Filomat 34:2 (2020), 631–638
https://doi.org/10.2298/FIL2002631O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Generalization of the Suborbital Graphs Generating Fibonacci Numbers for the Subgroup Γ^3

Seda Öztürk[a]

[a]*Karadeniz Technical University, Faculty of Sciences, Department of Mathematics, Trabzon-Turkey*

Abstract. The Modular group Γ is the most well-known discrete group with many applications. This work investigates some subgraphs of the subgroup Γ^3, defined by

$$\left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma : ab + cd \equiv 0 \pmod 3\right\}$$

In [1], the subgraph $F_{1,1}$ of the subgroup $\Gamma^3 \subset \Gamma$ is studied, and Fibonacci numbers are obtained by means of the subgraph of $F_{1,1}$. In this paper, we give a generalization of the subgraphs generating Fibonacci numbers for the subgroup Γ^3 and some subgraphs having special conditions.

1. Introduction and Preliminaries

The Modular group Γ is a subgroup of the automorphism group of the upper half plane, and defined as

$$\Gamma = PSL(2,\mathbb{Z}) = \{K : z \to \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{Z}, ad-bc = 1\}$$

The elements of the Modular group can also be taken as matrices $\mp\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,b,c,d \in \mathbb{Z}$ and $ad-bc = 1$, since the Modular group is isomorphic to $SL(2,\mathbb{Z})/\{\pm I\}$. It is generated by matrices $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ with defining relationships $U^2 = V^3 = I$, where I is the identity matrix. The Modular group acts on the extended rational numbers $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ with the action defined by

$$\mp\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \frac{x}{y} = \frac{ax+by}{cx+dy}$$

where $a,b,c,d \in \mathbb{Z}$, and $ad-bc = 1$ [2]. Some more information about the modular group Γ can be found in [3, 4, 5, 6].

2010 *Mathematics Subject Classification*. 05CXX, 11B39, 11F06, 20H05, 22E40
Keywords. Modular group, Suborbital graph, Fibonacci numbers, The subgroup Γ^3
Received: 19 March 2019; Revised: 11 May 2019; Accepted: 16 July 2019
Communicated by Yilmaz Simsek
Email address: seda.ozturk@ktu.edu.tr (Seda Öztürk)

The subgroup Γ^3 is defined as a subgroup of Γ generated by third power of all elements of Γ in [4, 5]. In ([6],p.33), it is shown that

$$\Gamma^3 = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma : ab + cd \equiv 0 \pmod 3 \right\}$$

In [7], it follows from the definition that the elements of Γ^3 are one of the forms $\left(\begin{smallmatrix} 3a & b \\ c & 3d \end{smallmatrix}\right), \left(\begin{smallmatrix} a & 3b \\ 3c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where, in third matrix, $a, b, c, d \not\equiv 0(\mathrm{mod}3)$. Hence, the subgroup Γ^3 acts transitively on the set of $\hat{\mathbb{Q}}$ and the stabilizer of ∞ is the group $\left\{ \mp\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) : m \in \mathbb{Z} \right\}$. We take the subgroup $\Gamma_0^3(n) = \{g \in \Gamma^3 : c \equiv 0 \pmod n\}$ and the stabilizer Γ_∞^3, and now we can set up Γ^3- invariant equivalence relation. Since the group Γ^3 is transitive, any reduced fraction $\dfrac{r}{s} \in \hat{\mathbb{Q}}$ equals $g(\infty)$ for some $g \in \Gamma^3$. The diagonal action, given by $g(\alpha, \beta) = (g\alpha, g\beta)$, of the group Γ^3 on $\hat{\mathbb{Q}} \times \hat{\mathbb{Q}}$ defines the suborbitals, which are actually orbits. The orbit $O^3(\alpha, \beta)$ containing (α, β) gives the suborbital graph $G^3(\alpha, \beta)$ defined as follows:

As in [2], the set of vertices is $\hat{\mathbb{Q}}$, and there is an edge $\gamma \to \delta$ in $G^3(\alpha, \beta)$ if and only if $(\gamma, \delta) \in O^3(\alpha, \beta)$. Due to the transitivity, every suborbital contains a pair $(\infty, \dfrac{u}{n})$ for some $\frac{u}{n} \in \hat{\mathbb{Q}}, (u, n) = 1, n > 0$. The congruence subgroup $\Gamma_0^3(n)$ defines the following equivalence relation on $\hat{\mathbb{Q}}$ by $g_1(\infty) \simeq g_2(\infty)$ for $g_1, g_2 \in \Gamma^3$ if and only if $g_1\Gamma_0^3(n) = g_2\Gamma_0^3(n)$. If $g_1(\infty) = \dfrac{r}{s}$ and $g_2(\infty) = \dfrac{x}{y}$, we have $\frac{r}{s} \simeq \frac{x}{y} \Leftrightarrow ry - sx \equiv 0(\mathrm{mod} n)$

We will denote the suborbital graph by $G_{u,n}^3$ for short. By virtue of the permuting the blocks transitively all subgraphs corresponding to the blocks are isomorphic. Hence, we will only consider the subgraph $F_{u,n}^3$ of $G_{u,n}^3$ whose vertice set is just the equivalence class or the block

$$[\infty] = \left\{ \frac{x}{y} \in \hat{\mathbb{Q}} : y \equiv 0 \pmod n \right\}$$

In [7], the author studied the connectivity properties of all subgraphs of the subgroup Γ^3 except of the subgraph $F_{1,1}$ and in [1], the authors showed that the subgraph $F_{1,1}$ is disconnected and for all natural numbers m, the natural numbers b that make the numbers $(9m^2-4)b^2+4$ square are $0, 1, 3m, 9m^2-1, 3m(9m^2-1) - 3m, \cdots, a, b, 3mb - a, \cdots$.

In this work, we will investigate some number theoretical problems and give a generalization of the subgraphs generating Fibonacci numbers for the subgroup Γ^3 and subgraphs with some special conditions by means of some special matrices.

2. Main Results

Theorem 1. *[7]* $F_{u,n}^3 = F_{u',n'}^3$ *if and only if* $n = n'$ *and* $u \equiv u'(\mathrm{mod}3n)$

Theorem 2. *[7] There is an edge* $\dfrac{r}{s} \to \dfrac{x}{y}$ *in* $F_{n,1}^3$ *if and only if either*

(i) if $r \equiv 0(\mathrm{mod}3)$, *then* $y \equiv \mp ns(\mathrm{mod}3)$ *and* $ry - sx = \mp 1$, *or*

(ii) if $s \equiv 0(\mathrm{mod}3)$, *then* $x \equiv \mp nr(\mathrm{mod}3)$ *and* $ry - sx = \mp 1$, *or*

(iii) if $r, s \not\equiv 0(\mathrm{mod}3)$, *then* $x \not\equiv \mp nr(\mathrm{mod}3), y \not\equiv \mp ns(\mathrm{mod}3)$ *and*
$ry - sx = \mp 1$

Theorem 3. *Let* $n \in \mathbb{N}$. *Then,* $A = \begin{pmatrix} -n & n^2 + 3n + 1 \\ -1 & n + 3 \end{pmatrix}$ *is in* Γ^3.

Proof. We have $-n.(n^2 + 3n + 1) - 1.(n + 3) \equiv -(n^3 + 2n) \bmod (3)$ for every $n \in \mathbb{N}$.To prove that the matrix A is in Γ^3, we must show that, for any n,

$$n^3 + 2n \equiv 0 \bmod (3) \tag{1}$$

If $n \equiv 0 \bmod (3)$, then (1) is true. Otherwise, we have that $n \equiv 1 \bmod (3)$ or $n \equiv -1 \bmod (3)$, and so $n^2 \equiv 1 \bmod (3)$. Therefore, in any case, $n^3 + 2n = n(n^2 + 2) \equiv 0 \bmod (3)$ for every natural number n. This gives the proof.

We give following straightforward corollary without proof.

Corollary 4. *If $m \in \mathbb{N}, m \not\equiv 0 \bmod (3)$, then there exists some natural number k such that $3k - 2 = m^2$. That is, the number $3k - 2 = m^2$ is a perfect square number. Some of the values of the number k are $1, 2, 6, 9, 17, 22, \cdots$.*

Theorem 5. *Let $n \in \mathbb{N}$ and $A = \begin{pmatrix} -n & n^2 + 3n + 1 \\ -1 & n + 3 \end{pmatrix}$. Then, for all $m \in \mathbb{N}$*

(i) $A^m(\frac{1}{0}) \to A^m(\frac{n}{1})$ in $F^3_{n,1}$,

(ii) $A^m(\frac{1}{0}) \to A^{m+1}(\frac{1}{0})$ in $F^3_{n,1}$.

Proof. (i) We will use mathematical induction principle. For $m = 1, A^1(\frac{1}{0}) = A(\frac{1}{0}) = \frac{n}{1} \to \frac{3n+1}{3} = A(\frac{n}{1}) = A^1(\frac{n}{1})$ is true. Let it be true for $m \in \mathbb{N}$. Hence, we must show that the hypothesis is true for $m + 1 \in \mathbb{N}$. From the assumption, we get that $A(A^m(\frac{1}{0})) = A^{m+1}(\frac{1}{0}) \to A(A^m(\frac{n}{1})) = A^{m+1}(\frac{n}{1})$.

(ii)Using (i), we have $A^m(\frac{1}{0}) \to A^m(\frac{n}{1}) = A^m(A(\frac{1}{0})) = A^{m+1}(\frac{1}{0})$.

Corollary 6. *The sequence $\{A^m\}_{m\in\mathbb{N}}$ is strictly monotone increasing and the path*

$$A(\frac{1}{0}) \to A^2(\frac{1}{0}) \to A^3(\frac{1}{0}) \to \cdots$$

is an infinite path.

Proof. For all $z \in \mathbb{R} \setminus \{n+3\}$, $A(z) = \dfrac{-nz + (n^2 + 3n + 1)}{-z + n + 3}$ and $A'(z) > 0$. This shows that A is strictly monotone increasing. Also, the path is an infinite. Because, if for some positive integers m and k such that $m > k$, $A^k(\frac{1}{0}) = A^m(\frac{1}{0})$, then put $m = k + l$ gives $A^l(\frac{1}{0}) = \frac{1}{0}$. In this case, the element A^l has three fixed points as $\dfrac{(2n+3) \mp \sqrt{5}}{2}$ and $\frac{1}{0}$, which gives A^l to be the identity. This gives a contradiction, since A is hyperbolic.

Theorem 7. *Let $A = \begin{pmatrix} -n & n^2 + 3n + 1 \\ -1 & n + 3 \end{pmatrix}$, which is in Γ^3 and $a, b \in \mathbb{N}$ such that $\frac{n}{1} \le \frac{a}{b} < \dfrac{(2n+3) - \sqrt{5}}{2}$. Then,*

(i) $\frac{a}{b} < A(\frac{a}{b}) < \dfrac{(2n+3) - \sqrt{5}}{2}$,

(ii) $\frac{a}{b} \to A(\frac{a}{b})$ is an edge in $F^3_{n,1}$ if and only if $a = \dfrac{(2n+3)b - \sqrt{5b^2+4}}{2}$ and there exists some $k \in \mathbb{N}$ such that $5b^2 + 4 = k^2$.

Proof.(i) Since $\frac{a}{b} < \dfrac{(2n+3) - \sqrt{5}}{2}$, we get that

$$2a < 2nb + 3b - \sqrt{5}b, 5b^2 < (2n+3)^2 b^2 + 4a^2 - 4ab(2n+3)$$

Then, $-a^2+abn+3ab<-anb+b^2(n^2+3n+1)$ and $\frac{a}{b}<\frac{-na+(n^2+3n+1)b}{-a+b(n+3)}=A(\frac{a}{b})$. Further, we show that $A(\frac{a}{b})<\frac{(2n+3)-\sqrt{5}}{2}$. From the above, $a^2-2abn-3ab+b^2(n^2+3n+1)>0$ and so $\sqrt{5}<\frac{a-(n+3)b}{3a-(3n+7)b}$ and

$$\sqrt{5}-(2n+3)<-2\frac{-an+b(n^2+3n+1)}{-a+(n+3)b}, \frac{(2n+3)-\sqrt{5}}{2}>A(\frac{a}{b})$$

Consequently, we have $\frac{a}{b}<A(\frac{a}{b})<\frac{(2n+3)-\sqrt{5}}{2}$.

(ii) Let $\frac{a}{b}\to A(\frac{a}{b})$be an edge in $F^3_{n,1}$. By (i) and by Theorem 2, we have

$$a^2-(2n+3)ab+(n^2+3n+1)b^2>0$$

and $a^2-(2n+3)ab+(n^2+3n+1)b^2=1$. Hence, we multiply the equation by 4 and add $5b^2$, we get $4a^2-4ab(2n+3)+4b^2(n^2+3n+1)+5b^2=4+5b^2$. Also, since $(2n+3)b-2a>0$, we have $a=\frac{(2n+3)b-\sqrt{5b^2+4}}{2}$. Furthermore, since $4+5b^2$ is a natural number, there exists some $k\in\mathbb{N}$ such that $5b^2+4=k^2$.

Conversely, let $a=\frac{(2n+3)b-\sqrt{5b^2+4}}{2}$. Then, after some calculations it is easily seen that

$$\frac{-b^2(n^2+3n+2)(2n+3)+nb(n+3)\sqrt{5b^2+4}}{2}\equiv 0 \bmod (3)$$

Therefore, the matrix

$$B=\begin{pmatrix}\frac{-(2n+3)b+\sqrt{5b^2+4}}{2} & (n^2+3n+1)b\\ -b & \frac{(2n+3)b+\sqrt{5b^2+4}}{2}\end{pmatrix}$$

is in Γ^3. Also, $B(\frac{1}{0})=\frac{a}{b}$, $B(\frac{n}{1})=\frac{a}{b}$. Hence by Theorem 5, we get that $\frac{a}{b}\to A(\frac{a}{b})$ is an edge in $F^3_{n,1}$.

Corollary 8. *Let $k,n\in\mathbb{N}$. Then,*

(i) The path

$$\frac{1}{0}\to n+\frac{0}{1}\to n+\frac{1}{3}\to\cdots\to n+\frac{a_k}{b_k}\to n+\frac{b_k}{3b_k-a_k}\to\cdots$$

is an infinite path under the matrix A.

(ii) All vertices in (i) are less than $\frac{(2n-3)+\sqrt{5}}{2}$.

(iii) For the numbers a_k,b_k in (i), $a_k=\frac{3b_k-\sqrt{5b_k^2+4}}{2}$ *and the numbers $5a_k^2+4, 5b_k^2+4$ are perfect squares.*

Proof. (i) From the Theorem 5, we get that $A^m(\frac{1}{0})=\frac{x_m}{y_m}\to A^{m+1}(\frac{1}{0})=n+\frac{y_m}{3y_m-(x_m-ny_m)}\to A^{m+2}(\frac{1}{0})=n+\frac{3y_m-(x_m-ny_m)}{3(3y_m-(x_m-ny_m))-y_m}$ for $m\in\mathbb{N}$, and so this gives the proof.

(ii) It is clear from (i) of the Theorem 7.

(iii) From (ii) of the Theorem 7, if $\frac{a_k}{b_k} \to A(\frac{a_k}{b_k})$, then $a_k = \frac{(2n+3)b_k - \sqrt{5b_k^2+4}}{2}$. Hence, we have

$$\frac{a_k}{b_k} = \frac{\frac{(2n+3)b_k - \sqrt{5b_k^2+4}}{2}}{b_k} \to n + \frac{b_k}{3b_k - \left(\frac{(2n+3)b_k - \sqrt{5b_k^2+4}}{2} - nb_k\right)} = n + \frac{b_k}{3b_k - \left(\frac{3b_k - \sqrt{5b_k^2+4}}{2}\right)}.$$

Therefore, by (i) of this Corollary, we get that $a_k = \frac{3b_k - \sqrt{5b_k^2+4}}{2}$.

Let $S = \begin{pmatrix} n+3 & -(n^2+3n+1) \\ 1 & -n \end{pmatrix}$ be the inverse matrix of the above matrix A.

Theorem 9. *Let $a, b \in \mathbb{N}$ such that $\frac{2n+1}{1} \le \frac{a}{b} < \frac{(2n+3)+\sqrt{5}}{2}$. Then,*

(i) $\frac{a}{b} < S(\frac{a}{b}) < \frac{(2n+3)+\sqrt{5}}{2}$,

(ii) $\frac{a}{b} \to S(\frac{a}{b})$ is an edge in $F^3_{n+3,1}$ if and only if $a = \frac{(2n+3)b + \sqrt{5b^2-4}}{2}$ and there exists $l \in \mathbb{N}$ such that $5b^2 - 4 = l^2$.

Proof. (i) Since $\frac{a}{b} < \frac{(2n+3)+\sqrt{5}}{2}$, $2a - b(2n+3) < \sqrt{5}b$. From this, $4a^2 - 4ab(2n+3) + (2n+3)b^2 < 5b^2$, and so $a^2 - (2n+3)ab + (n^2+3n+1)b^2 < 0$. Therefore,

$$\frac{a}{b} < \frac{(n+3)a - b(n^2+3n+1)}{a - nb} = S(\frac{a}{b}) \tag{2}$$

Also, since S is increasing on $\left[\frac{(2n+1)}{1}, \frac{(2n+3)+\sqrt{5}}{2}\right) \cap \mathbb{Q}$ and $S\left(\frac{(2n+3)+\sqrt{5}}{2}\right) = \frac{(2n+3)+\sqrt{5}}{2}$, we obtain that

$$S(\frac{a}{b}) < \frac{(2n+3)+\sqrt{5}}{2} \tag{3}$$

By (2) and (3), we get that $\frac{a}{b} < S(\frac{a}{b}) < \frac{(2n+3)+\sqrt{5}}{2}$.

(ii) Let $\frac{a}{b} \to S(\frac{a}{b})$ be an edge in $F^3_{n+3,1}$. So, $a^2 - 2nab - 3ab + n^2b^2 + 3nb^2 + b^2 < 0$ and from Theorem 5, $a^2 - (2n+3)ab + (n^2+3n+1)b^2 = -1$. Then, $(2a - (2n+3)b)^2 = -4 + 5b^2$ and taking square root, we get $|2a - (2n+3)b| = \sqrt{-4+5b^2}$. Since $2a - (2n+3)b > 0$, this shows that $a = \frac{(2n+3)b + \sqrt{5b^2-4}}{2}$. For $2a - (2n+3)b \in \mathbb{N}$, there exists some $w \in \mathbb{N}$ such that $5b^2 - 4 = w^2$. Conversely, let $a = \frac{(2n+3)b + \sqrt{5b^2-4}}{2}$ and let w be in $\mathbb{N}$ such that $5b^2 - 4 = w^2$. Then, $\frac{a}{b} = \frac{\frac{(2n+3)b + \sqrt{5b^2-4}}{2}}{b}$,

$$S(\frac{a}{b}) = \frac{(n+3)\frac{(2n+3)b + \sqrt{5b^2-4}}{2} - b(n^2+3n+1)}{\frac{3b + \sqrt{5b^2-4}}{2}}$$

From Theorem 5, we have that $\frac{a}{b} \to S(\frac{a}{b})$ is an edge in $F^3_{n+3,1}$.

Corollary 10. *Let $k \in \mathbb{N}$. Then,*

(i) The path $(2n+3) - \frac{1}{1} \to (2n+3) - \frac{1}{2} \to (2n+3) - \frac{2}{5} \to \cdots \to (2n+3) - \frac{a_k}{b_k} \to (2n+3) - \frac{b_k}{(3b_k - a_k)} \to \cdots$ is an infinite path under the matrix S.

(ii) The vertices in (i)are less than $\frac{(2n+3)+\sqrt{5}}{2}$.

(iii) For the numbers a_k, b_k in (i),$a_k = \frac{3b_k - \sqrt{5b_k^2 - 4}}{2}$ and the numbers $5a_k^2 - 4, 5b_k^2 - 4$ are perfect squares.

From the Corollaries 8-(i) and 10-(i), we get that the following two corollaries.

Corollary 11. *The numbers $b_k \in \mathbb{Z}^+$ making $5b_k^2 + 4$ perfect square are $0, 1, 3, 8, \cdots, x, y, 3y - x, \cdots$*

Corollary 12. *The numbers $b_k \in \mathbb{Z}^+$ making $5b_k^2 - 4$ perfect square are $1, 2, 5, \cdots, x, y, 3y - x, \cdots$*

Corollary 13. *Let the sequences $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}}$ be $(0, 1, 3, 8, \cdots, x, y, 3y - x, \cdots)$ and $(1, 2, 5, \cdots, r, s, 3s - r, \cdots)$, respectively. Then, the sequence $\{c_k\}_{k\in\mathbb{N}}$, defined by $(0, 1, 1, 2, 3, 5, 8, \cdots, a_k, b_k, a_{k+1}, b_{k+1}, \cdots)$ is the Fibonacci sequence.*

Proof. We must show that $a_k + b_k = a_{k+1}$ and $b_k + a_{k+1} = b_{k+1}$ for all $k \in \mathbb{N}$. By the mathematical induction principle, for $k = 1$, $a_1 + b_1 = a_2$ and $b_1 + a_2 = b_2$ are true. Let it true be for $k \in \mathbb{N}$. Let us see that $a_{k+1} + b_{k+1} = a_{k+2}$ and $b_{k+1} + a_{k+2} = b_{k+2}$. Since, $a_{k+1} = 3a_k - a_{k-1}$ and $b_{k+1} = 3b_k - b_{k-1}$, we get that $a_{k+1} + b_{k+1} = 3(a_k + b_k) - (a_{k-1} + b_{k-1}) = 3a_{k+1} - a_k = a_{k+2}$ and $b_{k+1} + a_{k+2} = 3(b_k + a_{k+1}) - (b_{k-1} + a_k) = 3b_{k+1} - b_k = b_{k+2}$.

Theorem 14. *[7] There is an edge $\frac{r}{s} \to \frac{x}{y}$ in $F^3_{1,n}$ if and only if either*

(i) if $r \equiv 0(\mathrm{mod}3)$, then $x \equiv \mp r(\mathrm{mod}3)$, $y \equiv \mp s(\mathrm{mod}3n)$ and $ry - sx = \mp n$, or

(ii) if $s \equiv 0(\mathrm{mod}3)$, then $x \equiv \mp r(\mathrm{mod}3n)$, $y \equiv \mp s(\mathrm{mod}n)$ and $ry - sx = \mp n$, or

(iii) if $r, s \not\equiv 0(\mathrm{mod}3)$, then $x \equiv \mp r(\mathrm{mod}n)$, $y \equiv \mp s(\mathrm{mod}n)$, $x \not\equiv \mp r(\mathrm{mod}3n)$, $y \not\equiv \mp s(\mathrm{mod}3n)$ and $ry - sx = \mp n$

Now, we consider a new matrix $K = \begin{pmatrix} -1 & 1 \\ -n & n-1 \end{pmatrix}$ for $n \in \mathbb{N}$, $n \geq 4$. It is easily proved that the matrix K is in Γ^3 if and only if $n \equiv 2 \bmod (3)$.

Theorem 15. *Let $n \in \mathbb{N}, n \geq 4$, and $K = \begin{pmatrix} -1 & 1 \\ -n & n-1 \end{pmatrix}$ be in Γ^3. Then,*

(i) $\forall m \in \mathbb{N}$, $K^m(\frac{1}{0}) \to K^m(\frac{1}{n})$ in $F^3_{1,n}$.

(ii) $\forall m \in \mathbb{N}$, $K^m(\frac{1}{0}) \to K^{m+1}(\frac{1}{0})$ in $F^3_{1,n}$.

(iii) The sequence $\{K^m\}_{m\in\mathbb{N}}$ is increasing and the path

$$K(\frac{1}{0}) \to K^2(\frac{1}{0}) \to K^3(\frac{1}{0}) \to \cdots$$

is infinite path.

(iv) The fixed points of K are $z_{1,2} = \frac{n \mp \sqrt{(n-4)n}}{2n}$.

From (ii) of Theorem 15, we obtain the following result
$K^m(\frac{1}{0}) := \frac{a_m}{nb_m}$, then $\frac{1}{n} + \frac{a_m - b_m}{nb_m} \to \frac{1}{n} + \frac{b_m}{n((n-2)b_m - (a_m - b_m))}$.

Theorem 16. *Let $n \in \mathbb{N}$, $n \geq 4$,and $K = \begin{pmatrix} -1 & 1 \\ -n & n-1 \end{pmatrix} \in \Gamma^3$ and $a, b \in \mathbb{N}$ such that $\frac{1}{n} \leq \frac{a}{nb} < \frac{n - \sqrt{(n-4)n}}{2n}$. Then,*

(i) $\frac{a}{nb} < K(\frac{a}{nb}) < \frac{n - \sqrt{(n-4)n}}{2n}$,

(ii) $\frac{a}{nb} \to K(\frac{a}{nb})$ is an edge in $F^3_{1,n}$ if and only if $a = \frac{nb - \sqrt{n(n-4)b^2 + 4}}{2}$ and there exists some $t \in \mathbb{N}$ such that $n(n-4)b^2 + 4 = t^2$.

Proof. (i) Since $\frac{a}{nb} < \frac{n - \sqrt{(n-4)n}}{2n}$, $a^2 - nab + nb^2 > 0$. From this, we have $na^2 - n^2ab + n^2b^2 > 0$, and so

$$\frac{a}{nb} < \frac{-a + nb}{-an + (n-1)b} = K(\frac{a}{nb}) \tag{4}$$

On the other hand, for the mapping K is increasing on $\left[\frac{1}{n}, \frac{n - \sqrt{n(n-4)}}{2n}\right) \cap \mathbb{Q}$ and $K\left(\frac{n - \sqrt{n(n-4)}}{2n}\right) = \frac{n - \sqrt{n(n-4)}}{2n}$, we get that

$$K(\frac{a}{nb}) < \frac{n - \sqrt{(n-4)n}}{2n} \tag{5}$$

From (4) and (5), we have $\frac{a}{nb} < K(\frac{a}{nb}) < \frac{n - \sqrt{(n-4)n}}{2n}$.

(ii) Let $\frac{a}{nb} \to K(\frac{a}{nb})$ be an edge in $F^3_{1,n}$. So, $a^2 - nab + nb^2 > 0$ and from Theorem 7, $a^2 - nab + nb^2 = 1$. Then, $(2a - nb)^2 = 4 + n(n-4)b^2$ and taking square root, we have $|2a - nb| = \sqrt{4 + n(n-4)b^2}$. Since $2a - nb < 0$, this shows that $a = \frac{nb - \sqrt{n(n-4)b^2 + 4}}{2}$. Also, since $nb - 2a \in \mathbb{N}$, there exists some $t \in \mathbb{N}$ such that $n(n-4)b^2 + 4 = t^2$.

Conversely, $a = \frac{nb - \sqrt{n(n-4)b^2 + 4}}{2}$ and there exists some $t \in \mathbb{N}$ such that $n(n-4)b^2 + 4 = t^2$. Then, $\frac{a}{nb} = \frac{\frac{nb - \sqrt{n(n-4)b^2 + 4}}{2}}{nb}$, $K(\frac{a}{nb}) = \frac{\frac{nb - \sqrt{n(n-4)b^2 + 4}}{2}}{n(\frac{(n-2)b - \sqrt{n(n-4)b^2 + 4}}{2})}$. From Theorem 7, we get that $\frac{a}{nb} \to K(\frac{a}{nb})$ is an edge in $F^3_{1,n}$.

Now, we give the following two corollaries without a proof.

Corollary 17. *Let $k, n \in \mathbb{N}; n \geq 4$. Then,*

(i)The path $\frac{1}{0} \to \frac{1}{n} \to \frac{1}{n} + \frac{1}{n(n-2)} \to \cdots \to \frac{1}{n} + \frac{a_k}{nb_k} \to \frac{1}{n} + \frac{b_k}{n((n-2)b_k - a_k)} \to \cdots$ is an infinite path under the matrix K.

(ii) All above vertices are less than $\dfrac{n-\sqrt{(n-4)n}}{2n}$.

(iii) For the numbers a_k, b_k *in (i),* $a_k = \dfrac{(n-2)b_k - \sqrt{n(n-4)b_k^2+4}}{2}$ *and the numbers* $n(n-4)a_k^2+4, n(n-4)b_k^2+4$ *are perfect squares.*

Corollary 18. *The integers* $b \in \mathbb{Z}^+ \cup \{0\}$ *in the equality* $n(n-4)b^2+4=t^2$ *are*

$$0, 1, (n-2), \cdots, x, y, (n-2)y-x, \cdots$$

Proof. It is easily seen from (i) of the Corollary 17.

Note. By the Corollary 18, we get that the number $(9m^2-4)b^2 4 = t^2$ in [1] for $n = 3m+2$.

Acknowledgements

The author thanks the referees for their valuable supports.

References

[1] M. Akbaş ,T. Kör and Y. Kesicioğlu, *Disconnectedness of the subgraph* F^3 *for the group* Γ^3, Journal of Inequalities and Applications, 2013.
[2] G. A. Jones, D. Singerman, and K. Wicks, *The Modular Group and Generalized Farey Graphs*, London Mathematical Society Lecture Note Series, Cambridge University Press, 160, 1991.
[3] G. A. Jones and D. Singerman, Complex Functions, An algebraic and geometric viewpoint, Cambridge university press, 1987.
[4] M. Newman, *The Structure of some subgroups of the modular group*, Illinois J. Math., 6,(1962) 480-487.
[5] M. Newman, *Classification of normal subgroups of the modular group*, Transactions of the American Math Society, Vol.126, 2,(1967) 267-277.
[6] R.A. Rankin, *Modular Forms and Functions*,Cambridge University Press, Cambridge, 1978.
[7] Y. Kesicioğlu, Suborbital graphs of Hecke groups Γ^3 and G_5, PhD thesis, Karadeniz Technical University, Trabzon, 2011.