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Identities and Relations for Special Numbers and Polynomials: An Approach to Trigonometric Functions

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Abstract. In this paper, by using trigonometric functions and generating functions, identities and relations associated with special numbers and polynomials are derived. Relations among the combinatorial numbers, the Bernoulli polynomials, the Euler numbers, the Stirling numbers and others special numbers and polynomials are given.

1. Introduction

Recently, mathematicians and other scientists have studied special functions, special numbers and polynomials. The motivation of this paper is to give some identities, formulas and relations of special numbers and polynomials with the help of special functions including generating functions and trigonometric identities.

The following definitions, relations and notations are used throughout this paper. Let $\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ denote the set of integers, $\mathbb{R}^+$ denote the set of positive real numbers and $\mathbb{C}$ denote the set of complex numbers. For $n \in \mathbb{N}$, $(\alpha)^{\underline{n}} = \alpha(\alpha-1)\ldots(\alpha-n+1) = \binom{\alpha}{n}n!$. We assume that $0^0 = 1$ and $i^2 = -1$.

Now we give some generating functions for very useful numbers and polynomials with their recurrence relations and other well-known properties.

Let $k \in \mathbb{Z}$. The Bernoulli polynomials of order k, are defined by means of the following generating function:

$$F_B(t, x; k) = \left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \tag{1}$$

where $|t| < 2\pi$ (*cf.* [9], [10], [18]; and the references therein).

We observe that $B_n^{(k)}(0) = B_n^{(k)}$, denoted the Bernoulli numbers of order k. The Bernoulli polynomials and numbers are given respectively by $B_n^{(1)}(x) = B_n(x)$ and $B_n(0) = B_n$ (*cf.* [5], [6], [7], [8], [9], [18]; and

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the references therein). Similarly, the Euler polynomials of order k, are defined by means of the following generating function:

$$F_E(t,x;k) = \left(\frac{2}{e^t+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}, \tag{2}$$

where $|t| < \pi$ (*cf.* [7], [9], [18]; and the references therein). We observe that $E_n^{(k)}(0) = E_n^{(k)}$, denoted the Euler numbers of order k. The Euler polynomials and numbers are given respectively by $E_n^{(1)}(x) = E_n(x)$ and $E_n(0) = E_n$ (*cf.* [5], [7], [8], [9], [18]; and the references therein).

The λ-array polynomials $S_k^n(x;\lambda)$ are defined by means of the following generating function:

$$F_A(t,x,k;\lambda) = \frac{(\lambda e^t-1)^k}{k!}e^{xt} = \sum_{n=0}^{\infty} S_k^n(x;\lambda)\frac{t^n}{n!}, \tag{3}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. When $x = 0$ and $\lambda = 1$, we have the Stirling numbers of the second kind: $S(n,k) = S_k^n(0;1)$ (*cf.* [1], [4], [7], [13], [15]; and the references therein).

The numbers $y_1(n,k;\lambda)$ are defined by means of the following generating function:

$$F_{y_1}(t,k;\lambda) = \frac{1}{k!}\left(\lambda e^t+1\right)^k = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}, \tag{4}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [16]).

The second author [16, Eq-(28)] gave

$$E_n^{(-k)} = k!2^{-k}y_1(n,k;1). \tag{5}$$

The numbers $y_2(n,k;\lambda)$ are defined by means of the following generating function:

$$F_{y_2}(t,k;\lambda) = \frac{1}{(2k)!}\left(\lambda e^t+\lambda^{-1}e^{-t}+2\right)^k = \sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!}, \tag{6}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [16]).

The numbers $y_3(n,k;\lambda;a,b)$ are defined by means of the following generating function:

$$F_{y_3}(t,k;\lambda;a,b) = \frac{e^{bkt}}{k!}\left(\lambda e^{(a-b)t}+1\right)^k = \sum_{n=0}^{\infty} y_3(n,k;\lambda;a,b)\frac{t^n}{n!}, \tag{7}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [15]).

The central factorial numbers $T(n,k)$ (of the second kind) are defined by means of the following generating function:

$$F_T(t,k) = \frac{1}{(2k)!}\left(e^t+e^{-t}-2\right)^k = \sum_{n=0}^{\infty} T(n,k)\frac{t^{2n}}{(2n)!} \tag{8}$$

(*cf.* [3], [5], [14], [15], [16], [17]; and the references therein).

2. Identity including sin t function and array polynomials

In this section, by using generating function, functional equation and sin t, we give an identity, including the array polynomials.

Theorem 2.1. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$S_{3n}^{m}\left(-\frac{3n}{2};1\right)=\sum_{j=0}^{n}\sum_{k=0}^{m}(-1)^{j}\binom{m}{k}\frac{3^{m+j-k}n!}{(3n)!}S_{j}^{k}\left(-\frac{j}{2};1\right)S_{n-j}^{m-k}\left(\frac{j-n}{2};1\right).$$

Proof. By combining (3) with the following well-known identity

$$(\sin t)^{3n}=4^{-n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(3\sin t)^{j}(\sin 3t)^{n-j}, \tag{9}$$

we have

$$(3n)!F_{A}\left(2it,-\frac{3n}{2},3n;1\right)=n!\sum_{j=0}^{n}(-1)^{j}3^{j}F_{A}\left(2it,-\frac{j}{2},j;1\right)F_{A}\left(6it,-\frac{n-j}{2},n-j;1\right).$$

By using the above functional equation, we get

$$\sum_{m=0}^{\infty}S_{3n}^{m}\left(-\frac{3n}{2};1\right)(2i)^{m}\frac{t^{m}}{m!}=\sum_{m=0}^{\infty}\sum_{j=0}^{n}(-1)^{j}\sum_{k=0}^{m}\binom{m}{k}\frac{3^{m+j-k}n!}{(3n)!}S_{j}^{k}\left(-\frac{j}{2};1\right)S_{n-j}^{m-k}\left(\frac{j-n}{2};1\right)(2i)^{m}\frac{t^{m}}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result. □

2.1. Identities and relations arised from De Moivre's formula and Euler identities

Here, we set $f_n(it,x)=e^{intx}$. Applying the De Moivre's formula and the Euler identities to this function and using generating functions for the Bernoulli polynomials of order $-k$, the Euler numbers and polynomials of order $-k$, the Stirling numbers of the second kind, the array polynomials, the numbers $y_1(n,k;\lambda)$, the numbers $y_2(n,k;\lambda)$, the numbers $y_3(n,k;\lambda;a,b)$, and the central factorial numbers, we derive some formulas and relations.

By the aid of the De Moivre's formula and Euler identities, we have

$$f_n(it,x)=\sum_{j=0}^{n}\binom{n}{j}(\cos tx)^{n-j}(i\sin tx)^{j}. \tag{10}$$

(*cf.* [11]). Using (10), we have the following well-known identities:

$$\mathrm{Re}\left(f_n(it,x)\right)=\sum_{l=0}^{\left[\frac{n}{2}\right]}(-1)^{l}\binom{n}{2l}(\sin tx)^{2l}(\cos tx)^{n-2l} \tag{11}$$

and

$$\mathrm{Im}\left(f_n(it,x)\right)=\sum_{l=0}^{\left[\frac{n-1}{2}\right]}(-1)^{l}\binom{n}{2l+1}(\sin tx)^{2l+1}(\cos tx)^{n-2l-1}. \tag{12}$$

(*cf.* [2], [11, Eq. (1.5)-(1.6)]).

By applying the Binomial theorem and (3) to the function $f_n(it,x)$, we obtain

$$f_n(it,x)=\sum_{m=0}^{\infty}\sum_{v=0}^{m}(x)^{\underline{v}}S(m,v)\frac{(int)^{m}}{m!}.$$

By substituting the Taylor expansion of the function $f_n(it,x)$ into the left-hand side of the above equation and comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the final equation, we have the following well-known identity for the Stirling numbers of the second kind:

$$x^m = \sum_{v=0}^{m} (x)^{\underline{v}} S(m,v). \tag{13}$$

Theorem 2.2. *Let $n,m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} (x)^{\underline{v}} S(m,v)\, n^m = x^m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{m-j} \binom{m-j}{k} (m)^{\underline{j}} 2^{m-j} E_{m-j-k}^{(j-n)}\left(\frac{j-n}{2}\right) B_k^{(-j)}\left(-\frac{j}{2}\right). \tag{14}$$

Proof. Combining (1), (2) and (10), we get the following functional equation:

$$f_n(it,x) = \sum_{j=0}^{n} \binom{n}{j} (itx)^j F_E\left(2itx, -\frac{n-j}{2}; -(n-j)\right) F_B\left(2itx, -\frac{j}{2}; -j\right).$$

By using the above functional equation, we get

$$\sum_{m=0}^{\infty}\sum_{v=0}^{m} (x)^{\underline{v}} S(m,v)\, n^m i^m \frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{m-j} \binom{m-j}{k} (m)^{\underline{j}} 2^{m-j} B_k^{(-j)}\left(-\frac{j}{2}\right) E_{m-j-k}^{(j-n)}\left(\frac{j-n}{2}\right) x^m i^m \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result. □

Integrating both sides of (14) from 0 to 1 with respect to x, and combining with following well-known the Cauchy numbers of the first kind (the Bernoulli numbers of the second kind)

$$b_n(0) = \int_0^1 (x)^{\underline{n}}\, dx \tag{15}$$

(*cf.* [5], [12]), we arrive at the following corollary:

Corollary 2.1. *Let $n,m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} b_v(0) S(m,v)\, n^m = \frac{1}{m+1} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{m-j} \binom{m-j}{k} (m)^{\underline{j}} 2^{m-j} E_{m-j-k}^{(j-n)}\left(\frac{j-n}{2}\right) B_k^{(-j)}\left(-\frac{j}{2}\right).$$

Theorem 2.3. *Let $n,m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} (x)^{\underline{v}} S(m,v)\, n^m = 2^{m-n} n! x^m \sum_{j=0}^{n}\sum_{k=0}^{m} \binom{m}{k} y_1(k, n-j; 1)\, S_j^{m-k}\left(-\frac{n}{2}; 1\right). \tag{16}$$

Proof. Combining (3), (4) and (10), we get the following functional equation:

$$f_n(it,x) = 2^{-n} n! \sum_{j=0}^{n} F_{y_1}(2itx, n-j; 1)\, F_A\left(2itx, -\frac{n}{2}, j; 1\right).$$

By using the above functional equation, we get

$$\sum_{m=0}^{\infty}\sum_{v=0}^{m} (x)^{\underline{v}} S(m,v)\, n^m i^m \frac{t^m}{m!} = n! \sum_{m=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{m} \binom{m}{k} 2^{m-n} y_1(k, n-j; 1)\, S_j^{m-k}\left(-\frac{n}{2}; 1\right) x^m i^m \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result. □

Integrating both sides of (16) from 0 to 1 with respect to x, and combining with (15), we arrive at the following corollary:

Corollary 2.2. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} b_v(0) S(m,v) n^m = \frac{2^{m-n} n!}{m+1} \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} y_1(k, n-j; 1) S_j^{m-k}\left(-\frac{n}{2}; 1\right).$$

Combining (5) with (16), we arrive at the following theorem:

Theorem 2.4. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} (x)^{\underline{v}} S(m,v) n^m = x^m \sum_{j=0}^{n} \binom{n}{j} j! 2^{m-j} \sum_{k=0}^{m} \binom{m}{k} E_k^{(j-n)} S_j^{m-k}\left(-\frac{n}{2}; 1\right). \tag{17}$$

Integrating both sides of (17) from 0 to 1 with respect to x, and combining with (15), we arrive at the following corollary:

Corollary 2.3. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} b_v(0) S(m,v) n^m = \frac{1}{m+1} \sum_{j=0}^{n} \binom{n}{j} j! 2^{m-j} \sum_{k=0}^{m} \binom{m}{k} E_k^{(j-n)} S_j^{m-k}\left(-\frac{n}{2}; 1\right).$$

Theorem 2.5. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} (x)^{\underline{v}} S(m,v) n^m = x^m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{m} \binom{m}{l} \frac{j!(2k)!(-1)^{n-j-k}}{2^{j+k+l-m}} S_j^{m-l}\left(-\frac{j}{2}; 1\right) y_2(l, k; 1). \tag{18}$$

Proof. Combining (3), (6) and (10), we get the following functional equation:

$$f_n(it, x) = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \frac{j!(-2)^{n-j-k}(2k)!}{2^n} F_A\left(2itx, -\frac{j}{2}, j; 1\right) F_{y_2}(itx, k; 1).$$

By using the above functional equation, we obtain

$$\sum_{m=0}^{\infty} \sum_{v=0}^{m} (x)^{\underline{v}} S(m,v) n^m i^m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{m} \binom{m}{l} \frac{j!(2k)!(-1)^{n-j-k}}{2^{j+k+l-m}} S_j^{m-l}\left(-\frac{j}{2}; 1\right) y_2(l, k; 1) x^m i^m \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result. □

Integrating both sides of (18) from 0 to 1 with respect to x, and combining with (15), we arrive at the following corollary:

Corollary 2.4. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} b_v(0) S(m,v) n^m = \frac{1}{m+1} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{m} \binom{m}{l} \frac{j!(2k)!(-1)^{n-j-k}}{2^{j+k+l-m}} S_j^{m-l}\left(-\frac{j}{2}; 1\right) y_2(l, k; 1).$$

Theorem 2.6. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} (x)^{\underline{v}} S(m,v) n^m = 2^{m-n} n! x^m \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} y_3\left(k, n-j; 1; \frac{1}{2}, \frac{-1}{2}\right) S_j^{m-k}\left(-\frac{j}{2}; 1\right). \tag{19}$$

Proof. Combining (3), (7) and (10), we get the following functional equation:

$$f_n(it,x) = 2^{-n} n! \sum_{j=0}^{n} F_{y_3}\left(2itx, n-j; 1; \frac{1}{2}, \frac{-1}{2}\right) F_A\left(2itx, -\frac{j}{2}, j; 1\right).$$

By using the above functional equation, we get

$$\sum_{m=0}^{\infty} \sum_{v=0}^{m} (x)^{\underline{v}} S(m,v) n^m i^m \frac{t^m}{m!} = n! \sum_{m=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} 2^{m-n} y_3\left(k, n-j; 1; \frac{1}{2}, \frac{-1}{2}\right) S_j^{m-k}\left(-\frac{j}{2}; 1\right) x^m i^m \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, after some elementary calculations we arrive at the desired result. □

Integrating both sides of (19) from 0 to 1 with respect to x, and combining with (15), we arrive at the following corollary:

Corollary 2.5. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} b_v(0) S(m,v) n^m = \frac{2^{m-n} n!}{m+1} \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} y_3\left(k, n-j; 1; \frac{1}{2}, \frac{-1}{2}\right) S_j^{m-k}\left(-\frac{j}{2}; 1\right).$$

Theorem 2.7. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} (x)^{\underline{v}} S(m,v) n^m = x^m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{\left[\frac{m}{2}\right]} \binom{m}{2l} \frac{j!(2k)!}{2^{j+k+2l-m}} S_j^{m-2l}\left(\frac{-j}{2}; 1\right) T(l,k). \tag{20}$$

Proof. Combining (3), (8) and (10), we get the following functional equation:

$$f_n(it,x) = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \frac{j!(2k)!}{2^{j+k}} F_A\left(2itx, -\frac{j}{2}, j; 1\right) F_T(itx, k).$$

By using the above functional equation, we have

$$\sum_{m=0}^{\infty} \sum_{v=0}^{m} (x)^{\underline{v}} S(m,v) n^m i^m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{\left[\frac{m}{2}\right]} \binom{m}{2l} \frac{j!(2k)!}{2^{j+k+2l-m}} S_j^{m-2l}\left(-\frac{j}{2}; 1\right) T(l,k) x^m i^m \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result. □

Integrating both sides of (20) from 0 to 1 with respect to x, and combining with (15), we arrive at the following corollary:

Corollary 2.6. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m} b_v(0) S(m,v) n^m = \frac{1}{m+1} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{\left[\frac{m}{2}\right]} \binom{m}{2l} \frac{j!(2k)!}{2^{j+k+2l-m}} S_j^{m-2l}\left(\frac{-j}{2}; 1\right) T(l,k).$$

Theorem 2.8. *Let $n, m \in \mathbb{N}_0$. Then we have*

$$E_m^{(-1)}\left(-\frac{1}{2}\right) = n^{-m} \sum_{l=0}^{\left[\frac{n}{2}\right]} \binom{n}{2l} \frac{(2l)!}{4^l} \sum_{k=0}^{m} \binom{m}{k} E_k^{(2l-n)} S_{2l}^{m-k}\left(-\frac{n}{2}; 1\right).$$

Proof. Combining (2), (3) and (11), we get the following functional equation:

$$\mathrm{Re}\left(f_n\left(it,x\right)\right)=\sum_{l=0}^{\left[\frac{n}{2}\right]}\binom{n}{2l}\frac{(2l)!}{4^l}F_A\left(2itx,-\frac{n}{2},2l;1\right)F_E\left(2itx,0;-(n-2l)\right).$$

By using the above functional equation, we get

$$\sum_{m=0}^{\infty}E_m^{(-1)}\left(-\frac{1}{2}\right)(2ix)^m n^m\frac{t^m}{m!}=\sum_{m=0}^{\infty}\sum_{l=0}^{\left[\frac{n}{2}\right]}\binom{n}{2l}\frac{(2l)!}{4^l}\sum_{k=0}^{m}\binom{m}{k}E_k^{(2l-n)}S_{2l}^{m-k}\left(-\frac{n}{2};1\right)(2ix)^m\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result. □

Theorem 2.9. *Let $n,m\in\mathbb{N}_0$. Then we have*

$$S_1^m\left(-\frac{1}{2};1\right)=n^{-m}\sum_{l=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2l+1}\frac{(2l+1)!}{4^l}\sum_{k=0}^{m}\binom{m}{k}E_k^{(2l+1-n)}S_{2l+1}^{m-k}\left(-\frac{n}{2};1\right).$$

Proof. Combining (2), (3) and (12), we get the following functional equation:

$$\mathrm{Im}\left(f_n\left(it,x\right)\right)=\sum_{l=0}^{\left[\frac{n-1}{2}\right]}(-1)^l\binom{n}{2l+1}\frac{(2l+1)!}{(2i)^{2l+1}}F_A\left(2itx,-\frac{n}{2},2l+1;1\right)F_E\left(2itx,0;-(n-(2l+1))\right).$$

By using the above functional equation, we get

$$\sum_{m=0}^{\infty}S_1^m\left(-\frac{1}{2};1\right)(2ix)^m n^m\frac{t^m}{m!}=\sum_{m=0}^{\infty}\sum_{l=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2l+1}\frac{(2l+1)!}{4^l}\sum_{k=0}^{m}\binom{m}{k}E_k^{(2l+1-n)}S_{2l+1}^{m-k}\left(-\frac{n}{2}\right)(2ix)^m\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result. □

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