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(A, φ)- Lacunary Statistical Convergence of Order α

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Abstract. In the present paper, we introduce and study (A, φ) -statistical convergence of order α, using the φ-function, infinite matrix and we establish some inclusion theorems.

1. Introduction and Background

By a φ-function we understood a continuous non-decreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0) = 0, \varphi(u) > 0$, for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$, (see, [15], [19]).

Let $A = (a_{nk})$ be an infinite matrix such that

(i) is nonnegative i.e. $a_{nv} \geq 0$ for $n, v = 1, 2, ...$

(ii) for an arbitrary positive integer n (or v) there exists a positive integers v_0 (or n_0) such that $a_{nv_0} \neq 0$ (or $a_{n_0v} \neq 0$, respectively,

(iii) there exist $\lim\limits_{n\to\infty} a_{nv} = 0$ for $v = 1, 2, ...,$

(iv) $sup_n \sum_n^\infty a_{nv} = K < \infty.$

(v) $sup_n a_{nv} \to \infty$ as $v \to \infty.$

Following Ruckle [13] and Maddox [8], we recall that a modulus f is a function $f : [0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0,$

(ii) $f(x + y) \leq f(x) + f(x)$ for all $x, y \geq 0,$

(iii) f increasing,

(iv) f is continuous from the right at zero.

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It follows that f must be continuous everywhere on $[0, \infty)$.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman at al. [5] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L|) = 0, \text{ for some } L \right\}.$$

In [3], Et and Sengül studied the Cesàro-type summability spaces of order α, $0 < \alpha \leq 1$ and also lacunary statistical convergence of order α where the notion of lacunary statistical convergence was introduced by replacing h_r by h_r^α in the denominator in the definition of lacunary statistical convergence. The idea of lacunary strong (A, φ) with respect to a modulus function was introduced and studied by Waszak [19]. Strongly almost lacunary statistical A-convergence defined by a Musielak-Orlicz function was studied by Savaş and Borgahain [16] and also lacunary statistical and sliding window convergence for measurable functions was present by Connor and Savaş [2].

In the present paper, we introduce and study (A, φ) -statistical convergence of order α, using the φ-function, infinite matrix and we establish some inclusion theorems.

2. Main Results

Let φ and f be given φ-function and modulus function, respectively and $p = (p_k)$ be a sequence of positive real numbers. Moreover, let $\mathbf{A}$ be the infinite matrix, a lacunary sequence $\theta = (k_r)$ and $0 < \alpha \leq 1$ be given. Then we define the following sequence spaces,

$$N_\theta^\alpha(\mathbf{A}, \varphi, f, p)_0 = \left\{ x = (x_k) : \lim_r \frac{1}{h_r^\alpha} \sum_{n \in I_r} f\Big(\left| \sum_{k=1}^{\infty} a_{nk} \varphi(|x_k|) \right| \Big)^{p_k} = 0 \right\},$$

where h_r^α denote the α th power $(h_r)^\alpha$ of h_r, that is $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, h_3^\alpha, ...)$.

The idea of statistical convergence was given by Zygmund [18] in the first editon of his monograph published in Warsaw in 1935. The notion of statistical convergence was introduced by Fast [4] and Schoenberg [17] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, and number theory. Moreover, statistical convergence is closely related to the concept of convergence in probability. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [6], Connor [1], Kolk [12], Šalát [14], Mursaleen [11], Savas [10], Maddox [9] and many others.

The idea of convergence of a real sequence was extended to statistical convergence by Fast [4] as follows : If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m, n)$ denotes the cardinality of the set $K \cap [m, n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n \to \infty} \sup \frac{K(1, n)}{n} \text{ and } \underline{d}(K) = \lim_{n \to \infty} \inf \frac{K(1, n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n \to \infty} \dfrac{K(1, n)}{n}$.

A sequence (x_k) of real numbers is said to be statistically convergent to L if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero.

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [7] as follows.

A sequence (x_k) of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L) if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [7] the relation between lacunary statistical convergence and statistical convergence was established among other things. Recently Et and Hacer [3] defined lacunary statistical convergence of order α as follows:

Let θ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The sequence (x_k) of real numbers is said to be lacunary statistically convergent of order α to L (or, S_θ-convergent of order α to L) if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_\theta^\alpha - \lim x_k = L$.

Assume that A is a non-negative regular summability matrix. Then the sequence $x = (x_n)$ is called A-statistically convergent to L provided that, for every $\varepsilon > 0$

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0$$

We denote this by $st_A - \lim_n x_n = L$.

Let θ be a lacunary sequence, and let $A = (a_{nk})$ be the infinite matrix and the sequence $x = (x_k)$, the φ-function $\varphi(u)$ and a positive number $\varepsilon > 0$ be given. We write, for all i

$$K_\theta^r((A,\varphi),\varepsilon) = \left\{ n \in I_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \geq \varepsilon \right\}.$$

The sequence x is said to be (A,φ)- lacunary statistically convergent of order α to a number zero if for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r^\alpha} \mu(K_\theta^r((A,\varphi),\varepsilon) = 0,$$

where $\mu(K_\theta^r((A,\varphi),\varepsilon)$ denotes the number of elements belonging to $K_\theta^r((A,\varphi),\varepsilon)$. We denote by $S_\theta^\alpha(A,\varphi)$, the set of sequences $x = (x_k)$ which are lacunary $(A,\varphi)-$ statistical convergent of order α to zero and we write

$$S_\theta^\alpha(A,\varphi)_0 = \left\{ x = (x_k) : \lim_r \frac{1}{h_r^\alpha} \mu(K_\theta^r((A,\varphi),\varepsilon) = 0 \right\}.$$

If we take $A = I$ and $\varphi(x) = x$ respectively, then $S_\theta^\alpha(A,\varphi)_0$ reduce to $(S_\theta^\alpha)_0$ which was defined as follows:

$$S_\lambda^0 = \left\{ x = (x_k) : \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k| \geq \varepsilon\}| = 0 \right\}.$$

Remark 2.1. *(i) If for all i,*

$$a_{nk} := \left\{ \begin{array}{lll} \frac{1}{n}, & if & n \geq k, \\ 0, & & otherwise. \end{array} \right.$$

then $S_\theta^\alpha(A,\varphi)_0$ reduce to $S_\theta^\alpha(C,\varphi)_0$,i.e., uniform $(C,\varphi)-$ statistical convergence. (ii) If for all i,

$$a_{nk} := \left\{ \begin{array}{lll} \frac{p_k}{P_n}, & if & n \geq k, \\ 0, & & otherwise. \end{array} \right.$$

then $S_\theta^\alpha(A,\varphi)_0$ *reduce to* $S_\theta^\alpha(N,p),\varphi)$*,i.e., uniform* $((N,p),\varphi)-$ *statistical convergence, where* $p=p_k$ *is a sequence of nonnegative numbers such that* $p_0>0$ *and*

$$P_i=\sum_{k=0}^{n}p_k\to\infty(n\to\infty).$$

Theorem 2.2. *If* $0<\alpha\leq\beta\leq 1$ *then* $S_\theta^\alpha(A,\varphi)_0\subset S_\theta^\beta(A,\varphi)_0$.

Proof. Let $0<\alpha\leq\beta\leq 1$. Then

$$\frac{1}{h_r^\beta}\mu(K_\theta^r((A,\varphi),\varepsilon)\leq\frac{1}{h_r^\alpha}\mu(K_\theta^r((A,\varphi),\varepsilon)$$

for every $\epsilon>0$ and finally we have that $S_\theta^\alpha(A,\varphi)_0\subset S_\theta^\beta(A,\varphi)_0$. This proves the theorem. □

Theorem 2.3. *Let* $\theta=(k_r)$ *and* $\theta'=(s_r)$ *be two lacunary sequences such that* $I_r\subset J_r$ *for all* $r\in\mathbb{N}$ *and let* α *and* β *be such that* $0<\alpha\leq\beta\leq 1$
(i) if

$$\liminf_{r\to\infty}\frac{h_r^\alpha}{l_r^\beta}>0 \tag{1}$$

then $S_{\theta_1}^\beta(A,\varphi)_0\subseteq S_\theta^\alpha(A,\varphi)_0$.
(ii) If

$$\lim_{r\to\infty}\frac{l_r}{h_r^\beta}>0 \tag{2}$$

then $S_\theta^\alpha(A,\varphi)_0\subseteq S_{\theta_1}^\beta(A,\varphi)_0$.

Proof. (i) Suppose that $I_r\subset J_r$ for all $r\in\mathbb{N}$ and let (1) be satisfied. For given $\varepsilon>0$, we have

$$\left\{k\in J_r:\sum_{k=1}^{\infty}a_{nk}\varphi(|x_k|)\geq\varepsilon\right\}\supseteq\left\{k\in I_r:\sum_{k=1}^{\infty}a_{nk}\varphi(|x_k|)\geq\varepsilon\right\}$$

and also

$$\frac{1}{\ell_r^\beta}\left|\left\{k\in J_r:\sum_{k=1}^{\infty}a_{nk}\varphi(|x_k|)\geq\varepsilon\right\}\right|\geq\frac{h_r^\alpha}{\ell_r^\beta}\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\sum_{k=1}^{\infty}a_{nk}\varphi(|x_k|)\geq\varepsilon\right\}\right|$$

for all $r\in\mathbb{N}$, where $I_r=(k_{r-1},k_r]$, $J_r=(s_{r-1},s_r]$, $h_r=k_r-k_{r-1}$ and $\ell_r=s_r-s_{r-1}$. Now taking the limit as $r\to\infty$ in the last inequality and using (1), we get $S_{\theta_1}^\beta(A,\varphi)_0\subseteq S_\theta^\alpha(A,\varphi)_0$.

(ii) Let $x = (x_k) \in S_\theta^\alpha$ and (2) be satisfied. Since $I_r \subset J_r$, for $\varepsilon > 0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^\beta}\left|\left\{k \in J_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \geq \varepsilon\right\}\right| &= \frac{1}{\ell_r^\beta}\left|\left\{s_{r-1} < k \leq k_{r-1} : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \geq \varepsilon\right\}\right| \\
&\quad + \frac{1}{\ell_r^\beta}\left|\left\{k_r < k \leq s_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \geq \varepsilon\right\}\right| \\
&\quad + \frac{1}{\ell_r^\beta}\left|\left\{k_{r-1} < k \leq k_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \geq \varepsilon\right\}\right| \\
&\leq \frac{k_{r-1} - s_{r-1}}{\ell_r^\beta} + \frac{s_r - k_r}{\ell_r^\beta} + \frac{1}{\ell_r^\beta}\left|\left\{k \in I_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \geq \varepsilon\right\}\right| \\
&= \frac{\ell_r - h_r}{\ell_r^\beta} + \frac{1}{\ell_r^\beta}\left|\left\{k \in I_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \geq \varepsilon\right\}\right| \\
&\leq \frac{\ell_r - h_r^\beta}{h_r^\beta} + \frac{1}{h_r^\beta}\left|\left\{k \in I_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \geq \varepsilon\right\}\right| \\
&\leq \left(\frac{\ell_r}{h_r^\beta} - 1\right) + \frac{1}{h_r^\alpha}\left|\left\{k \in I_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|) \geq \varepsilon\right\}\right|
\end{aligned}$$

for all $r \in \mathbb{N}$. Since $\lim_{r\to\infty} \frac{\ell_r}{h_r^\beta} = 1$ by (2) the first term and since $x = (x_k) \in S_\theta^\alpha(A, \varphi)_0$, the second term of right hand side of above inequality tend to 0 as $r \to \infty$. Note that $\left(\frac{\ell_r}{h_r^\beta} - 1\right) \geq 0$. This implies that $S_\theta^\alpha(A, \varphi)_0 \subseteq S_{\theta_1}^\beta(A, \varphi)_0$. $\square$

From the above, we have the following results.

Corollary 2.4. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$.*

If (2.1) holds, then

1. $S_{\theta_1}^\alpha(A, \varphi)_0 \subseteq S_\theta^\alpha(A, \varphi)_0$ for each $\alpha \in (0, 1]$,
2. $S_{\theta_1}(A, \varphi)_0 \subseteq S_\theta^\alpha(A, \varphi)_0$ for each $\alpha \in (0, 1]$,
3. $S_{\theta_1}(A, \varphi)_0 \subseteq S_\theta(A, \varphi)_0$.

If (2.2) holds, then

1. $S_\theta^\alpha(A, \varphi)_0 \subseteq S_{\theta_1}^\alpha(A, \varphi)_0$ for each $\alpha \in (0, 1]$,
2. $S_\theta^\alpha(A, \varphi)_0 \subseteq S_{\theta_1}(A, \varphi)_0$ for each $\alpha \in (0, 1]$,
3. $S_\theta(A, \varphi)_0 \subseteq S_{\theta_1}(A, \varphi)_0$.

Finally we conclude this paper by presenting inclusion relations between $N_\theta^0(A, \varphi, f, p)$ and $S_\theta^0(A, \varphi)$.

In the following theorem we assume that $0 < h = inf p_n \leq p_n \leq sup p_n \leq H < \infty$.

Theorem 2.5. *(a) If the matrix A and $\sup p_n = H$, the sequence θ and functions f and φ be given, then*

$$N_\theta^\alpha((A, \varphi), f, p)_0 \subset S_\theta^\alpha(A, \varphi)_0.$$

(b) If the φ- function $\varphi(u)$ and the matrix A are given, and if the modulus function f is bounded, then

$$S_\theta^\alpha(A, \varphi)_0 \subset N_\theta^\alpha(A, \varphi), f, p)_0.$$

Proof. (a) Let f be a modulus function and let ε be a positive numbers. We write the following inequalities,

$$\begin{aligned}
\frac{1}{h_r^\alpha}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} &= \frac{1}{h_r^\alpha}\sum_{n\in I_r^1} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\Big|\Big)^{p_n} \\
&\geq \frac{1}{h_r^\alpha}\sum_{n\in I_r^1}[f(\varepsilon)]^{p_n} \\
&\geq \frac{1}{h_r^\alpha}\sum_{n\in I_r^1}\min\Big([f(\varepsilon)]^{\inf p_n},[f(\varepsilon)]^H\Big) \\
&\geq \frac{1}{h_r^\alpha}\mu(K_\theta^r(A,\varphi),\varepsilon)\min\Big([f(\varepsilon)]^{\inf p_n},[f(\varepsilon)]^H\Big),
\end{aligned}$$

where

$$I_r^1 = \left\{ n\in I_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)\geq\varepsilon \right\}.$$

Finally, if $x\in N_\theta^0(A,\varphi,f,p)$ then $x\in S_\theta^0(A,\varphi,f)$.

(b) Let us suppose that $x\in S_\theta^\alpha(A,\varphi)_0$. If the modulus function f is a bounded function, then there exists an integer K such that $f(x)<K$ for $x\geq 0$. Let us take

$$I_r^2 = \left\{ n\in I_r : \sum_{k=1}^{\infty} a_{nk}\varphi(|x_k|)<\varepsilon \right\}.$$

Thus we have, for all i

$$\begin{aligned}
\frac{1}{h_r^\alpha}\sum_{n\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)\Big|\Big)^{p_n} &\leq \frac{1}{h_r^\alpha}\sum_{n\in I_r^1} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)\Big|\Big)^{p_n} \\
&+ \frac{1}{h_r^\alpha}\sum_{n\in I_r^2} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)\Big|\Big)^{p_n} \\
&\leq \frac{1}{h_r^\alpha}\sum_{n\in I_r^1} max(K^h,K^H)+\frac{1}{h_r}\sum_{n\in I_r^2}[f(\varepsilon)]^{p_n} \\
&\leq max(K^h,K^H)\frac{1}{h_r^\alpha}\mu(K_\theta^r((A,\varphi),\varepsilon))+max([f(\varepsilon)]^h,[f(\varepsilon)]^H).
\end{aligned}$$

Taking the limit as $\varepsilon\to 0$, we observe that $x\in N_\theta^\alpha(A,\varphi,f,p)_0$.
This completes the proof. □

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