Filomat 34:2 (2020), 323–338
https://doi.org/10.2298/FIL2002323S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Contraction Involving an Open Ball and Common Fixed Point of Multivalued Mappings in Ordered Dislocated Quasi Metric Spaces

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Abstract. The aim of this work is to obtain fixed point results for multivalued mappings satisfying generalized contractions on the intersection of an open ball and a sequence in left (right) K-sequentially complete ordered dislocated quasi metric space. An example has been built to demonstrate the novelty of results. Our results generalize and extend the results of Altun et al. (J. Funct. Spaces, Article ID 6759320, 2016)

1. Introduction and Preliminaries

By excluding one and a half condition, out of three conditions of a metric space, we obtain dislocated quasi metric space [19]. Complete dislocated quasi metric space is a generalization of 0-complete and complete quasi-partial metric space [12, 14]. Dislocated quasi metric also generalizes dislocated metric, partial metric and quasi metric [9]. Fixed point results in dislocated quasi metric space can be seen in [4, 17, 22, 23].

Ran and Reurings [16] gave a fixed point result in partially ordered sets and obtained solution to matrix equations as an application. Nieto et al. [15] gave an extension to the result in [16] for ordered mappings and used it to give a unique solution for ODE with periodic boundary conditions. Altun et al. [1] introduced a new approach to common fixed point of mappings, satisfying a generalized contraction with a new restriction of order, in a complete ordered metric space. For more results in ordered spaces see [2, 6, 8, 10, 11, 13].

Arshad et al. [3] observed that there was mappings which had fixed point but there were no results to ensure the existence of fixed point of such mappings. They introduced a contraction on closed ball to achieve common fixed point for such mappings, see also [20]. Fixed point results for multivalued mappings generalizes the results for single-valued mappings, for example, see [5, 7, 18].

In this paper, we extend the result of Altun et al. [1] in four different ways by using

(i) multivalued mappings instead of single-valued mappings;

(ii) open ball instead of whole space;

2010 *Mathematics Subject Classification*. Primary 47H10, 54H25; Secondary 46S40

Keywords. common fixed point; left (right) K-sequentially complete ordered dislocated quasi metric space; open ball; multivalued mappings; generalized contraction.

Received: 25 February 2019; Accepted: 03 June 2019

Communicated by Miodrag Spalević

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(iii) Ciric type contraction instead of Banach type contraction;

(iv) left (right) K-sequentially complete dislocated quasi metric space instead of complete metric space.

We give the following definitions and results which will be useful to understand the paper.

Definition 1.1 [1] Let Ψ denotes the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions:

(Ψ_1) ψ is non-decreasing.

(Ψ_2) For all $t > 0$, we have

$$\mu_0(t) = \sum_{k=0}^{\infty} \psi^k(t) < \infty,$$

where, ψ^k is the k^{th} iterate of ψ. The function $\psi \in \Psi$ is called comparison function.

Lemma 1.2 [1] Let $\psi \in \Psi$. Then

(i) $\psi(t) < t$, for all $t > 0$,

(ii) $\psi(0) = 0$.

Definition 1.3 [19] Let X be a nonempty set and let $d_q : X \times X \to [0, \infty)$ be a function satisfying the following axioms for all $x, y, z \in X$:

(i) If $d_q(x, y) = d_q(y, x) = 0$, then $x = y$; (ii) $d_q(x, y) \le d_q(x, z) + d_q(z, y)$.

Then the pair (X, d_q) is called a dislocated quasi metric space.

It is clear that if $d_q(x, y) = d_q(y, x) = 0$, then from (i), $x = y$. But if $x = y$, then $d_q(x, y)$ or $d_q(y, x)$ may not be 0. It is observed that if $d_q(x, y) = d_q(y, x)$ for all $x, y \in X$, then (X, d_q) becomes a dislocated metric space (metric-like space) (X, d_l). For $x \in X$ and $\varepsilon > 0$, $B_{d_q}(x, \varepsilon) = \{y \in X : d_q(x, y) < \varepsilon$ and $d_q(y, x) < \varepsilon\}$ and $\overline{B_{d_q}(x, \varepsilon)} = \{y \in X : d_q(x, y) \le \varepsilon$ and $d_q(y, x) \le \varepsilon\}$ are open and closed balls in (X, d_q) respectively.

Example 1.4 [19] Let $X = R^+ \cup \{0\}$ and $d_q(x, y) = x + \max\{x, y\}$ for any $x, y \in X$. Then (X, d_q) is a dislocated quasi metric space.

Definition 1.5 [19] Let (X, d_q) be a dislocated quasi metric space.

(i) A sequence $\{x_n\}$ in (X, d_q) is called left (right) K-Cauchy if $\forall\ \varepsilon > 0$, $\exists\ n_0 \in N$ such that $\forall\ n > m \ge n_0$ (respectively $\forall\ m > n \ge n_0$), $d_q(x_m, x_n) < \varepsilon$.

(ii) A sequence $\{x_n\}$ dislocated quasi-converges (for short d_q -converges) to x if $\lim_{n\to\infty} d_q(x_n, x) = \lim_{n\to\infty} d_q(x, x_n) = 0$ or for any $\varepsilon > 0$, there exists $n_0 \in N$, such that for all $n > n_0$, $d_q(x, x_n) < \varepsilon$ and $d_q(x_n, x) < \varepsilon$. In this case x is called a d_q-limit of $\{x_n\}$.

(iii) (X, d_q) is called left (right) K-sequentially complete if every left (right) K-Cauchy sequence in X converges to a point $x \in X$ such that $d_q(x, x) = 0$.

Definition 1.6 [19] $(X, \preceq, d_q)$ is called an ordered dislocated quasi metric space, if

(i) (X, d_q) is dislocated quasi metric space

(ii) $\preceq$ is a partial order on X.

Definition 1.7 [19] Let (X, d_q) be a dislocated quasi metric space. Let K be a nonempty subset of X and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$\begin{aligned} d_q(x, K) &= d_q(x, y_0), \text{ where } d_q(x, K) = \inf_{y\in K} d_q(x, y) \\ \text{and } d_q(K, x) &= d_q(y_0, x), \text{ where } d_q(K, x) = \inf_{y\in K} d_q(y, x). \end{aligned}$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set. We denote the set of all proximinal subsets of X by $P(X)$.

Definition 1.8 [19] The function $H_{d_q} : P(X) \times P(X) \to [0, \infty)$, defined by

$$H_{d_q}(A, B) = \max\{\sup_{a\in A} d_q(a, B), \sup_{b\in B} d_q(A, b)\}$$

is called dislocated quasi Hausdorff metric on $P(X)$. Also $(P(X), H_{d_q})$ is known as dislocated quasi Hausdorff metric space.

Lemma 1.9 [19] Let (X, d_q) be a dislocated quasi metric space. Let $(P(X), H_{d_q})$ be the dislocated quasi Hausdorff metric space on $P(X)$. Then, for all $A, B \in P(X)$ and for each $a \in A$, there exists $b_a \in B$, such that

$H_{d_q}(A, B) \geq d_q(a, b_a)$ and $H_{d_q}(B, A) \geq d_q(b_a, a)$.
Lemma 1.10 [19] Every closed ball Y in a left (right) K-sequentially complete dislocated quasi metric space X is left (right) K-sequentially complete.

2. Main Result

Let (X, d_q) be a dislocated quasi metric space, $x_0 \in X$ and $T : X \to P(X)$ be a multivalued mapping on X. As Tx_0 is a proximinal set, then there exists $x_1 \in Tx_0$ such that $d_q(x_0, Tx_0) = d_q(x_0, x_1)$ and $d_q(Tx_0, x_0) = d_q(x_1, x_0)$. Now, for $x_1 \in X$, there exist $x_2 \in Tx_1$ be such that $d_q(x_1, Tx_1) = d_q(x_1, x_2)$ and $d_q(Tx_1, x_1) = d_q(x_2, x_1)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in Tx_n$, $d_q(x_n, Tx_n) = d_q(x_n, x_{n+1})$ and $d_q(Tx_n, x_n) = d_q(x_{n+1}, x_n)$. We denote this iterative sequence $\{XT(x_n)\}$ and say that $\{XT(x_n)\}$ is a sequence in X generated by x_0.

Theorem 2.2 Let $(X, \preceq, d_q)$ be an ordered left (right) K-sequentially complete dislocated quasi metric space, $S, T : X \to P(X)$ be the multivalued mappings. Suppose that the following assertions hold:

(i) There exists a function $\mu \in \Psi$, $x_0 \in X$ and $r > 0$ such that for every $(x, y) \in X \times X$, we have

$$\max\{H_q(Tx, Ty), H_q(Ty, Tx)\} \leq \mu(D_q(x, y)),$$

for all $x, y \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$ with $x \succeq Sx$, $y \preceq Sy$, where

$$D_q(x, y) = \max\{d_q(x, y), d_q(x, Tx), d_q(y, Ty)\}.$$

(ii) If $x \in B_{d_q}(x_0, r)$, $d_q(x, Tx) = d_q(x, y)$ and $d_q(Tx, x) = d_q(y, x)$, then

$$(a)\ x \preceq Sx, \text{ implies } y \succeq Sy \quad (b)\ x \succeq Sx, \text{ implies } y \preceq Sy$$

(iii) The set $G(S) = \left\{x : x \preceq Sx \text{ and } x \in B_{d_q}(x_0, r)\right\}$ is closed and contains x_0.

(iv)

$$\sum_{i=0}^{n} \max\{\mu^i(d_q(x_1, x_0), \mu^i(d_q(x_0, x_1))\} < r \text{ for all } n \in \mathbb{N}.$$

Then the subsequence $\{x_{2n}\}$ of $\{XT(x_n)\}$ is a sequence in $G(S)$ and $\{x_{2n}\} \to x^* \in G(S)$ and $d_q(x^*, x^*) = 0$. Also, if the inequality (i) holds for x^*, then S and T have a common fixed point x^* in $B_{d_q}(x_0, r)$.

Proof: As x_0 be an element of $G(S)$, from condition (iii) $x_0 \preceq Sx_0$. Consider the sequence $\{XT(x_n)\}$. Then there exists $x_1 \in Tx_0$ such that

$$d\left(x_0,\ Tx_0\right) = d\left(x_0,\ x_1\right) \text{ and } d\left(Tx_0, x_0\right) = d\left(x_1, x_0\right).$$

From condition (ii) $x_1 \succeq Sx_1$. From condition (iv), we have

$$\max\{d_q(x_1, x_0), d_q(x_0, x_1)\} \leq \sum_{i=0}^{j} \max\{\mu^i(d_q(x_1, x_0), \mu^i(d_q(x_0, x_1))\} < r.$$

It follows that, $d_q(x_1, x_0) < r$ and $d_q(x_0, x_1) < r$. So, we have $x_1 \in B_{d_q}(x_0, r)$. Also,

$$d_q\left(x_1, Tx_1\right) = d_q\left(x_1, x_2\right) \text{ and } d_q\left(Tx_1, x_1\right) = d_q\left(x_2, x_1\right).$$

As $x_1 \succeq Sx_1$, so from condition (ii), we have $x_2 \preceq Sx_2$. By triangular inequality, we have

$$d_q\left(x_0, x_2\right) \leq d_q\left(x_0, x_1\right) + d_q\left(x_1, x_2\right). \tag{2.1}$$

Now, by Lemma 1.9, we have

$$\begin{aligned} d_q(x_1, x_2) &\leq H_{dq}(Tx_0, Tx_1) \\ &\leq \max\{H_{dq}(Tx_0, Tx_1), H_{dq}(Tx_1, Tx_0)\}. \end{aligned}$$

As $x_0, x_1 \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$, $x_1 \geq Sx_1$ and $x_0 \leq Sx_0$, then by (i), we have

$$\begin{aligned} d_q(x_1, x_2) &\leq \mu(D(x_1, x_0)) \\ &\leq \mu(\max\left\{d_q(x_1, x_0), d_q(x_1, x_2), d_q(x_0, x_1)\right\}). \end{aligned}$$

If $\max\left\{d_q(x_1, x_0), d_q(x_1, x_2), d_q(x_0, x_1)\right\} = d_q(x_1, x_2)$, then a contradiction arise by the fact $\mu(t) < t$, so, we have

$$d_q(x_1, x_2) \leq \mu(\max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\}). \tag{2.2}$$

Now, inequality (2.1) implies

$$\begin{aligned} d_q(x_0, x_2) &\leq d_q(x_0, x_1) + \mu(\max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\}) \\ &\leq \max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\} + \mu(\max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\}) \\ &= \sum_{i=0}^{1} \max\{\mu^i(d_q(x_1, x_0), \mu^i(d_q(x_0, x_1))\}. \end{aligned}$$

By using (iv), we have

$$d_q(x_0, x_2) < r. \tag{2.3}$$

Now, by triangular inequality, we have

$$d_q(x_2, x_0) \leq d_q(x_2, x_1) + d_q(x_1, x_0). \tag{2.4}$$

Now, by Lemma 1.9, we have

$$\begin{aligned} d_q(x_2, x_1) &\leq H_{dq}(Tx_1, Tx_0) \\ &\leq \max\{H_{dq}(Tx_0, Tx_1), H_{dq}(Tx_0, Tx_1)\}. \end{aligned}$$

As $x_1, x_0 \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$, $x_1 \geq Sx_1$ and $x_0 \leq Sx_0$, then by (i), we have

$$d_q(x_2, x_1) \leq \mu(\max\left\{d_q(x_1, x_0), d_q(x_1, x_2), d_q(x_0, x_1)\right\}).$$

If $\max\left\{d_q(x_1, x_0), d_q(x_1, x_2), d_q(x_0, x_1)\right\} = d_q(x_1, x_2)$, then by (2.2), we have

$$d_q(x_2, x_1) \leq \mu(\max\left\{d_q(x_1, x_0), \mu(\max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\}), d_q(x_0, x_1)\right\}).$$

If $\max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\} = d_q(x_0, x_1)$, then, we have

$$\begin{aligned} d_q(x_2, x_1) &\leq \mu(\max\left\{d_q(x_1, x_0), \mu(d_q(x_0, x_1)), d_q(x_0, x_1)\right\}) \\ &\leq \mu(\max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\}). \end{aligned}$$

Similarly, if $\max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\} = d_q(x_1, x_0)$, then, we have

$$d_q(x_2, x_1) \leq \mu(\max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\}).$$

Now, inequality (2.4) implies

$$\begin{aligned} d_q(x_2, x_0) &\leq d_q(x_1, x_0) + \mu(\max\left\{d_q(x_1, x_0), d_q(x_0, x_1)\right\}) \\ &\leq \sum_{i=0}^{1} \max\{\mu^i(d_q(x_1, x_0), \mu^i(d_q(x_0, x_1))\} < r. \end{aligned}$$

It follows that, $d_q(x_2, x_0) < r$. By (2.3), $d_q(x_0, x_2) < r$. So, $x_2 \in B_{d_q}(x_0, r)$. Also,

$$d_q(x_2, Tx_2) = d_q(x_2, x_3) \text{ and } d_q(Tx_2, x_2) = d_q(x_3, x_2).$$

As $x_2 \preceq Sx_2$, so from condition (ii), we have $x_3 \succeq Sx_3$. Let $x_3, \cdots, x_j \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$, $x_4 \preceq Sx_4$, $x_5 \succeq Sx_5$, $x_6 \preceq Sx_6$, $x_7 \succeq Sx_7$ up to $x_j \preceq Sx_j$ and $x_{j+1} \succeq Sx_{j+1}$ for some $j \in \mathbb{N}$, where $j = 2i$, $i = 2, 3 ... \frac{j}{2}$. Now, by Lemma 1.9, we obtain

$$\begin{aligned} d_q(x_{2i}, x_{2i+1}) &\leq H_{dq}(Tx_{2i-1,} Tx_{2i}) \\ &\leq \max\{H_{dq}(Tx_{2i-1}, Tx_{2i}), H_{dq}(Tx_{2i}, Tx_{2i-1})\}. \end{aligned}$$

As $x_{2i-1}, x_{2i} \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$, $x_{2i-1} \succeq Sx_{2i-1}$, $x_{2i} \preceq Sx_{2i}$, then by (i), we have

$$\begin{aligned} d_q(x_{2i}, x_{2i+1}) &\leq \mu(\max\left\{d_q(x_{2i-1}, x_{2i}), d_q(x_{2i-1}, x_{2i}), d_q(x_{2i}, x_{2i+1}\right\}) \\ &\leq \mu(\max\left\{d_q(x_{2i-1}, x_{2i}), d_q(x_{2i}, x_{2i+1})\right\}). \end{aligned}$$

If $\max\left\{d_q(x_{2i-1}, x_{2i}), d_q(x_{2i}, x_{2i+1})\right\} = d_q(x_{2i}, x_{2i+1})$, then $d_q(x_{2i}, x_{2i+1}) \leq \mu(d_q(x_{2i}, x_{2i+1}))$, which is contradiction to the fact $\mu(t) < t$. Therefore $\max\left\{d_q(x_{2i-1}, x_{2i}), d_q(x_{2i}, x_{2i+1})\right\} = d_q(x_{2i-1}, x_{2i})$. Then, we have

$$d_q(x_{2i}, x_{2i+1}) \leq \mu(d_q(x_{2i-1}, x_{2i})), \tag{2.5}$$

which implies that

$$d_q(x_{2i}, x_{2i+1}) \leq \max\left\{\mu(d_q(x_{2i-1}, x_{2i}), \mu(d_q(x_{2i}, x_{2i-1})\right\}. \tag{2.6}$$

Now, by Lemma 1.9

$$\begin{aligned} d_q(x_{2i-1}, x_{2i}) &\leq H_{dq}(Tx_{2i-2}, Tx_{2i-1}) \\ &\leq \max\left\{H_{dq}(Tx_{2i-2}, Tx_{2i-1}), H_{dq}(Tx_{2i-1}, Tx_{2i-2})\right\}. \end{aligned}$$

As $x_{2i-1}, x_{2i-2} \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$, $x_{2i-1} \succeq Sx_{2i-1}$ and $x_{2i-2} \preceq Sx_{2i-2}$, then by (i), we have

$$d_q(x_{2i-1}, x_{2i}) \leq \mu(\max d_q(x_{2i-1}, x_{2i-2}), (d_q(x_{2i-1}, x_{2i}), d_q(x_{2i-2}, x_{2i-1})\}.$$

If $\max\left\{d_q(x_{2i-1}, x_{2i-2}), (d_q(x_{2i-1}, x_{2i}), d_q(x_{2i-2}, x_{2i-1})\right\} = d_q(x_{2i-1}, x_{2i})$, then contradiction arise to the fact $\mu(t) < t$. Now,

$$d_q(x_{2i-1}, x_{2i}) \leq \mu(\max d_q(x_{2i-1}, x_{2i-2}), d_q(x_{2i-2}, x_{2i-1})\}.$$

As μ is non decreasing function, so

$$\mu\left(d_q(x_{2i-1}, x_{2i})\right) \leq \mu^2(\max d_q(x_{2i-1}, x_{2i-2}), d_q(x_{2i-2}, x_{2i-1})\}$$

$$\mu\left(d_q(x_{2i-1}, x_{2i})\right) \leq \max\left\{\mu^2(d_q(x_{2i-1}, x_{2i-2})), \mu^2(d_q(x_{2i-2}, x_{2i-1}))\right\}. \tag{2.7}$$

Now, using (2.7) in (2.5), we have

$$d_q(x_{2i}, x_{2i+1}) \leq \max\left\{\mu^2(d_q(x_{2i-1}, x_{2i-2})), \mu^2(d_q(x_{2i-2}, x_{2i-1}))\right\}, \tag{2.8}$$

Now, by Lemma 1.9

$$\begin{aligned} d_q(x_{2i-2}, x_{2i-1}) &\leq H_{dq}(Tx_{2i-3}, Tx_{2i-2}) \\ &\leq \max\left\{H_{dq}(Tx_{2i-3}, Tx_{2i-2}), H_{dq}(Tx_{2i-2}, Tx_{2i-3})\right\}. \end{aligned}$$

As $x_{2i-3}, x_{2i-2} \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$, $x_{2i-3} \geq Sx_{2i-3}$ and $x_{2i-2} \leq Sx_{2i-2}$, then by (i), we have

$$d_q(x_{2i-2}, x_{2i-1}) \leq \mu(\max\left\{d_q(x_{2i-3}, x_{2i-2}), d_q(x_{2i-2}, x_{2i-1})\right\}).$$

If $\max\left\{d_q(x_{2i-3}, x_{2i-2}), d_q(x_{2i-2}, x_{2i-1})\right\} = d_q(x_{2i-2}, x_{2i-1})$, then contradiction arise to the fact $\mu(t) < t$. Therefore

$$d_q(x_{2i-2}, x_{2i-1}) \leq \mu(d_q(x_{2i-3}, x_{2i-2})) \tag{2.9}$$

$$d_q(x_{2i-2}, x_{2i-1}) \leq \mu(\max\left\{d_q(x_{2i-3}, x_{2i-2}), d_q(x_{2i-2}, x_{2i-3})\right\})$$

$$\mu^2 d_q(x_{2i-2}, x_{2i-1}) \leq \mu^3(\max\left\{d_q(x_{2i-3}, x_{2i-2}), d_q(x_{2i-2}, x_{2i-3})\right\}). \tag{2.10}$$

Now, by Lemma 1.9

$$\begin{aligned} d_q(x_{2i-1}, x_{2i-2}) &\leq H_{dq}(Tx_{2i-2}, Tx_{2i-3}) \\ &\leq \max\left\{H_{dq}(Tx_{2i-3}, Tx_{2i-2}), H_{dq}(Tx_{2i-2}, Tx_{2i-3})\right\}. \end{aligned}$$

As $x_{2i-3}, x_{2i-2} \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$, $x_{2i-3} \geq Sx_{2i-3}$ and $x_{2i-2} \leq Sx_{2i-2}$, then by (i), we have

$$d_q(x_{2i-1}, x_{2i-2}) \leq \mu(\max\left\{d_q(x_{2i-3}, x_{2i-2}), d_q(x_{2i-2}, x_{2i-1})\right\}).$$

By using inequality (2.9), we have

$$\begin{aligned} d_q(x_{2i-1}, x_{2i-2}) &\leq \mu(\max\left\{d_q(x_{2i-3}, x_{2i-2}), \mu(d_q(x_{2i-3}, x_{2i-2}))\right\}) \\ &= \mu(d_q(x_{2i-3}, x_{2i-2})). \end{aligned}$$

Which implies that

$$\mu^2 d_q(x_{2i-1}, x_{2i-2}) \leq \mu^2(\mu(\max\left\{d_q(x_{2i-3}, x_{2i-2}), d_q(x_{2i-2}, x_{2i-3})\right\})). \tag{2.11}$$

Combining inequalities (2.8), (2.10) and (2.11), we have

$$d_q(x_{2i}, x_{2i+1}) \leq \max\left\{\mu^3(d_q(x_{2i-3}, x_{2i-2})), \mu^3(d_q(x_{2i-2}, x_{2i-3}))\right\}. \tag{2.12}$$

Following the patterns of inequalities (2.6), (2.8) and (2.12), we have

$$d_q(x_{2i}, x_{2i+1}) \leq \max\left\{\mu^{2i}(d_q(x_0, x_1)), \mu^{2i}(d_q(x_1, x_0))\right\}.$$

Similarly, we have

$$d_q(x_{2i-1}, x_{2i}) \leq \max\left\{\mu^{2i-1}(d_q(x_0, x_1)), \mu^{2i-1}(d_q(x_1, x_0))\right\}.$$

Combining the above two inequalities, we have

$$d_q(x_j, x_{j+1}) \leq \max\left\{\mu^j(d_q(x_0, x_1)), \mu^j(d_q(x_1, x_0))\right\}. \tag{2.13}$$

Now, by Lemma 1.9, we have

$$\begin{aligned} d_q(x_{2i+1}, x_{2i}) &\leq H_{dq}(Tx_{2i}, Tx_{2i-1}) \\ &\leq \max\left\{H_{dq}(Tx_{2i-1}, Tx_{2i}), H_{dq}(Tx_{2i}, Tx_{2i-1})\right\} \end{aligned}$$

As $x_{2i-1}, x_{2i} \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$, $x_{2i-1} \geq Sx_{2i-1}$ and $x_{2i} \leq Sx_{2i}$, then by (i), we have

$$d_q(x_{2i+1}, x_{2i}) \leq \mu(\max\left\{d_q(x_{2i-1}, x_{2i}), d_q(x_{2i}, x_{2i+1})\right\}). \tag{2.14}$$

By inequality (2.5), we have

$$d_q(x_{2i+1}, x_{2i}) \leq \mu(\max\left\{d_q(x_{2i-1}, x_{2i}), \mu(d_q(x_{2i-1}, x_{2i})\right\}).$$

As $\mu(t) < t$, we have

$$d_q(x_{2i+1}, x_{2i}) \leq \mu(d_q(x_{2i-1}, x_{2i})). \tag{2.15}$$

Now,

$$d_q(x_{2i+1}, x_{2i}) \leq \max\left\{\mu\left(d_q(x_{2i-1}, x_{2i})\right), \mu(d_q(x_{2i}, x_{2i-1})\right\}. \tag{2.16}$$

Now, using (2.7) in (2.15), we have

$$d_q(x_{2i+1}, x_{2i}) \leq \max\left\{\mu^2(d_q(x_{2i-1}, x_{2i-2})), \mu^2(d_q(x_{2i-2}, x_{2i-1}))\right\}. \tag{2.17}$$

Combining inequalities (2.10), (2.11) and (2.17), we have

$$d_q(x_{2i+1}, x_{2i}) \leq \max\left\{\mu^3(d_q(x_{2i-3}, x_{2i-2})), \mu^3(d_q(x_{2i-2}, x_{2i-3}))\right\}. \tag{2.18}$$

Following the patterns of inequalities (2.16), (2.17) and (2.18), we have

$$d_q(x_{2i+1}, x_{2i}) \leq \max\left\{\mu^{2i}(d_q(x_0, x_1)), \mu^{2i}(d_q(x_1, x_0))\right\}.$$

Similarly, we have

$$d_q(x_{2i}, x_{2i-1}) \leq \max\left\{\mu^{2i-1}(d_q(x_0, x_1)), \mu^{2i-1}(d_q(x_1, x_0))\right\}.$$

Combining the above two inequalities, we have

$$d_q(x_{j+1}, x_j) \leq \max\left\{\mu^j(d_q(x_0, x_1)), \mu^j(d_q(x_1, x_0))\right\}. \tag{2.19}$$

By using inequalities (2.13), (iv) and triangle inequality, we have

$$\begin{aligned} d_q(x_0, x_{j+1}) &\leq d_q(x_0, x_1) + ... + d_q(x_j, x_{j+1}) \\ &\leq d_q(x_0, x_1) + ... + \max\left\{\mu^j(d_q(x_1, x_0)), \mu^j(d_q(x_0, x_1))\right\} \end{aligned}$$

$$d_q(x_0, x_{j+1}) \leq \sum_{i=0}^{j} \max\{\mu^i(d_q(x_1, x_0), \mu^i(d_q(x_0, x_1))\} < r. \tag{2.20}$$

Similarly, by using inequalities (2.19), (iv) and triangle inequality, we have

$$d_q(x_{j+1}, x_0) \leq \sum_{i=0}^{j} \max\{\mu^i(d_q(x_1, x_0), \mu^i(d_q(x_0, x_1))\} < r. \tag{2.21}$$

By inequality (2.20) and (2.21), we have $x_{j+1} \in B_{dq}(x_0, r)$. Also,

$$d_q\left(x_{j+1}, Tx_{j+1}\right) = d_q\left(x_{j+1}, x_{j+2}\right) \text{ and } d_q\left(Tx_{j+1}, x_{j+1}\right) = d_q\left(x_{j+2}, x_{j+1}\right).$$

As $x_{j+1} \succeq Sx_{j+1}$, so from condition (ii), we have $x_{j+2} \preceq Sx_{j+2}$. Similarly, we get

$$d_q(x_{j+1}, x_{j+2}) \leq \max\left\{\mu^{j+1}(d_q(x_1, x_0))), \mu^{j+1}(d_q(x_0, x_1))\right\}, \tag{2.22}$$

and

$$d_q(x_{j+2}, x_{j+1}) \leq \max\left\{\mu^{j+1}(d_q(x_1, x_0))), \mu^{j+1}(d_q(x_0, x_1))\right\}. \tag{2.23}$$

Also,

$$d_q(x_0, x_{j+2}) < r \text{ and } d_q(x_{j+2}, x_0) < r.$$

It follows that $x_{j+2} \in B_{d_q}(x_0, r)$. Also,

$$d_q\left(x_{j+2}, Tx_{j+2}\right) = d_q\left(x_{j+2}, x_{j+3}\right) \text{ and } d_q\left(Tx_{j+2}, x_{j+2}\right) = d_q\left(x_{j+3}, x_{j+2}\right).$$

As $x_{j+2} \preceq Sx_{j+2}$, so from condition (ii), we have $x_{j+3} \succeq Sx_{j+3}$. Hence by mathematical induction $x_n \in B_{dq}(x_0, r)$, $x_{2n} \preceq Sx_{2n}$ and $x_{2n+1} \succeq Sx_{2n+1}$ for all $n \in \mathbb{N}$. Also $x_{2n} \in G(S)$. Now inequalities (2.13), (2.19), (2.22) and (2.23) can be merged as

$$d_q(x_n, x_{n+1}) \leq \max\left\{\mu^n(d_q(x_1, x_0)), \mu^n(d_q(x_0, x_1))\right\}, \tag{2.24}$$

$$d_q(x_{n+1}, x_n) \leq \max\left\{\mu^n(d_q(x_1, x_0)), \mu^n(d_q(x_0, x_1))\right\}, \tag{2.25}$$

for all $n \in \mathbb{N}$. Fix $\varepsilon > 0$ and let $k_1(\varepsilon) \in \mathbb{N}$ such that $\sum\limits_{k \geq k_1(\varepsilon)} \max\{\mu^k(d_q(x_1, x_0)), \mu^k(d_q(x_0, x_1))\} < \varepsilon$. Let $n, m \in \mathbb{N}$ with $m > n > k_1(\varepsilon)$, then

$$\begin{aligned} d_q(x_n, x_m) &\leq \sum_{k=n}^{m-1} d_q(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \max\{\mu^k(d_q(x_1, x_0)), \mu^k(d_q(x_0, x_1))\}, \text{ by (2.24)} \\ d_q(x_n, x_m) &\leq \sum_{k \geq k_1(\varepsilon)} \max\{\mu^n(d_q(x_1, x_0)), \mu^n(d_q(x_0, x_1))\} < \varepsilon. \end{aligned}$$

Thus, we proved that $\{XT(x_n)\}$ is a left K-Cauchy sequence in $(B_{d_q}(x_0, r), d_q)$. Similarly, by using (2.25), we have

$$d_q(x_m, x_n) \leq \sum_{k=n}^{m-1} d_q(x_{k+1}, x_k) < \varepsilon.$$

Hence, $\{XT(x_n)\}$ is a right K-Cauchy sequence in $(B_{d_q}(x_0, r), d_q)$. As every closed set in left(right) K-sequentially complete dislocated quasi metric space is left(right) K-sequentially complete and $G(S)$ is closed set, so $G(S)$ is left(right) K-sequentially complete. As $\{x_{2n}\}$ is a left(right) K-Cauchy sequence in $G(S)$, so there exists $x^* \in G(S)$ such that $\{x_{2n}\} \to x^*$, that is

$$\lim_{n\to\infty} d_q(x_{2n}, x^*) = \lim_{n\to\infty} d_q(x^*, x_{2n}) = 0. \tag{2.26}$$

Also,

$$x^* \preceq Sx^*. \tag{2.27}$$

Now,

$$d_q(x^*, x^*) \leq d_q(x^*, x_{2n}) + d_q(x_{2n}, x^*).$$

This implies $d_q(x^*, x^*) = 0$ as $n \to \infty$. Now

$$\begin{aligned} d_q(x^*, Tx^*) &\leq d_q(x^*, x_{2n+2}) + d_q(x_{2n+2}, Tx^*) \\ &\leq d_q(x^*, x_{2n+2}) + H_{d_q}(Tx_{2n+1}, Tx^*), \text{ (by Lemma 1.9)} \\ &\leq d_q(x^*, x_{2n+2}) + \max\left\{H_{d_q}(Tx_{2n+1}, Tx^*), H_{d_q}(Tx^*, Tx_{2n+1})\right\}. \end{aligned}$$

By assumption, inequality (i) holds for x^*. Also $x_{2n+1} \geq Sx_{2n+1}$ and $x^* \leq Sx^*$, so

$$\begin{aligned} d_q(x^*, Tx^*) &\leq d_q(x^*, x_{2n+2}) + \mu(\max\left\{d_q(x_{2n+1}, x^*), d_q(x_{2n+1}, x_{2n+2}), d_q(x^*, Tx^*)\right\}) \\ &\leq d_q(x^*, x_{2n+2}) + \mu(\max\{d_q(x_{2n+1}, x_{2n+2}) + d_q(x_{2n+2}, x^*), \\ &\quad d_q(x_{2n+1}, x_{2n+2}), d_q(x^*, Tx^*)\}). \end{aligned}$$

Letting $n \longrightarrow \infty$ and by using inequalities (2.24) and (2.26), we obtain

$$d_q(x^*, Tx^*) \leq \mu(d_q(x^*, Tx^*)).$$

This implies that

$$d_q(x^*, Tx^*) = 0. \tag{2.28}$$

Now,

$$\begin{aligned} d_q(Tx^*, x^*) &\leq d_q(Tx^*, x_{2n+2}) + d_q(x_{2n+2}, x^*) \\ &\leq H_{d_q}(Tx^*, Tx_{2n+1}) + d_q(x_{2n+2}, x^*) \\ &\leq \max\left\{H_{d_q}(Tx_{2n+1}, Tx^*), H_{d_q}(Tx^*, Tx_{2n+1})\right\} + d_q(x_{2n+2}, x^*). \end{aligned}$$

As inequality (i) holds for x^*, $x^* \leq Sx^*$ and $x_{2n+1} \geq Sx_{2n+1}$, then, we obtain

$$d_q(Tx^*, x^*) \leq \mu(\max\left\{d_q(x_{2n+1}, x^*), d_q(x_{2n+1}, x_{2n+2}), d_q(x^*, Tx^*)\right\}) + d_q(x_{2n+2}, x^*).$$

Taking $n \longrightarrow \infty$ and by using inequalities (2.24), (2.26) and (2.28), we have

$$d_q(Tx^*, x^*) = 0. \tag{2.29}$$

From inequalities (2.28) and (2.29), we have $x^* \in Tx^*$. As $x^* \leq Sx^*$ and $d_q(x^*, Tx^*) = d_q(Tx^*, x^*) = 0 = d_q(x^*, x^*)$, then from (ii)

$$x^* \geq Sx^*. \tag{2.30}$$

From (2.27) and (2.30), we have $x^* \leq Sx^* \leq x^*$. This implies $x^* \leq y \leq x^*$, for all $y \in Sx^*$. Therefore $x^* = y$, for all $y \in Sx^*$ or $Sx^* = \{x^*\}$. Hence, x^* is a common fixed point for S and T.

Example 2.2 Let $X = [0, \infty)$, $\mu(t) = \frac{5t}{8}$ and

$$d_q(x, y) = x + 2y, \ (x, y) \in X \times X.$$

Then $(X, \preceq, d_q)$ be an ordered left(right) K sequentially complete dislocated quasi metric space. Let $\mathcal{R}$ be the binary relation on X defined by

$$\begin{aligned} \mathcal{R} &= \{(x, x) : x \in X\} \cup \left\{(x, \frac{x}{3}) : x \in \{1, \frac{1}{9}, \frac{1}{81}, \frac{1}{729}, ...\}\right\} \\ &\quad \cup \left\{(\frac{x}{3}, x) : x \in \{\frac{1}{3}, \frac{1}{27}, \frac{1}{243}, ...\}\right\}. \end{aligned}$$

Consider the partial order on X defined by

$$(x,y) \in X \times X,\ x \preceq y \text{ if and only } (x,y) \in R.$$

Define the pair of mapping $T, S : X \to P(X)$ by

$$Tx = \left[\frac{x}{3}, \frac{x}{2}\right],\ Sx = \begin{cases} \left\{\frac{x}{3}\right\} : x \in [0,1] \\ \{x+3\} : x \geq 1 \end{cases}.$$

Let

$$\begin{aligned} A &= \{x : x \preceq Sx\} = \left\{0, 1, \frac{1}{9}, \frac{1}{81}, \frac{1}{729}, ...\right\}, \\ B &= \{y : y \succeq Sy\} = \left\{0, \frac{1}{3}, \frac{1}{27}, \frac{1}{243}, ...\right\}. \end{aligned}$$

Let $x_0 = 1$ and $r = 7$, then

$$B_{dq}(x_0, r) = \left\{y : d_q(1,y) < 7 \wedge d_q(y,1) < 7\right\} = [0,3).$$

Then

$$\begin{aligned} G(S) &= \left\{x : x \preceq Sx \text{ and } x \in B_{dq}(x_0, r)\right\} \\ &= \left\{0, 1, \frac{1}{9}, \frac{1}{81}, \frac{1}{729}, ...\right\}. \end{aligned}$$

Now, as $\frac{1}{9^{n-1}} \in B_{dq}(x_0, r)$,

$$d_q(\frac{1}{9^{n-1}}, T\frac{1}{9^{n-1}}) = d_q(\frac{1}{9^{n-1}}, \frac{1}{3 \times 9^{n-1}})$$

and

$$d_q(T\frac{1}{9^{n-1}}, \frac{1}{9^{n-1}}) = d_q(\frac{1}{3 \times 9^{n-1}}, \frac{1}{9^{n-1}}).$$

Also, $(\frac{1}{9^{n-1}}, \frac{1}{3\times 9^{n-1}}) \in \mathcal{R}$, so $\frac{1}{9^{n-1}} \preceq S\frac{1}{9^{n-1}}$. As $(\frac{1}{9\times 9^{n-1}}, \frac{1}{3\times 9^{n-1}}) \in \mathcal{R}$, so $\frac{1}{3\times 9^{n-1}} \succeq S\frac{1}{3\times 9^{n-1}}$. Hence, condition (ii)(a) is satisfied. Now, as $\frac{1}{3\times 9^{n-1}} \in B_{dq}(x_0, r)$,

$$d_q(\frac{1}{3 \times 9^{n-1}}, T\frac{1}{3 \times 9^{n-1}}) = d_q(\frac{1}{3 \times 9^{n-1}}, \frac{1}{9 \times 9^{n-1}})$$

and

$$d_q(T\frac{1}{3 \times 9^{n-1}}, \frac{1}{3 \times 9^{n-1}}) = d_q(\frac{1}{9 \times 9^{n-1}}, \frac{1}{3 \times 9^{n-1}}).$$

Also, $\frac{1}{3\times 9^{n-1}} \succeq S\frac{1}{3\times 9^{n-1}}$, implies $\frac{1}{9\times 9^{n-1}} \preceq S\frac{1}{9\times 9^{n-1}}$. Hence, condition (ii)(b) is satisfied. Also, condition (ii)(a) and (ii)(b) are trivially satisfied for $0 \in B_{dq}(x_0, r)$. Now

$$B_{dq}(x_0, r) \cap \{XT(x_n)\} = \left\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \frac{1}{729}, \frac{1}{2187}, ...\right\}.$$

Now for $x, y \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$ with $x \succeq Sx$ and $y \preceq Sy$, then $x \in B$ and $y \in A$. In general for some $n, m \in \mathbb{N}$

$$x = \frac{1}{3 \times 9^{m-1}},\ y = \frac{1}{9^{n-1}}.$$

Case i: Let $n \leq m$, we have

$$H(Ty,Tx) = H\left(\left[\frac{1}{3\times 9^{n-1}}, \frac{1}{2\times 9^{n-1}}\right], \left[\frac{1}{9\times 9^{m-1}}, \frac{1}{6\times 9^{m-1}}\right]\right)$$

$$H(Ty,Tx) = \max\left\{\frac{1}{2\times 9^{n-1}} + \frac{2}{9\times 9^{m-1}}, \frac{1}{3\times 9^{n-1}} + \frac{1}{3\times 9^{m-1}}\right\} \tag{2.31}$$

$$= \max\left\{\frac{9\times 9^{m-n}+4}{18\times 9^{m-1}}, \frac{9^{m-n}+1}{3\times 9^{m-1}}\right\} = \frac{9\times 9^{m-n}+4}{18\times 9^{m-1}}.$$

Now,

$$H(Tx,Ty) = H\left(\left[\frac{1}{9\times 9^{m-1}}, \frac{1}{6\times 9^{m-1}}\right], \left[\frac{1}{3\times 9^{n-1}}, \frac{1}{2\times 9^{n-1}}\right]\right)$$

$$H(Tx,Ty) = \max\left\{\frac{1}{6\times 9^{m-1}} + \frac{2}{3\times 9^{n-1}}, \frac{1}{9\times 9^{m-1}} + \frac{1}{9^{n-1}}\right\} \tag{2.32}$$

$$= \max\left\{\frac{3+12\times 9^{m-n}}{18\times 9^{m-1}}, \frac{1+9\times 9^{m-n}}{9\times 9^{m-1}}\right\} = \frac{2+18\times 9^{m-n}}{18\times 9^{m-1}}.$$

Also

$$\begin{aligned}\max\left\{H_q(Ty,Tx), H_q(Tx,Ty)\right\} &= \max\left\{\frac{9\times 9^{m-n}+4}{18\times 9^{m-1}}, \frac{2+18\times 9^{m-n}}{18\times 9^{m-1}}\right\} \\ &= \frac{2+18\times 9^{m-n}}{18\times 9^{m-1}}.\end{aligned}$$

Now, we have

$$D_q(x,y) = \max\left\{d_q\left(\frac{1}{3\times 9^{m-1}}, \frac{1}{9^{n-1}}\right), d_q\left(\frac{1}{3\times 9^{m-1}}, T\frac{1}{3\times 9^{m-1}}\right), d_q\left(\frac{1}{9^{n-1}}, T\frac{1}{9^{n-1}}\right)\right\}$$

$$D_q(x,y) = \max\left\{\frac{1+6\times 9^{m-n}}{3\times 9^{m-1}}, \frac{1}{3\times 9^{m-1}} + \frac{2}{9\times 9^{m-1}}, \frac{1}{9^{n-1}} + \frac{2}{3\times 9^{n-1}}\right\} \tag{2.33}$$

$$= \max\left\{\frac{1+6\times 9^{m-n}}{3\times 9^{m-1}}, \frac{3+2}{9\times 9^{m-1}}, \frac{3+2}{3\times 9^{n-1}}\right\} = \frac{1+6\times 9^{m-n}}{3\times 9^{m-1}}.$$

As

$$\begin{aligned}\frac{24}{30}\left(\frac{1+6\times 9^{m-n}}{3\times 9^{m-1}}\right) &< \frac{1+6\times 9^{m-n}}{3\times 9^{m-1}} \\ \text{or} \quad \frac{3+18\times 9^{m-n}}{18\times 9^{m-1}} &< \frac{5}{8}\left(\frac{1+6\times 9^{m-n}}{3\times 9^{m-1}}\right) \\ \text{or} \quad \frac{2+18\times 9^{m-n}}{18\times 9^{m-1}} &< \frac{5}{8}\left(\frac{1+6\times 9^{m-n}}{3\times 9^{m-1}}\right) \\ \text{or } \max\left\{H_q(Tx,Ty), H_q(Ty,Tx)\right\} &< \mu\left(D_q(x,y)\right).\end{aligned}$$

Case ii: Let $n > m$, then by using (2.31), we have

$$\begin{aligned}H_q(Ty,Tx) &= \max\left\{\frac{9+4\times 9^{n-m}}{18\times 9^{n-1}}, \frac{1+9^{n-m}}{3\times 9^{n-1}}\right\} \\ &= \frac{6+6\times 9^{n-m}}{18\times 9^{n-1}}.\end{aligned}$$

Similarly, by using (2.32), we have

$$\begin{aligned} H(Tx,Ty) &= \max\left\{\frac{3\times 9^{n-m}+12}{18\times 9^{n-1}}, \frac{9^{n-m}+9}{9\times 9^{n-1}}\right\} \\ &= \frac{3\times 9^{n-m}+12}{18\times 9^{n-1}}. \end{aligned}$$

Now,

$$\begin{aligned} \max\left\{H_q(Ty,Tx), H_q(Tx,Ty)\right\} &= \max\left\{\frac{6+6\times 9^{n-m}}{18\times 9^{n-1}}, \frac{3\times 9^{n-m}+12}{18\times 9^{n-1}}\right\} \\ &= \frac{6+6\times 9^{n-m}}{18\times 9^{n-1}}. \end{aligned}$$

Now, by (2.33), we have

$$\begin{aligned} D_q(x,y) &= \max\left\{\frac{1+6\times 9^{n-m}}{3\times 9^{n-1}}, \frac{3+2}{9\times 9^{m-1}}, \frac{3+2}{3\times 9^{n-1}}\right\} \\ &= \frac{1+6\times 9^{n-m}}{3\times 9^{n-1}}. \end{aligned}$$

As

$$\begin{aligned} \frac{40+40\times 9^{n-m}}{18\times 9^{n-1}} &< \frac{30+180\times 9^{n-m}}{18\times 9^{n-1}} \\ \frac{6+6\times 9^{n-m}}{18\times 9^{n-1}} &< \frac{5}{8}\left(\frac{1+6\times 9^{n-m}}{3\times 9^{n-1}}\right) \\ \max\left\{H_q(Tx,Ty), H_q(Ty,Tx)\right\} &< \mu\left(D_q(x,y)\right). \end{aligned}$$

Case iii: Let

$$x=0,\ y=\frac{1}{9^{n-1}}.$$

We have

$$\begin{aligned} H(Tx,Ty) &= \max\left\{[0,0], \left[\frac{1}{3\times 9^{n-1}}, \frac{1}{2\times 9^{n-1}}\right]\right\} \\ &= \max\left\{\frac{2}{3\times 9^{n-1}}, \frac{1}{9^{n-1}}\right\} = \frac{1}{9^{n-1}}. \end{aligned}$$

Also,

$$H(Ty,Tx) = \max\left\{\left[\frac{1}{3\times 9^{n-1}}, \frac{1}{2\times 9^{n-1}}\right], [0,0]\right\} = \frac{1}{2\times 9^{n-1}}.$$

Now,

$$\max\left\{H_q(Tx,Ty), H_q(Ty,Tx)\right\} = \frac{1}{9^{n-1}}.$$

Also,

$$\begin{aligned} D_q(x,y) &= \max\left\{d_q\left(0, \frac{1}{9^{n-1}}\right), d_q(0,T0), d_q\left(\frac{1}{9^{n-1}}, T\frac{1}{9^{n-1}}\right)\right\} \\ &= \max\left\{\frac{2}{9^{n-1}}, \frac{3+2}{3\times 9^{n-1}}\right\} = \frac{2}{9^{n-1}}. \end{aligned}$$

Clearly

$$\max\left\{H_q\left(Tx,Ty\right),H_q\left(Ty,Tx\right)\right\}<\mu\left(D_q(x,y)\right).$$

Case iv: Let

$$x=\frac{1}{3\times 9^{n-1}},\ y=0.$$

We have

$$H(Tx,Ty)=\max\left\{\frac{1}{6\times 9^{n-1}},\frac{1}{9\times 9^{n-1}}\right\}=\frac{1}{6\times 9^{n-1}}.$$

Also,

$$H(Ty,Tx)=\max\left\{\frac{2}{9\times 9^{n-1}},\frac{1}{3\times 9^{n-1}}\right\}=\frac{1}{3\times 9^{n-1}}.$$

Now,

$$\max\left\{H_q\left(Tx,Ty\right),H_q\left(Ty,Tx\right)\right\}=\frac{1}{3\times 9^{n-1}}.$$

Also,

$$D_q(x,y)=\frac{5}{3\times 9^{n-1}}.$$

Clearly

$$\max\left\{H_q\left(Tx,Ty\right),H_q\left(Ty,Tx\right)\right\}\leq\mu\left(D_q(x,y)\right).$$

Case v: The contraction trivially holds for $x=0$ and $y=0$. Also

$$\begin{aligned}&\sum_{i=0}^{n}\max\{\mu^i(d_q(x_1,x_0)),\mu^i(d_q(x_0,x_1))\}\\=&\sum_{i=0}^{n}\max\{\mu^i(\frac{7}{3}),\mu^i(\frac{5}{3})\}\\=&\frac{7}{3}+\frac{35}{24}+\frac{175}{192}+........\\=&\frac{56}{9}<7=r\end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied. Hence S and T have a common fixed point 0 in $B_{d_q}(x_0,r)$.

By taking $D_q(x,y)=d_q(x,y)$, we obtain the following result.

Corollary 2.3 Let $(X,\preceq,d_q)$ be an ordered left (right) K-sequentially complete dislocated quasi metric space, $S,T:X\to P(X)$ be the multivalued mappings. Suppose that the following assertions hold:

(i) There exists a function $\mu\in\Psi$, $x_0\in X$ and $r>0$ such that for every $(x,y)\in X\times X$, we have

$$\max\{H_q(Tx,Ty),H_q(Ty,Tx)\}\leq\mu(d_q(x,y)),$$

for all $x,y\in B_{d_q}(x_0,r)\cap\{XT(x_n)\}$ with $x\succeq Sx$, $y\preceq Sy$.

(ii) If $x \in B_{d_q}(x_0, r)$, $d_q(x, Tx) = d_q(x, y)$ and $d_q(Tx, x) = d_q(y, x)$, then

$$(a)\ x \preceq Sx, \text{ implies } y \succeq Sy\ (b)\ x \succeq Sx, \text{ implies } y \preceq Sy.$$

(iii) The set $G(S) = \left\{x : x \preceq Sx \text{ and } x \in B_{d_q}(x_0, r)\right\}$ is closed and contains x_0.
(iv)

$$\sum_{i=0}^{n} \max\{\mu^i(d_q(x_1, x_0), \mu^i(d_q(x_0, x_1))\} < r \text{ for all } n \in \mathbb{N}.$$

Then the subsequence $\{x_{2n}\}$ of $\{XT(x_n)\}$ is a sequence in $G(S)$ and $\{x_{2n}\} \to x^* \in G(S)$ and $d_q(x^*, x^*) = 0$. Also, if the inequality (i) holds for x^*, then S and T have a common fixed point x^* in $B_{d_q}(x_0, r)$.

By taking complete metric space instead of left (right) K-sequentially complete dislocated quasi metric space, we obtain the following result.
Corollary 2.4 Let $(X, \preceq, d_q)$ be an ordered complete metric space, $S, T : X \to P(X)$ be the multivalued mappings. Suppose that the following assertions hold:
(i) There exists a function $\mu \in \Psi$, $x_0 \in X$ and $r > 0$ such that for every $(x, y) \in X \times X$, we have

$$H_q(Tx, Ty) \leq \mu(\max\{d_q(x, y), d_q(x, Tx), d_q(y, Ty)\}),$$

for all $x, y \in B_{d_q}(x_0, r) \cap \{XT(x_n)\}$ with $x \succeq Sx$, $y \preceq Sy$.
(ii) If $x \in B_{d_q}(x_0, r)$, $d_q(x, Tx) = d_q(x, y)$ and $d_q(Tx, x) = d_q(y, x)$, then

$$(a)\ x \preceq Sx, \text{ implies } y \succeq Sy\ \ (b)\ x \succeq Sx, \text{ implies } y \preceq Sy.$$

(iii) The set $G(S) = \left\{x : x \preceq Sx \text{ and } x \in B_{d_q}(x_0, r)\right\}$ is closed and contains x_0.
(iv)

$$\sum_{i=0}^{n} \mu^i(d_q(x_0, x_1)) < r \text{ for all } n \in \mathbb{N}.$$

Then the subsequence $\{x_{2n}\}$ of $\{XT(x_n)\}$ is a sequence in $G(S)$ and $\{x_{2n}\} \to x^* \in G(S)$ and $d_q(x^*, x^*) = 0$. Also, if the inequality (i) holds for x^*, then S and T have a common fixed point x^* in $B_{d_q}(x_0, r)$.

By excluding open ball, we obtain the following result.
Corollary 2.5 Let $(X, \preceq, d_q)$ be an ordered left (right) K-sequentially complete dislocated quasi metric space, $S, T : X \to P(X)$ be the multivalued mappings. Suppose that the following assertions hold:
(i) There exists a function $\mu \in \Psi$, $x_0 \in X$ such that for every $(x, y) \in X \times X$, we have

$$\max\{H_q(Tx, Ty), H_q(Ty, Tx)\} \leq \mu(D_q(x, y)),$$

for all $x, y \in \{XT(x_n)\}$ with $x \succeq Sx$, $y \preceq Sy$, where

$$D_q(x, y) = \max\{d_q(x, y), d_q(x, Tx), d_q(y, Ty)\}.$$

(ii) If $d_q(x, Tx) = d_q(x, y)$ and $d_q(Tx, x) = d_q(y, x)$, then

$$(a)\ x \preceq Sx, \text{ implies } y \succeq Sy\ \ (b)\ x \succeq Sx, \text{ implies } y \preceq Sy.$$

(iii) The set $G(S) = \{x : x \preceq Sx\}$ is closed and contains x_0.
Then the subsequence $\{x_{2n}\}$ of $\{XT(x_n)\}$ is a sequence in $G(S)$ and $\{x_{2n}\} \to x^* \in G(S)$ and $d_q(x^*, x^*) = 0$. Also, if the inequality (i) holds for x^*, then S and T have a common fixed point x^* in X.

By taking self mappings, we obtain the following result.

Corollary 2.6 Let $(X, \preceq, d_q)$ be an ordered left (right) K-sequentially complete dislocated quasi metric space, $S, T : X \to X$ be the self mappings. Suppose that the following assertions hold:

(i) There exists a function $\mu \in \Psi$, $x_0 \in X$, $r > 0$ and $x_n = Tx_{n-1}$ such that for every $(x, y) \in X \times X$, we have

$$\max\{d_q(Tx, Ty), d_q(Ty, Tx)\} \leq \mu(D_q(x, y)),$$

for all $x, y \in B_{d_q}(x_0, r) \cap \{x_n\}$ with $x \succeq Sx$, $y \preceq Sy$, where

$$D_q(x, y) = \max\{d_q(x, y), d_q(x, Tx), d_q(y, Ty)\}.$$

(ii) If $x \in B_{d_q}(x_0, r)$, then

$$(a)\ x \preceq Sx, \text{ implies } Tx \succeq STx \ \ (b)\ x \succeq Sx, \text{ implies } Tx \preceq STx.$$

(iii) The set $G(S) = \left\{x : x \preceq Sx \text{ and } x \in B_{d_q}(x_0, r)\right\}$ is closed and contains x_0.

(iv)

$$\sum_{i=0}^{n} \max\{\mu^i(d_q(x_1, x_0)), \mu^i(d_q(x_0, x_1))\} < r \text{ for all } n \in \mathbb{N}.$$

Then the subsequence $\{x_{2n}\}$ of $\{x_n\}$ is a sequence in $G(S)$ and $\{x_{2n}\} \to x^* \in G(S)$ and $d_q(x^*, x^*) = 0$. Also, if the inequality (i) holds for x^*, then S and T have a common fixed point x^* in $B_{d_q}(x_0, r)$.

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