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Influence of θ-Metric Spaces on the Diameter of Rough Weighted $\mathcal{I}_2$-Limit Set

Sanjoy Ghosal[a], M. C. Listán-García[b], Manasi Mandal[c], Mandobi Banerjee[c]

[a]*Department of Mathematics, University of North Bengal, Raja Rammohunpur, Darjeeling-734013, West Bengal, India.*
[b]*Dpto. de Matemáticas, Universidad de Cádiz, Apdo. 40, 11510-Puerto Real (Cádiz), Spain.*
[c]*Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India.*

Abstract. In this paper we continue our investigation of the recent summability notion introduced in [Math. Slovaca 69 (4) (2019) 871-890] (where rough weighted statistical convergence for double sequences is discussed over norm linear spaces) and introduce the notion of rough weighted $\mathcal{I}_2$-convergence over θ-metric spaces. Also we exercise the behavior of weighted $\mathcal{I}_2$-cluster points set over θ-metric spaces. Based on the new notion we vividly discuss some important results and perceive how the existing results are vacillating.

1. Introduction

Before we assert what we have done in this paper it is necessary to understand the history behind this investigation. The idea of convergence of a real sequence had been extended to statistical convergence by Fast [13] (see also Schoenberg [33], Steinhaus [34]) as follows: Let $\mathbb{N}$ denote the set of all natural numbers and $A \subseteq \mathbb{N}$. The upper and lower natural densities of the subset A are defined by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|\{k \in A : k \leq n\}|}{n} \text{ and } \underline{d}(A) = \liminf_{n\to\infty} \frac{|\{k \in A : k \leq n\}|}{n}.$$

If $\overline{d}(A) = \underline{d}(A)$ then we say that the natural density of A exists and it is denoted simply by $d(A)$. Clearly $d(A) = \lim_{n\to\infty} \frac{1}{n}|\{k \in A : k \leq n\}|$.

A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to a real number ℓ if for every $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [14] and Šalát [31]. More initial works on this convergence can be found from [15, 16].

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Email addresses: sanjoykumarghosal@nbu.ac.in, sanjoyghosalju@gmail.com (Sanjoy Ghosal), mariadelcarmen.listan@uca.es (M. C. Listán-García), manasi_ju@yahoo.in (Manasi Mandal), banerjeeju@rediffmail.com, mandobibanerjee@gmail.com (Mandobi Banerjee)

The idea of statistical convergence for double sequences was introduced by Mursaleen and Edely [25] as follows: Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(m,n)$ be the numbers of $(i,j) \in K$ such that $i \leq m$ and $j \leq n$. Then the two-dimensional analogue of natural density can be defined by

$$d_2(K) = \lim_{m,n\to\infty} \frac{K(m,n)}{mn}.$$

Some authors use the notation δ_2 in place of d_2.

A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in a norm linear space $(X, \|.\|)$ is said to be statistically convergent to $\ell \in X$ if for every $\varepsilon > 0$,

$$d_2(\{(i,j) \in \mathbb{N} \times \mathbb{N} : i \leq m, j \leq n, \|x_{ij} - \ell\| \geq \varepsilon\}) = 0$$

and we write $x_{ij} \xrightarrow{st_2} c$. Several works on this convergence are done later which can be seen from [1, 4, 10, 32].

The concept of $\mathcal{I}$-convergence was first introduced as a generalization of statistical converence for single sequence by Kostyrko et al. [21, 22] and during the same period Nuray et al. [26] also developed the same concept independently as generalized statistical convergence. After that Dems [11] introduced $\mathcal{I}_2$-convergence for double sequences as follows: A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}_2$-convergent to a real number ℓ if for every $\varepsilon > 0$, $\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \ell| \geq \varepsilon\} \in \mathcal{I}_2$ and we write $x_{ij} \xrightarrow{\mathcal{I}_2} \ell$. Some other results on this convergence are found in [8, 23].

A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in a norm linear space $(X, \|.\|)$ is said to be $\mathcal{I}_2$-bounded [12] if there exists a positive real number M such that $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|x_{ij}\| \geq M\} \in \mathcal{I}_2$.

An element $c \in X$ is called a $\mathcal{I}_2$-cluster point [9, 12] of a double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ if for every $\varepsilon > 0$, the set $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|x_{ij} - c\| < \varepsilon\} \notin \mathcal{I}_2$. We denote the set of all $\mathcal{I}_2$-cluster points of the double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ by $\mathcal{I}_2(\Gamma_x)$.

On one hand, as recently as in the year 2019, Ghosal et al. [18] developed rough weighted statistical convergence for single sequence [7] to rough weighted statistical convergence of double sequence, defined as: Let $p = \{p_m\}_{m\in\mathbb{N}}$ and $q = \{q_n\}_{n\in\mathbb{N}}$ be sequences of real numbers such that $p_m > \delta, q_n > \delta$ for all $m, n \in \mathbb{N}$ (where δ is a fixed positive real number) and $P_m = \sum_{i=1}^{m} p_i$, $Q_n = \sum_{j=1}^{n} q_j$ for all $m, n \in \mathbb{N}$. A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in a norm linear space $(X, \|.\|)$ is said to be rough weighted statistically convergent to x_* w.r.t the roughness of degree r if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{P_m Q_n} |\{(i,j) \in \mathbb{N} \times \mathbb{N} : i \leq P_m, j \leq Q_n, p_i q_j \|x_{ij} - x_*\| \geq r + \varepsilon\}| = 0.$$

and we write $x_{ij} \xrightarrow[r]{W^2_{pq}st} x_*$. The set $W^2_{pq}st - LIM^r x = \{x_* \in \mathbb{R} : x_{ij} \xrightarrow[r]{W^2_{pq}st} x_*\}$ is called the rough weighted statistical limit set of the double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ with degree of roughness r. The sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ is said to be rough weighted statistically convergent w.r.t the roughness of degree r provided $W^2_{pq}st - LIM^r x \neq \emptyset$. On this topic lot of works are done which are followed from [2, 24, 27, 29, 30].

On the other hand, the idea of θ-metric space is a generalization of metric space, which was first defined by Khojasteh et al. [20] in the year 2013 and most recently in 2017 related to the subject Chanda et al. [5] proved some important results on fixed point theorem related to θ-metric space.

In this article, we combine the approaches of rough weighted statistical convergence for double sequences, rough $\mathcal{I}_2$-limit set and $\mathcal{I}_2$-cluster points and introduce new and more advanced summability methods, namely, rough weighted $\mathcal{I}_2$-limit set and weighted $\mathcal{I}_2$-cluster points set of a sequence in a θ-metric space. Some new examples are constructed to ensure the deviation from basic results such as:

Result 1.1 [18, Theorem 2.2]. For a double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in a normed space X and the weighted sequences $p = \{p_m\}_{m\in\mathbb{N}}$, $q = \{q_n\}_{n\in\mathbb{N}}$ we have

$$0 \leq \operatorname{diam}(W^2_{pq}st - LIM^r x) \leq \begin{cases} \frac{2r}{(\liminf\limits_{m\in A} p_m)(\liminf\limits_{n\in B} q_n)}, & \text{if both the} \\ \text{weighted sequences are statistically bounded,} \\ 0, & \text{if any one of the weighted sequences} \\ \text{are statistically unbounded.} \end{cases}$$

where $W^2_{pq}st - LIM^r x$ denotes the rough weighted statistical limit set, $A = \{k \in \mathbb{N} : q_k < M_1\}$ and $B = \{k \in \mathbb{N} : q_k < M_2\}$ for some positive real numbers M_1, M_2. In general it has no smaller bound than $\frac{2r}{(\liminf\limits_{n\in A} p_m)(\liminf\limits_{n\in B} q_n)}$ if both the weighted sequences are statistically bounded.

Result 1.2 [12, Theorem 2.3]. The diameter of the rough $\mathcal{I}_2$-limit set i.e., $\operatorname{diam}(\mathcal{I}_2 - LIM^r x) \leq 2r$, for any double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$. In general, the diameter has no smaller bound.

Result 1.3 [12, Theorem 2.6]. Suppose $r > 0$. Then a double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ is $x_{ij} \xrightarrow[r]{\mathcal{I}_2} x_*$ if there exists a sequence $y = \{y_{ij}\}_{i,j\in\mathbb{N}}$ such that $y_{ij} \xrightarrow{\mathcal{I}_2} x_*$ and $\|x_{ij} - y_{ij}\| \leq r$ for every $i, j \in \mathbb{N}$.

Result 1.4 [9, Theorem 2 (i)]. Let $\mathcal{I}_2$ be a strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$. The set $\mathcal{I}_2(\Gamma_x)$ for any double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in a metric space is closed.

Result 1.5 [28, Corollary 1]. If a sequence in a normed space is statistically bounded then the set of statistical cluster points is non-empty.

Result 1.6 [12, Lemma 2.7]. For an arbitrary $c \in \mathcal{I}_2(\Gamma_x)$ of a double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$, $\|x_* - c\| \leq r$ for every $x_* \in \mathcal{I}_2 - LIM^r x$.

So our main objective is to analyze the different behaviors of these new convergences and characterize both sets under θ-metric spaces.

2. Preliminaries

In this section we recall some of the basic concepts of ideal, filter, rough $\mathcal{I}_2$-convergence, weighted statistical convergence for double sequence and θ-metric space. Interested readers can look into [5, 6, 17, 19] for details.

Definition 2.1. [22]. Let $Y \neq \emptyset$. A class $\mathcal{I}$ of subsets of Y is said to be an ideal in Y if
(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$.
The ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\emptyset\}$ and $Y \notin \mathcal{I}$.

Definition 2.2. [22]. Let $Y \neq \emptyset$. A non-empty class $\mathcal{F}$ of subsets of Y is said to be a filter in Y if
(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}, A \subset B$ implies $B \in \mathcal{F}$.
If $\mathcal{I}$ is a non-trivial ideal of Y and $Y \neq \emptyset$, then the class $\mathcal{F}(\mathcal{I}) = \{K \subset Y : Y \setminus K \in \mathcal{I}\}$ is a filter on Y, called the filter associated with $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all singleton sets.

Definition 2.3. [8, 22]. A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i\in\mathbb{N}$. It is evident that a strongly admissible ideal is admissible too.

Theorem 2.4. [9]. Let $x=\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a sequence of real numbers then
(i) $\mathcal{I}_2-\limsup x=\alpha$ (finite) iff for any $\varepsilon>0$,

$$\{(i,j)\in\mathbb{N}\times\mathbb{N}: x_{ij}>\alpha-\varepsilon\}\notin\mathcal{I}_2 \text{ and } \{(i,j)\in\mathbb{N}\times\mathbb{N}: x_{ij}>\alpha+\varepsilon\}\in\mathcal{I}_2.$$

(i) $\mathcal{I}_2-\liminf x=\beta$ (finite) iff for any $\varepsilon>0$,

$$\{(i,j)\in\mathbb{N}\times\mathbb{N}: x_{ij}<\beta+\varepsilon\}\notin\mathcal{I}_2 \text{ and } \{(i,j)\in\mathbb{N}\times\mathbb{N}: x_{ij}<\beta-\varepsilon\}\in\mathcal{I}_2.$$

Definition 2.5. [12]. Let r be a non-negative real number. A double sequence $x=\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $(X,\|.\|)$ is said to be rough $\mathcal{I}_2$-convergent to x_* w.r.t. the roughness of degree r if for every $\varepsilon>0$, the set $\{(i,j)\in\mathbb{N}\times\mathbb{N}:\|x_{ij}-x_*\|\geq r+\varepsilon\}\in\mathcal{I}_2$, which is denoted by $x_{ij}\xrightarrow[r]{\mathcal{I}_2}x_*$. If we take $r=0$, then we obtain the ordinary $\mathcal{I}_2$-convergence [11]. The set $\mathcal{I}_2-LIM^r x=\{x_*\in X: x_{ij}\xrightarrow[r]{\mathcal{I}_2}x_*\}$ is called the rough $\mathcal{I}_2$-limit set w.r.t. the roughness of degree r of the double sequence $x=\{x_{ij}\}_{i,j\in\mathbb{N}}$. A double sequence $x=\{x_{ij}\}_{i,j\in\mathbb{N}}$ is said to be rough $\mathcal{I}_2$-convergent if $\mathcal{I}_2-LIM^r x\neq\emptyset$ for some $r>0$.

Definition 2.6. [6]. Let $p=\{p_m\}_{m\in\mathbb{N}}$ and $q=\{q_n\}_{n\in\mathbb{N}}$ be sequences of real numbers such that $\liminf\limits_{m\to\infty}p_m>0$, $\liminf\limits_{n\to\infty}q_n>0$ and $P_m=\sum\limits_{i=1}^{m}p_i,\ Q_n=\sum\limits_{j=1}^{n}q_j$ for every $m,n\in\mathbb{N}$. The double weighted density of $K\subseteq\mathbb{N}\times\mathbb{N}$ is

$$w\delta_2(K)=\lim_{n,m\to\infty}\frac{K(P_m,Q_n)}{P_mQ_n}, \text{ provided the limit exists,}$$

where $K(P_m,Q_n)=\{(i,j)\in K: i\leq P_m, j\leq Q_n\}$. Then the double sequence $x=\{x_{ij}\}_{i,j\in\mathbb{N}}$ in a normed linear space is said to be weighted statistically convergent to $\ell\in X$ if for every $\varepsilon>0$,

$$\lim_{m,n\to\infty}\frac{1}{P_mQ_n}|\{(i,j)\in\mathbb{N}\times\mathbb{N}: i\leq P_m, j\leq Q_n, p_iq_j\|x_{ij}-\ell\|\geq\varepsilon\}|=0.$$

Definition 2.7. [20]. Let $\theta:[0,\infty)\times[0,\infty)\to[0,\infty)$ be a continuous function with respect to each variable. Let $Im(\theta)=\{\theta(s,t):s,t\geq 0\}$. The mapping θ is called a B-action iff the following conditions hold:
(B1) $\theta(0,0)=0$ and $\theta(s,t)=\theta(t,s)$ for every $s,t\geq 0$,
(B2)

$$\theta(s,t)<\theta(u,v) \text{ if } \begin{cases}\text{either } s<u, t\leq v,\\ \text{or } s=u, t<v,\end{cases}$$

(B3) for each $r\in Im(\theta)$ and for each $s\in[0,r]$, there exists $t\in[0,r]$ such that $\theta(s,t)=r$,
(B4) $\theta(s,0)\leq s$ for every $s>0$.

The set of all B-actions is denoted by Υ. In some of the following theorems we will need to require that θ is a continuous B-action, i.e., continuous in both variables simultaneously.

Example 2.8. [20]. The following functions are examples of B-action:
(θ_1) $\theta(s,t)=k(s+t)$, where $k\in(0,1]$,
(θ_2) $\theta(s,t)=k(s+t+ts)$, where $k\in(0,1]$,
(θ_3) $\theta(s,t)=\frac{st}{1+st}$,

(θ_4) $\theta(s,t) = s + t + \sqrt{st}$,
(θ_5) $\theta(s,t) = \sqrt{s^2 + t^2}$.

Definition 2.9. [20]. Let X be a non-empty set. A function $d_\theta : X \times X \to [0,\infty)$ is called a θ-metric on X with respect to B-action $\theta \in \Upsilon$ if d_θ satisfies the following conditions:
(A1) $d_\theta(a,b) = 0$ iff $a = b$,
(A2) $d_\theta(a,b) = d_\theta(b,a)$ for every $a, b \in X$,
(A3) $d_\theta(a,b) \leq \theta(d_\theta(a,c), d_\theta(c,b))$ for every $a, b, c \in X$.

The pair (X, d_θ) is called a θ-metric space.

In [20] it is said that, if (X, d_θ) is a θ-metric space, $\theta(s,t) = k(s+t)$ and $k \in (0,1]$, then (X, d_θ) is a metric space.

However, in the case $k < 1$ this cannot happen unless X is singleton. For example, take $k = \frac{2}{3}$, and a space $X = \{a, b\}$ with $a \neq b$; then

$$0 \leq d_\theta(a,b) \leq \frac{2}{3}\left(d_\theta(a,b) + d_\theta(b,b)\right) = \frac{2}{3} d_\theta(a,b).$$

This implies that $d_\theta(a,b) = 0$, so $a = b$ and this is a contradiction. Also we mention that metric spaces are included in the class of all θ-metric spaces if we consider the θ-metric as $\theta(s,t) = s + t$ for every $s, t \geq 0$. Indeed it is included if $\theta(s,t) \leq s + t$, since

$$d_\theta(a,b) \leq \theta(d_\theta(a,c), d_\theta(c,b)) \leq d_\theta(a,c) + d_\theta(c,b).$$

Example 2.10. [20]. Let $X = \{a, b, c\}$ be a three-point set and $d_\theta : X \times X \to [0,\infty)$ defined by

$$d_\theta(a,b) = 2, d_\theta(a,c) = 6, d_\theta(c,b) = 10, d_\theta(a,a) = d_\theta(b,b) = d_\theta(c,c) = 0,$$

$$d_\theta(a,b) = d_\theta(b,a), d_\theta(a,c) = d_\theta(c,a), d_\theta(b,c) = d_\theta(c,b).$$

For $\theta(s,t) = s + t + st$ for every $s, t \geq 0$, the pair (X, d_θ) is a θ-metric space but not a metric space.

If we consider the following example: Let $X = \{x, y, z\}$ and $d_\theta : X \times X \to [0,\infty)$ defined as:

$$d_\theta(x,y) = 5, d_\theta(y,z) = 12, d_\theta(z,x) = 13, d_\theta(x,y) = d_\theta(y,x),$$

$$d_\theta(y,z) = d_\theta(z,y), d_\theta(z,x) = d_\theta(x,y), d_\theta(x,x) = d_\theta(y,y) = d_\theta(z,z) = 0.$$

For $\theta(s,t) = \sqrt{s^2 + t^2}$ for every $s, t \geq 0$, the pair (X, d_θ) apparently looks like a θ-metric space but not a metric space. But this example cannot be correct: note that $\theta(s,t) = \sqrt{s^2 + t^2} \leq \sqrt{s^2} + \sqrt{t^2} = s + t$ since $s, t \geq 0$.

Next, we can give an example in a quasi-normed space if we consider $(\mathbb{R}^2, \|\cdot\|_{\frac{1}{2}})$, the induced mapping $d_\theta(a,b) = \|a - b\|_{\frac{1}{2}} = |a_1 - b_1| + |a_2 - b_2| + 2\sqrt{|a_1 - b_1||a_2 - b_2|}$, which is well known not to be a metric; and $\theta(s,t) = s + t + 2\sqrt{st}$. The first two properties are trivial to prove, we will see the third. Assuming x, y, z are arbitrary elements of $\mathbb{R}^2$ and using the triangle inequality for real numbers, we have

$$|x_1 - y_1| + |x_2 - y_2| + 2\sqrt{|x_1 - y_1||x_2 - y_2|} \leq |x_1 - z_1| + |z_2 - x_2| + 2\sqrt{|x_1 - z_1||z_2 - x_2|} + |z_1 - y_1| + |z_2 - y_2| +$$

$$+ 2\sqrt{|z_1 - y_1||z_2 - y_2|} + 2\sqrt{|x_1 - z_1||z_2 - y_2| + |z_1 - y_1||x_2 - z_2|} \leq$$

$$\leq |x_1 - z_1| + |z_2 - x_2| + 2\sqrt{|x_1 - z_1||z_2 - x_2|} + |z_1 - y_1| + |z_2 - y_2| + 2\sqrt{|z_1 - y_1||z_2 - y_2|} +$$

$$+ 2\sqrt{(|x_1 - z_1| + |z_2 - x_2| + 2\sqrt{|x_1 - z_1||z_2 - x_2|})(|z_1 - y_1| + |z_2 - y_2| + 2\sqrt{|z_1 - y_1||z_2 - y_2|})}$$

These inequalities prove

$$d_\theta(x, y) \leq \theta(d_\theta(x, z), d_\theta(z, y)).$$

Definition 2.11. [20]. Let (X, d_θ) be a θ-metric space. An open ball $B_{d_\theta}(a, r)$ at a center $a \in X$ with a radius $r \in Im(\theta)$ is defined as follows: $B_{d_\theta}(a, r) = \{y \in X : d_\theta(y, a) < r\}$.

3. Rough weighted $\mathcal{I}_2$-limit set in θ-metric space

In this section, we introduce the concept of rough weighted $\mathcal{I}_2$-convergence and rough weighted $\mathcal{I}_2$-limit set for double sequences in θ-metric space.

Definition 3.1. Let r be a non-negative real number, $p = \{p_m\}_{m\in\mathbb{N}}$ and $q = \{q_n\}_{n\in\mathbb{N}}$ be sequences of real numbers such that $p_m > \delta, q_n > \delta$ for all $m, n \in \mathbb{N}$ (where δ is a fixed positive real number). Then the double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in (X, d_θ) is said to be rough weighted $\mathcal{I}_2$-convergent to $x_* \in X$ w.r.t the roughness of degree r if for every $\varepsilon > 0$,

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : p_i q_j d_\theta(x_{ij}, x_*) \geq r + \varepsilon\} \in \mathcal{I}_2.$$

In this case we write $x_n \xrightarrow[r]{W\mathcal{I}_2} x_*$. The set $W\mathcal{I}_2 - LIM^r x = \{x_* \in X : x_{ij} \xrightarrow[r]{W\mathcal{I}_2} x_*\}$ is called the rough weighted $\mathcal{I}_2$-limit set of the sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ with degree of roughness r. The sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ is said to be r-rough weighted $\mathcal{I}_2$-convergent provided $W\mathcal{I}_2 - LIM^r x \neq \emptyset$.

Throughout the paper we take $\mathcal{I}_2$ as a non-trivial strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A natural question may arise that: Is the set $W\mathcal{I}_2 - LIM^r x$ in (X, d_θ) always finite? To answer this question we demonstrate an example below.

Example 3.2. Let us consider the function $d_\theta : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

$$d_\theta(\alpha, \beta) = d_\theta(\beta, \alpha) = \begin{cases} 0, \text{ for } \alpha = \beta, \\ \alpha + \beta + \alpha\beta \text{ for } \alpha \neq \beta, \end{cases}$$

where $\alpha, \beta \in [0, \infty)$. For $\theta(s, t) = s + t + st$ for every $s, t \geq 0$, the function d_θ forms a θ-metric and hence the pair $([0, \infty), d_\theta)$ is a θ-metric space. Further it can be observed that $d_\theta(1, 2) \nleq d_\theta(1, 0) + d_\theta(0, 2)$.

Now we define a sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in the θ-metric space $[0, \infty)$ and the weighted sequences $p = \{p_i\}_{i\in\mathbb{N}}, q = \{q_j\}_{j\in\mathbb{N}}$ in the following manner.

$$x_{ij} = \begin{cases} ij, \text{ if } i = k^2, j = l^2 \text{ for some } k, l \in \mathbb{N}, \\ 1, \text{ otherwise}, \end{cases}$$

$$p_i = \begin{cases} i, \text{ if } i = k^2 \text{ for some } k \in \mathbb{N}, \\ 1 + \frac{1}{i}, \text{ otherwise} \end{cases}$$

and

$$q_j = \begin{cases} j, \text{ if } j = l^2 \text{ for some } l \in \mathbb{N}, \\ 1 + \frac{1}{j}, \text{ otherwise}. \end{cases}$$

Let $r = 3$ and $x_* \in [0, 1]$, then for any $0 < \varepsilon < 1$,

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : p_i q_j d_\theta(x_{ij}, x_*) \geq r + \varepsilon\} \in \mathcal{I}_{w\delta_2},$$

where $\mathcal{I}_{w\delta_2} = \{A \subset \mathbb{N} \times \mathbb{N} : w\delta_2(A) = 0\}$. This shows that the set $W\mathcal{I}_{w\delta_2} - LIM^r x$ contains infinitely many points.

Dündar [12, Theorem 2.3] has shown that for a double sequence the diameter of the set $\mathcal{I}_2 - LIM^r x$ is $\leq 2r$ (where r is the roughness of the convergence) and in general the diameter has no smaller bound. In case of rough weighted $\mathcal{I}_2$-convergence in θ-metric space the diameter of rough weighted $\mathcal{I}_2$-limit set may be strictly greater than $2r$ which does not follow the generalization of Result 1.1 & 1.2 We show this in our next example.

Example 3.3. Let (X, d_θ) be a θ-metric space as in Example 2.10. Define $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$, $p = \{p_i\}_{i\in\mathbb{N}}$ and $q = \{q_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}$ by,

$$x_{ij} = \begin{cases} b, \text{ if } i \neq k^2, j \neq l^2 \text{ for every } k, l \in \mathbb{N}, \\ c, \text{ otherwise,} \end{cases}$$

$$p_i = \begin{cases} \frac{1}{2} + \frac{1}{i}, \text{ if } i \neq k^2 \text{ for every } k \in \mathbb{N}, \\ i, \text{ otherwise} \end{cases}$$

and

$$q_j = \begin{cases} \frac{1}{2} + \frac{1}{j}, \text{ if } j \neq l^2 \text{ for every } l \in \mathbb{N}, \\ j, \text{ otherwise.} \end{cases}$$

Let $r = \frac{1}{2}$. Then $W\mathcal{I}_{w\delta_2} - LIM^r x = \{a, b\}$ and

$$2r < 2 = \text{diam}(W\mathcal{I}_{w\delta_2} - LIM^r x) < \frac{2r}{(\liminf_{m\in A} p_m)(\liminf_{n\in B} q_n)} = 4,$$

where $A = B = \mathbb{N} \setminus \{1^2, 2^2, 3^2, ...\}$. Hence, the result.

Our next theorem shows the conclusion we get regarding diameter of the set $W\mathcal{I}_2 - LIM^r x$.

Theorem 3.4. For a sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$, we have if θ is a continuous B-action,

$$0 \leq \text{diam}(W\mathcal{I}_2 - LIM^r x) \leq \begin{cases} \theta\left(\frac{r}{\mathcal{I}_2\text{-} \liminf p_i q_j}, \frac{r}{\mathcal{I}_2\text{-} \liminf p_i q_j}\right), \text{ if } \{p_i q_j\}_{i,j\in\mathbb{N}} \text{ is } \mathcal{I}_2\text{-bounded,} \\ 0, \text{ otherwise.} \end{cases}$$

In general it has no smaller bound than $\theta\left(\frac{r}{\mathcal{I}_2\text{-} \liminf p_i q_j}, \frac{r}{\mathcal{I}_2\text{-} \liminf p_i q_j}\right)$ if the sequence $\{p_i q_j\}_{i,j\in\mathbb{N}}$ is $\mathcal{I}_2$-bounded.

Proof. Case 1: Let $\{p_i q_j\}_{i,j\in\mathbb{N}}$ be $\mathcal{I}_2$-bounded and $\lambda \in (\delta^2, \mathcal{I}_2 - \liminf p_i q_j)$. Also let x_*, $y_* \in W\mathcal{I}_2 - LIM^r x$. Choose $B = \{(i,j) \in \mathbb{N} \times \mathbb{N} : p_i q_j \geq \lambda\}$, $C = \{(i,j) \in \mathbb{N} \times \mathbb{N} : p_i q_j d_\theta(x_{ij}, x_*) < r + \varepsilon\}$ and $D = \{(i,j) \in \mathbb{N} \times \mathbb{N} : p_i q_j d_\theta(x_{ij}, y_*) < r + \varepsilon\}$. Then $B, C, D \in \mathcal{F}(\mathcal{I}_2)$ and $\emptyset \notin \mathcal{F}(\mathcal{I}_2)$ then $B \cap C \cap D \in \mathcal{F}(\mathcal{I}_2)$ and $B \cap C \cap D \neq \emptyset$.

Now we choose any $(i_0, j_0) \in B \cap C \cap D$. Then

$$d_\theta(x_*, y_*) \leq \theta(d_\theta(x_{i_0 j_0}, x_*), d_\theta(x_{i_0 j_0}, y_*)) \leq \theta\left(\frac{r+\varepsilon}{\lambda}, \frac{r+\varepsilon}{\lambda}\right),$$

$$\Rightarrow d_\theta(x_*, y_*) \leq \lim_{\varepsilon\to 0+} \theta\left(\frac{r+\varepsilon}{\lambda}, \frac{r+\varepsilon}{\lambda}\right) = \theta\left(\frac{r}{\lambda}, \frac{r}{\lambda}\right),$$

$$\Rightarrow d_\theta(x_*, y_*) \leq \lim_{\lambda \to (\mathcal{I}_2 - \liminf p_i q_j)} \theta\left(\frac{r}{\lambda}, \frac{r}{\lambda}\right) = \theta\left(\frac{r}{\mathcal{I}_2 - \liminf p_i q_j}, \frac{r}{\mathcal{I}_2 - \liminf p_i q_j}\right).$$

Hence the proof of case 1 is completed.

Case 2: If possible let there exist two distinct elements x_*, $y_* \in X$ such that x_*, $y_* \in W\mathcal{I}_2 - LIM^r x$. Then $C = \{(i,j) \in \mathbb{N} \times \mathbb{N} : p_i q_j d_\theta(x_{ij}, x_*) < r + 1\} \in \mathcal{F}(\mathcal{I}_2)$ and $D = \{(i,j) \in \mathbb{N} \times \mathbb{N} : p_i q_j d_\theta(x_{ij}, y_*) < r + 1\} \in \mathcal{F}(\mathcal{I}_2)$.
Since $C, D \in \mathcal{F}(\mathcal{I}_2)$ and $\emptyset \notin \mathcal{F}(\mathcal{I}_2)$ then $A = C \cap D \in \mathcal{F}(\mathcal{I}_2)$ and $A \neq \emptyset$.

Let $\varepsilon > 0$. Since θ is continuous at $(0,0)$, then there exists a positive real number G (however large) such that $\theta(\frac{r+1}{G}, \frac{r+1}{G}) < \varepsilon$. Since the sequence $\{p_i q_j\}_{i,j \in \mathbb{N}}$ is not $\mathcal{I}_2$-bounded, so the set $E = \{(i,j) \in \mathbb{N} \times \mathbb{N} : p_i q_j > G\} \notin \mathcal{I}_2$.
Then obviously the set $A \cap E \neq \emptyset$. Otherwise $E \subseteq (\mathbb{N} \times \mathbb{N}) \setminus A \in \mathcal{I}_2$.

Then any integer $(i_0, j_0) \in A \cap E$ implies

$$d_\theta(x_*, y_*) \leq \theta(d_\theta(x_{i_0 j_0}, x_*), d_\theta(x_{i_0 j_0}, y_*)) < \theta\left(\frac{r+1}{p_{i_0} q_{j_0}}, \frac{r+1}{p_{i_0} q_{j_0}}\right) < \theta\left(\frac{r+1}{G}, \frac{r+1}{G}\right) < \varepsilon.$$

Hence, the proof of case 2 is completed.

Before going to the second part of the theorem, we first show that in general the diameter of $B_{d_\theta}(c, r) = \{y \in X : d_\theta(y, c) < r\}$ has no smaller bound than $\theta(r, r)$.
If possible, let the diameter of $B_{d_\theta}(c, r)$ be $> \theta(r, r)$. For any y_1, $y_2 \in B_{d_\theta}(c, r)$ we get $d_\theta(y_1, c) < r$ and $d_\theta(y_2, c) < r$. This implies

$$d_\theta(y_1, y_2) \leq \theta(d_\theta(c, y_1), d_\theta(c, y_2)) < \theta(r, r),$$

which is a contradiction. Therefore the diameter of $B_{d_\theta}(c, r)$ is always $\leq \theta(r, r)$. Equality may occur. To prove this choose a θ-metric space $(\mathbb{R}, d_\theta)$ which is not a metric space such that $d_\theta : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$d_\theta(\alpha, \beta) = d_\theta(\beta, \alpha) = \begin{cases} 0, \text{ for } \alpha = \beta, \\ |\alpha| + |\beta| + |\alpha\beta| \text{ for } \alpha \neq \beta, \end{cases}$$

where α, $\beta \in \mathbb{R}$ and $\theta(s, t) = s + t + st$ for every s, $t \geq 0$. So the diameter of the set $B_{d_\theta}(0, r) = (-r, r)$ is $2r + r^2 = \theta(r, r)$. So in general, the diameter of $B_{d_\theta}(c, r)$ has no smaller bound than $\theta(r, r)$. Therefore equality holds.

Now we consider a double sequence $x = \{x_{ij}\}_{i,j \in \mathbb{N}}$ in $(\mathbb{R}, d_\theta)$, such that

$$x_{ij} = \begin{cases} 4, \text{ if } i = u^2, j = v^2 \text{ for some } u, v \in \mathbb{N}, \\ 0, \text{ otherwise,} \end{cases}$$

$$p_i = 3 - \frac{1}{i} \text{ and } q_j = 1 - \frac{1}{j} \text{ for every } i, j \in \mathbb{N}.$$

Then p_m, $q_n < 3 = \mathcal{I}_{w\delta_2} - \liminf p_i q_j$ for every $m, n \in \mathbb{N}$. Let r be a fixed positive real number, then for any $\varepsilon > 0$,

$$\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : i \leq P_m, j \leq Q_n, p_i q_j d_\theta(x_{ij}, 0) \geq \frac{3\varepsilon}{3 + r}\right\} \in \mathcal{I}_{w\delta_2}.$$

Let $y \in B_{\frac{r}{3}}(0)$, then

$$p_i q_j d_\theta(x_{ij}, y) \leq p_i q_j d_\theta(d_\theta(x_{ij}, 0), d_\theta(0, y)) = p_i q_j d_\theta(x_{ij}, 0) + p_i q_j d_\theta(0, y) + p_i q_j d_\theta(x_{ij}, 0) d_\theta(0, y)$$

$$= p_iq_jd_\theta(x_{ij},0)(1+d_\theta(0,y)) + p_iq_jd_\theta(0,y) \leq p_iq_jd_\theta(x_{ij},0)(1+\frac{r}{3}) + p_iq_jd_\theta(0,y) < \varepsilon + 3\frac{r}{3} = r+\varepsilon$$

$$\text{for every } (i,j) \in \mathbb{N}\times\mathbb{N} \setminus \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : p_iq_jd_\theta(x_{ij},0) \geq \frac{3\varepsilon}{3+r}\right\}.$$

Therefore we get $y \in W\mathcal{I}_{w\delta_2} - LIM^r x$. So $B_{\frac{r}{3}(0)} \subseteq W\mathcal{I}_{w\delta_2} - LIM^r x$. From the first part of Theorem 3.4, we get $\text{diam}(W\mathcal{I}_{w\delta_2} - LIM^r x) \leq \theta(\frac{r}{3}, \frac{r}{3})$ since $\{p_iq_j\}_{i,j\in\mathbb{N}}$ is $\mathcal{I}_{w\delta_2}$-bounded and $\mathcal{I}_{w\delta_2} - \liminf p_iq_j = 3$. Also $\text{diam}(B_{\frac{r}{3}(0)}) = \theta(\frac{r}{3}, \frac{r}{3})$, so

$$\text{diam}(W\mathcal{I}_{w\delta_2} - LIM^r x) = \theta(\frac{r}{3}, \frac{r}{3}) = \frac{2r}{3} + (\frac{r}{3})^2 > \frac{2r}{(\liminf_{m\in\mathbb{N}} p_m)(\liminf_{n\in\mathbb{N}} q_n)},$$

which contradicts the Result 1.1.

This shows that in general, the upper bound $\theta\left(\frac{r}{\mathcal{I}_2 - \liminf p_iq_j}, \frac{r}{\mathcal{I}_2 - \liminf p_iq_j}\right)$ of the diameter of the set $W\mathcal{I}_2 - LIM^r x$ can't be decreased anymore. □

Therefore we come to a conclusion that Theorem 3.4 is the non-trivial extension of the results obtained by different authors in the past [12, 18], because if we take $\theta(s,t) = s+t$ for every $s, t \geq 0$, $p_i = q_j = 1$ for every $i, j \in \mathbb{N}$ and X is a norm space, then $\mathcal{I}_2 - \liminf p_iq_j = 1$,

$$\theta\left(\frac{r}{\mathcal{I}_2 - \liminf p_iq_j}, \frac{r}{\mathcal{I}_2 - \liminf p_iq_j}\right) = 2r$$

and $W\mathcal{I}_2 - LIM^r x = \mathcal{I}_2 - LIM^r x$. So $\text{diam}(\mathcal{I}_2 - LIM^r x) \leq 2r$. In general it has no smaller bound than $2r$. So Theorem 3.4 reduces to Theorem 2.3 of [12] and Theorem 3.4 reduces to Theorem 2.2 of [18] (here we assume $\theta(s,t) = s+t$ for every $s, t \geq 0$ and $\mathcal{I}_2 = \mathcal{I}_{\delta_2} = \{K \subset \mathbb{N}\times\mathbb{N} : \delta_2(K) = 0\}$ where $\delta_2(K)$ denotes the double natural density of the set $K \subset \mathbb{N}\times\mathbb{N}$).

4. Weighted $\mathcal{I}_2$-cluster points set in θ-metric space

We begin with a definition of weighted $\mathcal{I}_2$-cluster points set for double sequences in θ-metric space.

Definition 4.1. Let $p = \{p_i\}_{i\in\mathbb{N}}$ and $q = \{q_j\}_{j\in\mathbb{N}}$ be sequences of real numbers such that $p_i, q_j > \mu$ for every i, $j \in \mathbb{N}$ (where μ is a fixed positive real number). An element $c \in X$ is called a weighted $\mathcal{I}_2$-cluster point of a double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in (X, d_θ) if for every $\varepsilon > 0$,

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : p_iq_jd_\theta(x_{ij}, c) < \varepsilon\} \notin \mathcal{I}_2.$$

We denote the set of all weighted $\mathcal{I}_2$-cluster points of the sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ by $W\mathcal{I}_2(\Gamma_x)$. Here the sequences $p = \{p_i\}_{i\in\mathbb{N}}$ and $q = \{q_j\}_{j\in\mathbb{N}}$ are called the weighted sequences.

The following example shows that in general, the weighted $\mathcal{I}_2$-cluster points set in a θ-metric space may not be closed. So the next example gives us the required answer of not closedness of $W\mathcal{I}_2(\Gamma_x)$ i.e., the set does not follow the generalization of Result 1.4.

Example 4.2. First of all we find out a suitable θ-metric space which satisfies the required aim but not a metric space.

Let $\theta(s,t) = s + t + 2\sqrt{st}$ for every $s, t \geq 0$, then the mapping θ satisfies all the properties of B-action. We proof only $(B3)$: let $r \in Im(\theta)$ and $s \in [0, r]$ such that $\theta(t,s) = r \Rightarrow s + t + 2\sqrt{st} = r$ then $(\sqrt{t})^2 + (2\sqrt{s})\sqrt{t} - (r - s) = 0 \Rightarrow \sqrt{t} = \sqrt{r} - \sqrt{s}$. So $t \in [0, r]$.

Next we assume that $d_\theta : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ such that $d_\theta(a, b) = (a - b)^2$ for every a, $b \in \mathbb{R}$. For any a, b, $c \in \mathbb{R}$ we get $d_\theta(a, b) = (a - b)^2$, $d_\theta(a, c) = (a - c)^2$, $d_\theta(c, b) = (c - b)^2$. So,

$$d_\theta(a, b) = (a - b)^2 = (a - c + c - b)^2 \leq (a - c)^2 + (c - b)^2 + 2|(a - c)(c - b)| = \theta(d_\theta(a, c), d_\theta(c, b)).$$

For $\theta(s, t) = s + t + 2\sqrt{st}$ for every $s, t \geq 0$, the function d_θ forms a θ-metric and hence the pair $(\mathbb{R}, d_\theta)$ is a θ-metric space. Further it can be observed that $d_\theta(2, 10) = 64 \nleq 9 + 25 = d_\theta(2, 5) + d_\theta(5, 10)$. This shows that it is an unbounded θ-metric space but not a metric space. This shows that a θ-metric space and a metric space are totally different.

Choose $\mathbb{N} = \bigcup_{j=1}^{\infty} \Delta_j$ be a decomposition of $\mathbb{N}$ (i.e., $\Delta_m \cap \Delta_n = \emptyset$ for $m \neq n$). Assume that $\Delta_j = \{2^{j-1}(2s - 1) : s \in \mathbb{N}\}$ for every $j \in \mathbb{N}$.

Next, let $x = \{x_{kl}\}_{k \in \mathbb{N}}$ is a sequence in $(\mathbb{R}, d_\theta)$ such that

$$p_k = k^2, q_l = 1 \text{ for every } k, l \in \mathbb{N} \text{ and } x_{kl} = \frac{1}{j} + \frac{1}{k^2} \text{ if } k \in \Delta_j \text{ where } j \in \mathbb{N}, l \in \mathbb{N}.$$

Let $0 < \varepsilon < \frac{1}{4}$. Then for each $j \in \mathbb{N}$, we get $\{(k, l) \in \mathbb{N} \times \mathbb{N} : p_k q_l d_\theta(x_{kl}, \frac{1}{j}) < \varepsilon\} \notin \mathcal{I}_{w\delta_2}$.
This shows that $\frac{1}{j} \in W\mathcal{I}_{w\delta_2}(\Gamma_x)$ for every $j \in \mathbb{N}$.

Next we assume $k \in \mathbb{N}$. Then there exists an integer $j \in \mathbb{N}$ such that $k \in \Delta_j$ for some $j \in \mathbb{N}$. This implies k is of the form $k = 2^{j-1}(2s - 1)$ for some $s \in \mathbb{N}$.

For each $k, l \in \mathbb{N}$,

$$p_k q_l d_\theta(x_{kl}, 0) = \left\{2^{j-1}(2s - 1)\right\}^2 \left\{\frac{1}{j} + \frac{1}{2^{j-1}(2s - 1)}\right\}^2 > \left\{\frac{1}{2} \cdot \frac{2^j}{j}(2s - 1)\right\}^2 \geq \frac{1}{4}.$$

Then $\{(k, l) \in \mathbb{N} \times \mathbb{N} : p_k q_l d_\theta(x_{kl}, 0) < \varepsilon\} \in \mathcal{I}_{wd_2}$. This implies $0 \notin W\mathcal{I}_{w\delta_2}(\Gamma_x)$. So $W\mathcal{I}_{w\delta_2}(\Gamma_x)$ is not a closed set.

So a natural question arises that under which conditions the weighted $\mathcal{I}_2$-cluster points set in θ-metric space is closed. The following theorem gives the required answer.

Theorem 4.3. The set $W\mathcal{I}_2(\Gamma_x)$ is closed if the sequence $\{p_i q_j\}_{i,j \in \mathbb{N}}$ is $\mathcal{I}_2$-bounded.

Proof. Without loss of any generality we assume that $W\mathcal{I}_2(\Gamma_x) \neq \emptyset$. Let $x_* \in \overline{W\mathcal{I}_2(\Gamma_x)}$. Then there exists a sequence $\{y_n\}_{n \in \mathbb{N}}$ in $W\mathcal{I}_2(\Gamma_x)$ such that $y_n \to x_*$ as $n \to \infty$. We have to show that $x_* \in W\mathcal{I}_2(\Gamma_x)$. Since $\{p_i q_j\}_{i,j \in \mathbb{N}}$ is $\mathcal{I}_2$-bounded so there exists a $M > 0$ such that $A = \{(i, j) \in \mathbb{N} \times \mathbb{N} : p_i q_j < M\} \in \mathcal{F}(\mathcal{I}_2)$.

Take $\varepsilon > 0$. Since $y_n \to x_*$ as $n \to \infty$, so there exists a $k_0 \in \mathbb{N}$ such that if $n > k_0$ then $d_\theta(y_n, x_*) < \varepsilon$. Let $B = \{(i, j) \in \mathbb{N} \times \mathbb{N} : p_i q_j d_\theta(x_{ij}, y_{k_0}) < \varepsilon\} \notin \mathcal{I}_2$. Then $A \cap B \neq \emptyset$, otherwise $B \subseteq (\mathbb{N} \times \mathbb{N}) \setminus A \in \mathcal{I}_2$, which is a contradiction.
Let $(i_0, j_0) \in A \cap B$. Then,

$$d_\theta(x_{i_0 j_0}, x_*) \leq \theta(d_\theta(x_{i_0 j_0}, y_{k_0}), d_\theta(y_{k_0}, x_*)) \leq \theta(d_\theta(x_{i_0 j_0}, y_{k_0}), \varepsilon),$$

$$\Rightarrow d_\theta(x_{i_0 j_0}, x_*) \leq \lim_{\varepsilon \to 0+} \theta(d_\theta(x_{i_0 j_0}, y_{k_0}), \varepsilon) = \theta(d_\theta(x_{i_0 j_0}, y_{k_0}), 0) \leq d_\theta(x_{i_0 j_0}, y_{k_0}).$$

This implies $A \cap B \subseteq \{(i, j) \in \mathbb{N} \times \mathbb{N} : p_i q_j d_\theta(x_{ij}, x_*) < \varepsilon\}$. Now we prove that $A \cap B \notin \mathcal{I}_2$.

Then three cases arise.

Case 1: If $B \subseteq A$, then $A \cap B = B \notin \mathcal{I}_2$.

Case 2: If $A \subset B$, then $A \cap B = A \notin \mathcal{I}_2$.

Case 3: If $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$ then $B \setminus (A \cap B) \subseteq (\mathbb{N} \times \mathbb{N}) \setminus A \in \mathcal{I}_2$.

Also $B = [B \setminus (A \cap B)] \cup (A \cap B)$ and $B \notin \mathcal{I}_2$. This implies $A \cap B \notin \mathcal{I}_2$. This shows that $x_* \in W\mathcal{I}(\Gamma_x)$. Hence the result. □

Note 4.4. From the Result 1.5, Pehlivan et. al. [28], had shown that if a sequence in a finite dimensional normed space is statistically bounded then the statistical cluster points set is non-empty. For the case of weighted statistical convergence the weighted statistical cluster point set may be empty even if the space is finite dimensional and the sequence is statistically bounded.

To prove this important fact, we consider $(\mathbb{R}, d_\theta)$ be a θ-metric space as in Example 4.2 and the sequence of real numbers $x_{ij} = \frac{1}{ij}$ for every $i, j \in \mathbb{N}$ and $p_m = m^2$, $q_n = n^2$ for every $m, n \in \mathbb{N}$. Then the sequence $x = \{x_{ij}\}_{i,j \in \mathbb{N}}$ is $\mathcal{I}_{\delta_2}$-bounded but $W\mathcal{I}_{w\delta_2}(\Gamma_x) = \emptyset$.

In the case of rough weighted $\mathcal{I}_2$-convergence in θ-metric space the Result 1.3 is not true, which is established in the following example.

Example 4.5. Let $(\mathbb{R}, d_\theta)$ be a θ-metric space as in Example 4.2 and $0 < r < 1$. Define

$$y_{ij} = \begin{cases} 3 + \frac{1}{(ij)^2}, & \text{if } i \neq m^2, j \neq n^2 \text{ for every } m, n \in \mathbb{N}, \\ (ij)^2, & \text{otherwise,} \end{cases}$$

$$p_i = i, q_j = j, i, j \in \mathbb{N} \text{ and}$$

$$x_{ij} = \begin{cases} 3 + \frac{r}{2} + \frac{1}{(ij)^2}, & \text{if } i \neq m^2, j \neq n^2 \text{ for every } m, n \in \mathbb{N}, \\ \frac{r}{2} + (ij)^2, & \text{otherwise.} \end{cases}$$

Then $y_{ij} \xrightarrow{W\mathcal{I}_{w\delta_2}} 3$ and $d_\theta(x_{ij}, y_{ij}) < r$ but $\{(i,j) \in \mathbb{N} \times \mathbb{N} : p_i q_j d_\theta(x_{ij}, 3) \geq r + \varepsilon\} \notin \mathcal{I}_{w\delta_2}$. So $3 \notin W\mathcal{I}_{w\delta_2} - LIM^r x$.

5. Comparison between weighted $\mathcal{I}_2$-cluster points set and rough weighted $\mathcal{I}_2$-limit set

Now we intend to study the weighted $\mathcal{I}_2$-cluster points set and rough weighted $\mathcal{I}_2$-limit set and relate them with classical limit points.

In case of rough weighted $\mathcal{I}_2$-limit set and weighted $\mathcal{I}_2$-cluster points set, $d_\theta(x_*, c)$ may be strictly greater than r for some c in weighted $\mathcal{I}_2$-cluster points set and x_* in rough weighted $\mathcal{I}_2$-limit set, which does not follow the generalization of Result 1.6. We show this in our next example.

Examples 5.1. Let (X, d_θ) be a θ-metric space as in Example 2.10 and

$$x_{ij} = \begin{cases} a, & \text{if } i = m^2 \text{ and } j = n^2 \text{ for some } m, n \in \mathbb{N}, \\ b, & \text{otherwise.} \end{cases}$$

Now we define the weighted sequences $\{p_i\}_{i \in \mathbb{N}}$ and $\{q_j\}_{j \in \mathbb{N}}$ such that

$$p_iq_j = \begin{cases} ij, \text{ if } i = m^2 \text{ and } j = n^2 \text{ for some } m, n \in \mathbb{N}, \\ \frac{1}{6} + \frac{1}{ij}, \text{ otherwise.} \end{cases}$$

Then,

$$p_iq_jd_\theta(x_{ij}, a) = \begin{cases} 0, \text{ if } i = m^2 \text{ and } j = n^2 \text{ for some } m, n \in \mathbb{N}, \\ \frac{1}{3} + \frac{2}{ij}, \text{ otherwise,} \end{cases}$$

$$p_iq_jd_\theta(x_{ij}, b) = \begin{cases} 2ij, \text{ if } i = m^2 \text{ and } j = n^2 \text{ for some } m, n \in \mathbb{N}, \\ 0, \text{ otherwise} \end{cases}$$

and

$$p_iq_jd_\theta(x_{ij}, c) = \begin{cases} 6ij, \text{ if } i = m^2 \text{ and } j = n^2 \text{ for some } m, n \in \mathbb{N}, \\ \frac{5}{3} + \frac{10}{ij}, \text{ otherwise .} \end{cases}$$

Then we get $W\mathcal{I}_{w\delta_2}(\Gamma_x) = \{b\}$ and $W\mathcal{I}_{w\delta_2} - LIM^r x = \{a, b\}$, where $r = \frac{1}{2}$. Choose $x_* = a$, $c^* = b$ then $d_\theta(x_*, c^*) = 2 > r$.

An important relationship between the set of weighted $\mathcal{I}_2$-cluster points and the set of rough weighted $\mathcal{I}_2$-limit points of a double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ is explained below.

Theorem 5.2. For an arbitrary $c^* \in W\mathcal{I}_2(\Gamma_x)$ of a sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in X, and θ continuous B-action, we have

$$d_\theta(x_*, c^*) \le \begin{cases} \frac{r}{\mathcal{I}_2 - \liminf p_iq_j}, \text{ if } \{p_iq_j\}_{i,j\in\mathbb{N}} \text{ is } \mathcal{I}_2\text{-bounded,} \\ \frac{r}{\inf\limits_{i,j\in\mathbb{N}} p_iq_j}, \text{ otherwise,} \end{cases}$$

for every $x_* \in W\mathcal{I}_2 - LIM^r x$.

Proof. Case 1: Let $\{p_iq_j\}_{i,j\in\mathbb{N}}$ be $\mathcal{I}_2$-bounded. By contradiction we assume that there exists a point $c^* \in W\mathcal{I}_2(\Gamma_x)$ and $x_* \in W\mathcal{I}_2 - LIM^r x$ such that $d_\theta(x_*, c^*) > \frac{r}{\mathcal{I}_2 - \liminf p_iq_j} > 0$.

Then there exists a positive real number $\lambda \in (0, \mathcal{I}_2 - \liminf p_iq_j)$ such that

$$d_\theta(x_*, c^*) > \frac{r}{\lambda} > \frac{r}{\mathcal{I}_2 - \liminf p_iq_j}.$$

Let $A = \{(i, j) \in \mathbb{N} \times \mathbb{N} : p_iq_j \ge \lambda\}$, $B = \{(i, j) \in \mathbb{N} \times \mathbb{N} : p_iq_jd_\theta(x_{ij}, x_*) < r + \varepsilon\}$ and $C = \{(i, j) \in \mathbb{N} \times \mathbb{N} : p_iq_jd_\theta(x_{ij}, c^*) < \varepsilon\}$. Since $A \cap B \in \mathcal{F}(\mathcal{I}_2)$ and $C \notin \mathcal{I}_2$ then it is obvious that $A \cap B \cap C \neq \emptyset$. Now we choose any $(i_0, j_0) \in A \cap B \cap C$. Then,

$$d_\theta(x_*, c^*) \le \theta(d_\theta(x_{i_0j_0}, x_*), d_\theta(x_{i_0j_0}, c^*)) \le \theta\left(\frac{r+\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}\right),$$

$$\Rightarrow d_\theta(x_*, c^*) \le \lim_{\varepsilon\to 0+} \theta\left(\frac{r+\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}\right) = \theta\left(\frac{r}{\lambda}, 0\right) \le \frac{r}{\lambda},$$

which is a contradiction. Hence, the proof is completed.

Case 2: Let $\{p_iq_j\}_{i,j\in\mathbb{N}}$ be not $\mathcal{I}_2$-bounded. If possible, let there exist a point $c^* \in W\mathcal{I}_2(\Gamma_x)$ and $x_* \in W\mathcal{I}_2 - LIM^r x$ such that $d_\theta(x_*, c^*) > \frac{r}{\inf\limits_{i,j\in\mathbb{N}} p_iq_j} > 0$. Then there exists a positive real number $\lambda_0 \in (0, \inf\limits_{i,j\in\mathbb{N}} p_iq_j)$ such that

$$d_\theta(x_*, c^*) > \frac{r}{\lambda_0} > \frac{r}{\inf\limits_{i,j\in\mathbb{N}} p_iq_j}.$$

Since $c^* \in W\mathcal{I}_2(\Gamma_x)$ so for every $\varepsilon > 0$ we have $A = \{(i,j) \in \mathbb{N}\times\mathbb{N} : p_iq_jd_\theta(x_{ij}, c^*) < \varepsilon\} \notin \mathcal{I}_2$.
This is obvious that $A \cap B \neq \emptyset$, where $B = \{(i,j) \in \mathbb{N}\times\mathbb{N} : p_iq_jd_\theta(x_{ij}, c^*) < r + \varepsilon\}$.
Now we choose any $(i_0, j_0) \in A \cap B$. Then,

$$d_\theta(x_*, c^*) \leq \theta(d_\theta(x_{i_0j_0}, x_*), d_\theta(x_{i_0j_0}, c^*)) \leq \theta\left(\frac{r+\varepsilon}{\lambda_0}, \frac{\varepsilon}{\lambda_0}\right),$$

$$\Rightarrow d_\theta(x_*, c^*) \leq \lim_{\varepsilon\to 0+} \theta\left(\frac{r+\varepsilon}{\lambda_0}, \frac{\varepsilon}{\lambda_0}\right) = \theta\left(\frac{r}{\lambda_0}, 0\right) \leq \frac{r}{\lambda_0},$$

which is a contradiction. Hence, the result. □

Theorem 5.3. For a sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$, we have

$$W\mathcal{I}_2 - LIM^r x \subseteq \begin{cases} \bigcap\limits_{c\in W\mathcal{I}_2(\Gamma_x)} \overline{B}_{d_\theta}\left(c, \frac{r}{p}\right), & \text{if } \{p_iq_j\}_{i,j\in\mathbb{N}} \text{ is } \mathcal{I}_2\text{-bounded}, \\ \bigcap\limits_{c\in W\mathcal{I}_2(\Gamma_x)} \overline{B}_{d_\theta}\left(c, \frac{r}{q}\right), & \text{otherwise}, \end{cases}$$

where $p = \mathcal{I}_2 - \liminf p_iq_j$, $q = \inf\limits_{i,j\in\mathbb{N}} p_iq_j$ and $\overline{B}_{d_\theta}(c, r) = \{y \in X : d_\theta(c, y) \leq r\}$.

Proof. From the Theorem 2.8 [12] the results are obvious. □

The following example shows that equalities of the above Theorem 5.3 may not occur in some cases.

Example 5.4. Let $[0, \infty)$ be a θ-metric space as in Example 4.2 and

$$x_{ij} = \begin{cases} 1, & \text{if } i = m^2 \text{ and } j = n^2 \text{ for some } m, n \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

and $p_iq_j = ij$ for every $n \in \mathbb{N}$ and $r = 1$. Then $W\mathcal{I}_2(\Gamma_x) = \{0\}$, $q = 1$, $\overline{B}_{\frac{r}{q}}(0) = [0, r]$ and $W\mathcal{I}_2 - LIM^r x = \{0\}$. This shows that $W\mathcal{I}_2 - LIM^r x \subsetneq \overline{B}_{\frac{r}{q}}(0)$.

Note 5.5. If both of the sets $W\mathcal{I}_2(\Gamma_x)$ and $W\mathcal{I}_2 - LIM^r x$ are non-empty then from Theorem 3.4 and Theorem 5.2, we get the set $W\mathcal{I}_2(\Gamma_x)$ is bounded. But interestingly the set $W\mathcal{I}_2(\Gamma_x)$ may be unbounded which follows from Example 4.2.

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