies Are Spectral

Let M be a left R-module and $Y \subseteq u.Spec^s(M)$. We denote the closure of Y with respect to the upper dual Zariski topology by $Cl^{udz}(Y)$.

Proposition 4.1. *Let M be a left R-module and $Y \subseteq u.Spec^s(M)$.Then $Cl^{udz}(Y) = V^u(T^u(Y))$ where $T^u(Y)$ is the sum of all elements in Y. In particular, $Cl^{udz}(\{S\}) = V^u(S)$ for every $S \in u.Spec^s(M)$.*

Proof. Clearly, $Y \subseteq V^u(T^u(Y))$. Let $V^u(N)$ be a closed subset of $u.Spec^s(M)$ containing Y where N is a submodule of M. Then $ann_R(N) \subseteq ann_R(S)$ for every $S \in Y$ so that $ann_R(N) \subseteq \cap_{S\in Y} ann_R(S) = ann_R(\sum_{S\in Y} S) = ann_R(T^u(Y))$. Hence for every $Q \in V^u(T^u(Y))$, $ann_R(N) \subseteq ann_R(T^u(Y)) \subseteq ann_R(Q)$ and so $Q \in V^u(N)$. Therefore, $V^u(T^u(Y)) \subseteq V^u(N)$. This shows that $Cl^{udz}(Y) = V^u(T^u(Y))$. □

Lemma 4.2. *Let M be a left R-module and $S, Q \in u.Spec^s(M)$. If $V^u(S) = V^u(Q)$, then $S = Q$.*

Proof. $V^u(S) \subseteq V^u(Q)$ implies $S \in V^u(Q)$ and so $ann_R(Q) \subseteq ann_R(S)$. Similarly, we get $ann_R(S) \subseteq ann_R(Q)$. Therefore, $ann_R(Q) = ann_R(S)$ and hence $S = Q$. □

Proposition 4.3. *Let M be a left R-module Then $u.Spec^s(M)$ is a T_0-space with the upper dual Zariski topology.*

Proof. The result follows from Proposition 4.1, Lemma 4.2 and the fact that a topological space is T_0 if and only if the closures of distinct points are distinct. □

Lemma 4.4. *Let M be a left R-module. Then for each $S \in Spec^s(M)$, $V^u(S)$ is irreducible with the upper dual Zariski topology.*

Proof. Let $V^u(S) \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are closed sets. Then there exists submodules N_1 and N_2 of M such that $Y_1 = V^u(N_1)$ and $Y_2 = V^u(N_2)$. There exists an upper second submodule Q of M such that $ann_R(Q) = ann_R(S)$. Since $Q \in V^u(S)$, either $Q \in Y_1$ or $Q \in Y_2$. Without loss of generality we may assume that $Q \in Y_1 = V^u(N_1)$. Then $ann_R(N_1) \subseteq ann_R(Q) = ann_R(S)$. This implies that $V^u(S) \subseteq V^u(N_1) = Y_1$. Thus $V^u(S)$ is irreducible. □

Corollary 4.5. *Let M be a left R-module and $S \in Spec^s(M)$. If $Q \in u.Spec^s(M)$ such that $ann_R(Q) = ann_R(S)$, then Q is a generic point of the irreducible closed subset $V^u(S)$ of $u.Spec^s(M)$ with the dual Zariski topology..*

Proof. By Lemma 4.4, $V^u(S)$ is an irreducible closed subset of $u.Spec^s(M)$. On the other hand $Cl^{udz}(\{Q\}) = V^u(Q) = V^u(S)$ by Proposition 4.1. Thus Q is a generic point of $V^u(S)$. □

Proposition 4.6. *Let M be a left R-module and $Y \subseteq u.Spec^s(M)$. If $T^u(Y)$ is a second submodule of M, then Y is irreducible with the dual Zariski topology.*

Proof. Suppose that $S := T^u(Y)$ is a second submodule of M. By Proposition 4.1, $Cl^{udz}(Y) = V^u(S)$. Now let $Y \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are closed subsets of $u.Spec^s(M)$. Then $Cl^{udz} = V^u(S) \subseteq Y_1 \cup Y_2$. By Lemma 4.4, $V^u(S)$ is irreducible. Therefore we obtain $Y \subseteq Y_1$ or $Y \subseteq Y_2$. Thus Y is irreducible. □

Proposition 4.7. *Let M be a left R-module. If $u.Spec^s(M)$ is quasi-compact with the finer upper patch topology, then every irreducible closed subset of $u.Spec^s(M)$ with the upper dual Zariski topology has a generic point.*

Proof. Let Y be an irreducible closed subset of $u.Spec^s(M)$. First, we show that $Y = \cup_{S\in Y} V^u(S)$. Clearly, $Y \subseteq \cup_{S\in Y} V^u(S)$. By Proposition 4.1, for each $S \in Y$, we have $V^u(S) = Cl^{udz}(\{S\}) \subseteq Cl^{udz}(Y)$, and since $Cl^{udz}(Y) = Y$, $\cup_{S\in Y} V^u(S) \subseteq Y$. Thus $Y = \cup_{S\in Y} V^u(S)$. For each $S \in Y$, $V^u(S) = V^u(S) \cap U^u(0)$ is an open subset of $u.Spec^s(M)$ with the finer upper patch topology. Since Y is closed with the upper dual Zariski topology, $Y = V^u(N)$ for some submodule N of M. Since $U^u(N) = U^u(N) \cap V^u(M)$ is open with the finer upper patch topology, $V^u(N)$ is closed in $u.Spec^s(M)$ with the finer upper patch topology. Therefore, Y is compact with the finer upper patch topology. Thus there exists a finite subset Y' of Y such that $Y = \cup_{S\in Y'} V^u(S)$. Also, since Y is irreducible, $Y = V^u(S) = Cl^{udz}(\{S\})$ for some $S \in Y'$ and so Y has a generic point with the upper dual Zariski topology. □

Theorem 4.8. *Let M be a left R-module. If $u.Spec^s(M)$ is quasi-compact with the finer upper patch topology, then $u.Spec^s(M)$ is a spectral space with the upper dual Zariski topology.*

Proof. By Proposition 4.3, $u.Spec^s(M)$ is a T_0-space and by Theorem 3.8, $u.Spec^s(M)$ is quasi-compact and has a basis of quasi-compact open subsets which are closed under finite intersections. Finally, by Proposition 4.7, every irreducible closed subset of $u.Spec^s(M)$ has a generic point. Thus, $u.Spec^s(M)$ is a spectral space by Hochster's characterization. □

Corollary 4.9. *Let M be a left R-module. Then the following hold.*

(1) *If $u.Spec^s(M)$ is finite, then $u.Spec^s(M)$ is a spectral space with the upper dual Zariski topology.*

(2) *If M is an Artinian R-module or M is a quasi-secondful R-module such that $R/ann_R(M)$ satisfies ascending chain condition on ideals, then $u.Spec^s(M)$ is a spectral space with the upper dual Zariski topology.*

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