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# On Douglas-Shapiro-Shields Factorizations

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**Abstract.** In this note we consider the kernels of vectorial Hankel operators and examine a question which functions are admitted to canonical 'pseudo'-Douglas-Shapiro-Shields factorizations.

## 1. Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. For a separable complex Hilbert space $E$, let $L^2_E$ be the set of all strongly measurable functions $f : \mathbb{T} \to E$ such that

$$\|f\|_2 := \left( \int_{\mathbb{T}} \|f(z)\|_E^2 dm(z) \right)^{\frac{1}{2}} < \infty.$$

For $f \in L^2_E$, the $n$-th Fourier coefficient of $f$, denoted by $\widehat{f}(n)$, is defined by

$$\widehat{f}(n) := \int_{\mathbb{T}} \overline{z}^n f(z) dm(z) \quad (n \in \mathbb{Z}).$$

Then $H^2_E$ denotes the corresponding $E$-valued Hardy space, i.e., the set of $f \in L^2_E$ with $\widehat{f}(n) = 0$ for $n < 0$. Let $\mathcal{B}(D, E)$ denote the set of all bounded linear operators between separable complex Hilbert spaces $D$ and $E$, and abbreviate $\mathcal{B}(E, E)$ to $\mathcal{B}(E)$. A function $\Phi : \mathbb{T} \to \mathcal{B}(D, E)$ is called WOT measurable if $z \mapsto \Phi(z)x$ is weakly measurable for every $x \in D$. Let $L^\infty(\mathcal{B}(D, E))$ denote the set of all bounded WOT measurable $\mathcal{B}(D, E)$-valued functions on $\mathbb{T}$. Define $H^\infty(\mathcal{B}(D, E))$ by the set of functions $\Phi \in L^\infty(\mathcal{B}(D, E))$ whose Fourier coefficients $\widehat{\Phi}(n) = 0$ for $n < 0$. A function $\Delta \in H^\infty(\mathcal{B}(D, E))$ is called an *inner* function if $\Delta^*\Delta = I_D$ a.e. on $\mathbb{T}$ and is called *two-sided inner* function if $\Delta$ is inner and $\Delta\Delta^* = I_E$ a.e. on $\mathbb{T}$. For a function $\Phi \in H^\infty(\mathcal{B}(D, E))$, an inner function $\Delta$ with values in $\mathcal{B}(D', E)$ is called a *left inner divisor* of $\Phi$ if $\Phi = \Delta A$ for $A \in H^\infty(\mathcal{B}(D, D'))$. For $\Phi \in H^\infty(\mathcal{B}(D_1, E))$ and $\Psi \in H^\infty(\mathcal{B}(D_2, E))$, we say that $\Phi$ and $\Psi$ are *left coprime* if the only common left

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inner divisor of both $\Phi$ and $\Psi$ is a unitary operator. Also, for $\Phi \in H^\infty(\mathcal{B}(E, D_1))$ and $\Psi \in H^\infty(\mathcal{B}(E, D_2))$, we say that $\Phi$ and $\Psi$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime, where $\widetilde{\Phi}(z) := \Phi(\overline{z})^*$.

A Hankel operator with symbol $\Phi \in L^\infty(\mathcal{B}(D, E))$ is an operator $H_\Phi : H^2_D \to H^2_E$ defined by

$$H_\Phi f := JP^\perp(\Phi f) \quad \text{for } f \in H^2_D,$$

where $P^\perp$ is the orthogonal projection of $L^2_E$ onto $(H^2_E)^\perp$ and $J$ denotes the unitary operator from $L^2_E$ onto $L^2_E$ given by $J(f)(z) := \overline{z} f(\overline{z})$ for $f \in L^2_E$. A *shift* operator $S_E$ on $H^2_E$ is defined by

$$(S_E f)(z) := z f(z) \quad \text{for each } f \in H^2_E.$$

We can see that the kernel of a Hankel operator $H_{\Phi^*}$ is an invariant subspace of the shift operator on $H^2_E$. Thus by the Beurling-Lax-Halmos Theorem (cf. [2], [15], [14], [17]),

$$\ker H_{\Phi^*} = \Delta H^2_{E'}, \tag{1}$$

for some inner function $\Delta \in H^\infty(\mathcal{B}(E', E))$. Some kernels of products of Hankel operators with scalar symbols are also invariant subspaces of the shift operator on $H^2$ (cf. [11] [8], [9]).

Related to this is the notion of Douglas-Shapiro-Shields (DSS) factorization. For a function $\Phi \in L^\infty(\mathcal{B}(E', E))$, the *Douglas-Shapiro-Shields (briefly, DSS) factorization* of $\Phi$ is (cf. [4], [6], [7], [12]):

$$\Phi = \Delta A^*, \tag{2}$$

where $\Delta \in H^\infty(\mathcal{B}(E))$ is two-sided inner and $A \in H^\infty(\mathcal{B}(E, E'))$. It is known (cf. [4], [7], [12]) that if $\Phi \in L^\infty(\mathcal{B}(E', E))$ admits a DSS factorization of the form (2), then $\Delta$ can be obtained from the equation

$$\ker H_{\Phi^*} = \Delta H^2_E : \tag{3}$$

in this case, $\Delta$ and $A$ are right coprime. The DSS factorization satisfying (3) is called *canonical*. Consequently, each function that admits a DSS factorization can be arranged in a canonical form.

We recall (cf. [1], [16]) that for a scalar function $\varphi$ defined on $\mathbb{T}$, $\varphi$ is said to be of bounded type if

$$\varphi = h_1/h_2 \quad \text{a.e. on } \mathbb{T}$$

for some $h_1, h_2 \in H^\infty$. If $\Phi$ is a matrix-valued $L^\infty$-function then $\Phi$ is said to be of bounded type if each entry of $\Phi$ is of bounded type. It is also known that if $\Phi$ is a matrix-valued function then (cf. [3], [12])

$$\Phi^* \text{ is of bounded type} \iff \Phi \text{ admits a (canonical) DSS factorization.} \tag{4}$$

If the condition "$\Delta$ is two-sided" is dropped in (2), what can we say about a DSS factorization ? More concretely, we would like to ask:

**Question 1.1.** *If $\Phi \in L^\infty(\mathcal{B}(E', E))$ is expressed as*

$$\Phi = \Delta A^*, \tag{5}$$

*where $\Delta \in H^\infty(\mathcal{B}(D, E))$ is inner and $A \in H^\infty(\mathcal{B}(D, E'))$, does it follows that $\Delta$ can be obtained from the equation* $\ker H_{\Phi^*} = \Delta H^2_E$ ?

In this note we consider Question 1.1.

We remark that the kernels of Hankel operators with operator-valued symbols are studied recently in [4] where the degree of cyclicity of the set obtained by the analytic part of the symbol is shown to be connected with the size of the inner matrix $\Delta$ as in (5) (the case of matrix-valued symbol is studied in [13] where an index of the adjoint of the symbol is also connected with the same thing). We will use the degree of cyclicity to give a more explicit answer to Question 1.1 for matrix-valued symbols. The following inverse question is investigated in [10]: Given an (nonsquare) inner matrix $\Delta$, find all matrix-valued $\Phi$ in $L^\infty(B(D, E))$ such that $\ker H_{\Phi^*} = \Delta H^2_{E'}$. A complete answer to this inverse question is given in the case $\Delta$ is a $2 \times 1$ inner matrix or $\Delta$ is an inner matrix such that $\Delta^*$ is of bounded type.

## 2. The main results

For an inner function $\Delta \in H^\infty(\mathcal{B}(D,E))$, $\mathcal{H}(\Delta)$ denotes the orthogonal complement of the subspace $\Delta H^2_D$ in $H^2_E$, i.e.,

$$\mathcal{H}(\Delta) := H^2_E \ominus \Delta H^2_D.$$

For a function $\Phi : \mathbb{T} \to \mathcal{B}(D,E)$, write $\breve{\Phi}(z) := \Phi(\overline{z})$.

We now answer Question 1.1 affirmatively.

**Theorem 2.1.** *If $\Phi \in L^\infty(\mathcal{B}(E',E))$ is expressed as*

$$\Phi = \Delta A^*, \tag{6}$$

*where $\Delta \in H^\infty(\mathcal{B}(D,E))$ is inner and $A \in H^\infty(\mathcal{B}(D,E'))$, then we can write*

$$\Phi = \Delta_A B_0^*, \tag{7}$$

*where $B_0 \in H^\infty(\mathcal{B}(E_0,E'))$ and $\Delta_A \in H^\infty(\mathcal{B}(E_0,E))$ is an inner function which comes from the equation*

$$\ker H_{\Phi^*} = \Delta_A H^2_{E_0} \tag{8}$$

*for some Hilbert space $E_0$. Moreover, in the factorization (7), $\Delta_A$ and $B_0$ are right coprime.*

*Proof.* Suppose that $\Phi \in L^\infty(\mathcal{B}(E',E))$ can be written as

$$\Phi = \Delta A^*, \tag{9}$$

where $\Delta \in H^\infty(\mathcal{B}(D,E))$ is inner and $A \in H^\infty(\mathcal{B}(D,E'))$. Define

$$\Delta_A := \text{left-g.c.d.}\Big\{\Theta : \Phi = \Theta B^* \text{ with } \Theta \in H^\infty(\mathcal{B}(D,E)) \text{ inner and } B \in H^\infty(D,E')\Big\}, \tag{10}$$

where left-g.c.d. means the greatest common left inner divisor. If $\Phi = \Theta B^*$ for some inner function $\Theta \in H^\infty(D,E)$ and $B \in H^\infty(D,E')$. Then $\Theta H^2_D \subseteq \ker H_{\Phi^*}$. We thus have

$$\Delta_A H^2_{E_0} \subseteq \ker H_{\Phi^*} \quad \text{for some Hilbert space } E_0. \tag{11}$$

For the reverse inclusion, suppose $\ker H_{\Phi^*} \neq \{0\}$. Then in view of the Beurling-Lax-Halmos Theorem that $\ker H_{\Phi^*} = \Delta_1 H^2_{E_1}$ for some nonzero inner function $\Delta_1$ with values in $\mathcal{B}(E_1,E)$. Thus we have $\Delta H^2_D \subseteq \Delta_1 H^2_{E_1}$, which implies that $\Delta_1$ is a left inner divisor of $\Delta$. Write

$$\Delta = \Delta_1 \Omega,$$

where $\Omega$ is inner function with values in $\mathcal{B}(D,E_1)$. Since $\ker H_{\Phi^*} = \Delta_1 H^2_{E_1}$, it follows that for all $f \in H^2_{E_1}$,

$$A\Omega^* f = \Phi^* \Delta_1 f \in H^2_{E'}. \tag{12}$$

Put $B := A\Omega^*$. Then $B \in L^\infty(\mathcal{B}(E_1,E'))$. It thus follows from (12) that for all $x \in E_1$ and $n = 1,2,3,\cdots$,

$$\widehat{B}(-n)x = \int_{\mathbb{T}} z^n B(z) x \, dm(z) = 0.$$

Thus $B$ belongs to $H^\infty(\mathcal{B}(E_1,E'))$. Since $\Delta_1 B^* = \Phi$, it follows that $\Delta_1 H^2_{E_1} \subseteq \Delta_A H^2_{E_0}$, which together with (11) gives $\Delta_1 H^2_{E_1} = \Delta_A H^2_{E_0}$. Thus $\Delta_1 = \Delta_A U$ for some unitary operator $U \in \mathcal{B}(E_1,E_0)$. Put $B_0 := BU^* \in H^\infty(\mathcal{B}(E_0,E'))$. Then

$$\Phi = \Delta_A B_0^* \quad \text{and} \quad \ker H_{\Phi^*} = \Delta_A H^2_{E_0}. \tag{13}$$

We now claim that $\Delta_A$ and $B_0$ are right coprime. To see this we assume that $\Omega$ is a common left inner divisor of $\widetilde{\Delta}_A$ and $\widetilde{B}_0$. Then we can write

$$\widetilde{\Delta}_A = \Omega\Delta_2 \quad \text{and} \quad \widetilde{B}_0 = \Omega B_2,$$

where $\Delta_2 \in H^\infty(\mathcal{B}(E, E_1))$ and $B_2 \in H^\infty((E', E_1))$. Then $\widetilde{\Delta}_2$ is a left inner divisor of $\Delta_A$, and we have that

$$\Phi = \Delta_A B_0^* = \widetilde{\Delta}_2\widetilde{\Omega}\widetilde{\Omega}^*\widetilde{B}_2^* = \widetilde{\Delta}_2\widetilde{B}_2^*.$$

Thus

$$\widetilde{\Delta}_2 H^2_{E_1} \subseteq \ker H_{\Phi^*} = \Delta_A H^2_{E_0}$$

which implies that $\Delta_A$ is a left inner divisor of $\widetilde{\Delta}_2$. It thus follows that $\widetilde{\Omega}$ is a unitary operator and so is $\Omega$. Therefore $\Delta_A$ and $B_0$ are right coprime. This completes the proof. $\square$

**Remark 2.2.** *The expression (6) will be called a pseudo-DSS factorization and the expression (7) will be called a canonical pseudo-DSS factorization. Thus Theorem 2.1 says that if a function $\Phi \in L^\infty(\mathcal{B}(E', E))$ admits a pseudo-DSS factorization then we can always arrange the pseudo-DSS factorization of $\Phi$ in a canonical form.*

For an inner function $\Delta \in H^\infty(\mathcal{B}(D, E))$, define the kernel of $\Delta^*$ by

$$\ker \Delta^* := \{f \in H^2_E : \ \Delta^*(z)f(z) = 0 \ \text{ for almost all } z \in \mathbb{T}\}.$$

Since $\ker \Delta^*$ is an invariant subspace for the shift operator $S_D$, it follows from the Beurling-Lax-Halmos Theorem that $\ker \Delta^* = \Omega H^2_{D'}$ for some inner function $\Omega \in H^\infty(D', E)$.

The following lemma gives a concrete description for the kernel of $\Delta^*$.

**Lemma 2.3.** *[4] [10] Let $\Delta$ be an inner function with values in $\mathcal{B}(D, E)$. Then we may write* $\ker \Delta^* = \Omega H^2_{D'}$ *for some inner function $\Omega \in H^\infty(D', E)$. Put*

$$\Delta_c := \textit{left-g.c.d.}\Big\{ [g]^i : g \in \ker \Delta^* \Big\}, \tag{14}$$

*where $[g] : \mathbb{T} \to \mathcal{B}(\mathbb{C}, E)$ is defined by $[g](z)\alpha := \alpha g(z)$ $(\alpha \in \mathbb{C})$ and $[g]^i$ denotes the inner part of $[g]$. Then,*

*(a)* $\Omega = \Delta_c$;

*(b)* $[\Delta, \Delta_c]$ *is an inner function with values in* $\mathcal{B}(D \oplus D', E)$;

*(c)* $\ker H_{\Delta^*} = [\Delta, \Delta_c]H^2_{D\oplus D'} \equiv \Delta H^2_D \bigoplus \Delta_c H^2_{D'}$.

**Definition 2.4.** $\Delta_c$ *is called the complementary factor of an inner function* $\Delta$.

We then have:

**Corollary 2.5.** *Suppose $\Delta$ is an inner function with values in $H^\infty(\mathcal{B}(D, E))$ and $A \in H^\infty(\mathcal{B}(D, E'))$. If $\Delta$ admits a DSS factorization then*

$$\ker H_{A\Delta^*} = \Theta H^2_{E},$$

*where $\Theta \equiv [\Delta, \Delta_c]\breve{\Omega}$ is two-sided inner with*

$$\Omega := \textit{left-g.c.d.}\Big(\widetilde{[\Delta, \Delta_c]}, \widetilde{[A, 0]}\Big) \quad (\textit{where } [A, 0] \in H^\infty(\mathcal{B}(D \oplus D', E'))).$$

*Proof.* Let

$$\Omega := \text{left-g.c.d.}\Big(\widetilde{[\Delta, \Delta_c]}, \widetilde{[A, 0]}\Big).$$

Since $\Delta$ admits a DSS factorization, it follows from Lemma 2.3 that $[\Delta, \Delta_c]$ is two-sided inner, and so is $\widetilde{[\Delta, \Delta_c]}$. Thus $\Omega$ is two-sided inner, and hence we may write

$$[\Delta, \Delta_c] = \Theta\widetilde{\Omega} \quad \text{and} \quad [A, 0] = B\widetilde{\Omega} \quad (\Theta \in H^\infty(\mathcal{B}(E)),\ B \in H^\infty(\mathcal{B}(E, E'))),$$

where $\Theta$ and $B$ are right coprime. Thus we have that

$$\Delta A^* = [\Delta, \Delta_c][A, 0]^* = \Theta B^*.$$

But since $\widetilde{\Omega}$ is two-sided inner, so is $\Theta$, and hence $\ker H_{A\Delta^*} = \Theta H^2_E$. This completes the proof. $\square$

The following example shows that Corollary 2.5 may fail if the condition "$\Delta$ admits a DSS factorization" is dropped.

**Example 2.6.** *Let $h(z) := e^{\frac{1}{z-3}} \in H^\infty$. Put*

$$f(z) := \frac{h(z)}{\sqrt{2}\|h\|_\infty}.$$

*Clearly, $\overline{f}$ is not of bounded type. Let $h_1(z) := \sqrt{1 - |f(z)|^2}$. Then $h_1 \in L^\infty$ and $|h_1| \geq \frac{1}{\sqrt{2}}$. Thus there exists an outer function $g$ such that $|h_1| = |g|$ a.e. on $\mathbb{T}$ (cf. [5, Corollary 6.25], [4]). Let*

$$\Delta := \begin{bmatrix} f & 0 \\ g & 0 \\ 0 & \frac{z}{\sqrt{2}} \\ 0 & \frac{z}{\sqrt{2}} \end{bmatrix} \quad \textit{and} \quad A := \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

*Then $\Delta$ is inner and $\Delta^*$ is not of bounded type, so that by (4), $\Delta$ does not admit a DSS factorization. Write*

$$\Delta_1 := \begin{bmatrix} f \\ g \end{bmatrix} \quad \textit{and} \quad \Delta_2 := \frac{1}{\sqrt{2}} \begin{bmatrix} z \\ z \end{bmatrix}.$$

*Then it follows from Lemma 2.3 that*

$$\ker H_{\Delta^*} = \ker H_{\Delta_1^*} \oplus \ker H_{\Delta_2^*} = \Delta_1 H^2 \oplus \begin{bmatrix} \frac{z}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{z}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} H^2_{\mathbb{C}^2}$$

*and hence,*

$$[\Delta, \Delta_c] = \begin{bmatrix} f & 0 & 0 \\ g & 0 & 0 \\ 0 & \frac{z}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{z}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

*Since $\ker H_{A\Delta^*} = H^2 \oplus H^2 \oplus \ker H_{\Delta_2^*}$, it follows that*

$$\ker H_{A\Delta^*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{z}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{z}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} H^2_{\mathbb{C}^4} \equiv \Theta H^2_{\mathbb{C}^4}.$$

*Since $\widetilde{f}$ is invertible in $H^\infty$, it follows that $A$ and $\Delta$ are right coprime. On the other hand, since*

$$[\widetilde{\Delta, \Delta_c}]H^2_{\mathbb{C}^4} \bigvee [\widetilde{A, 0}]H^2 = \begin{bmatrix} \widetilde{f} & \widetilde{g} & 0 & 0 \\ 0 & 0 & \frac{z}{\sqrt{2}} & \frac{z}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} H^2_{\mathbb{C}^4} \bigvee \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} H^2 = H^2_{\mathbb{C}^3},$$

*it follows that*

$$\Omega \equiv \textit{left-g.c.d.}([\widetilde{\Delta, \Delta_c}], [\widetilde{A, 0}]) = I_3.$$

*We thus have $\Theta \neq [\Delta, \Delta_c]\check{\Omega}$.*

Let $M_{n\times m}$ denote the set of all $n \times m$ complex matrices and write $M_n \equiv M_{n\times n}$.

**Remark 2.7.** *It is clear that if $\Phi \in L^\infty(\mathcal{B}(E', E))$ is such that $\ker H_{\Phi^*} = \{0\}$ then, by Theorem 2.1, $\Phi$ does not admit a pseudo-DSS factorization. We next give a less trivial example in the sense that $\ker H_{\Phi^*} \neq \{0\}$, but $\Phi$ still does not admit a pseudo-DSS factorization. Suppose that $\theta_1$ and $\theta_2$ are coprime inner functions. Consider*

$$\Phi := \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & a \end{bmatrix} \in H^\infty_{M_{3\times3}},$$

*where $a \in H^\infty$ is such that $\overline{a}$ is not of bounded type. Then a direct calculation shows that*

$$\mathit{ker} H_{\Phi^*} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{bmatrix} H^2_{\mathbb{C}^2} \equiv \Theta H^2_{\mathbb{C}^2}.$$

*Assume that $\Phi$ admits a pseudo-DSS factorization. Then, by Theorem 2.1, we may write*

$$\Phi = \Theta B^*$$

*for some $B \in H^\infty_{M_{3\times2}}$. However, for any $B \in H^\infty_{M_{3\times2}}$,*

$$\Phi = \Theta B^* = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{bmatrix} B^* = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix},$$

*a contradiction.*

## 3. When $\Phi$ is a matrix-valued symbol

Theorem 2.1 gives a satisfactory answer to Question 1.1, however as we have seen in a previous example, it is not a simple matter to find the canonical pseudo-DSS factorization of $\Phi$. Equivalently, we need to find $\ker H_{\Phi^*}$. In the case when $\Delta$ admits a DSS factorization, Corollary 2.5 gives a practical way of finding $\ker H_{\Phi^*}$. Here we extend Corollary 2.5 to more general situations when $\Phi$ is a matrix-valued symbol. Since Corollary 2.5 covers the case when $\Phi \equiv \Delta A^*$ admits a DSS factorization, here we will assume $\Phi$ does not admit a DSS factorization.

**Proposition 3.1.** Suppose $\Phi \in L^\infty_{M_{n\times m}}$ does not admit a DSS factorization. Let

$$\Phi = \Delta A^* \quad \text{(pseudo-DSS factorization).}$$

If $[\Delta, \Delta_c]$ is in $H^\infty_{M_{n\times(n-1)}}$ and $\Omega \equiv$ left-g.c.d$\left([\widetilde{\Delta, \Delta_c}], [\widetilde{A, 0}]\right)$ is two-sided inner, then $\Phi = \Theta B^*$ is a canonical pseudo-DSS factorization for some $B \in H^\infty_{M_{m\times(n-1)}}$, where

$$\Theta = [\Delta, \Delta_c]\,\check{\Omega}. \tag{15}$$

*Proof.* By Theorem 2.1, we need to show $\Delta_A$ given by (8) is the same as the $\Theta$ given by (15). By the definition of $\Omega$,

$$[\Delta, \Delta_c] = \Theta\widetilde{\Omega} \quad \text{and} \quad [A, 0] = B\widetilde{\Omega}$$

for some $B \in H^\infty_{M_{m\times(n-1)}}$ and $\Theta$ and $B$ are right coprime. Since

$$\Phi = \Delta A^* = [\Delta, \Delta_c]\,[A, 0]^* = \Theta\widetilde{\Omega}\widetilde{\Omega}^* B^* = \Theta B^*,$$

it follows that $\Theta H^2_{\mathbb{C}^{n-1}} \subseteq \ker H_{\Phi^*} \equiv \Delta_A H^2_{\mathbb{C}^r}$. Thus $\Theta = \Delta_A\Gamma$ for some inner function $\Gamma \in H^\infty_{M_{r\times(n-1)}}$. Since $\Phi \in L^\infty_{M_{n\times m}}$ does not admit a DSS factorization, it follows that $r < n$, and hence $r = n-1$. It thus follows from Theorem 2.1 that

$$\Delta_A\Gamma B^* = \Theta B^* = \Phi = \Delta_A B_0^* \quad \text{for some } B_0 \in H_{M_{m\times(n-1)}}.$$

Thus $\Gamma B^* = B_0^*$, and hence $B = B_0\Gamma$. Hence, the fact that $\Theta$ and $B$ are right coprime implies that $\Gamma$ is a unitary constant, and therefore $\Theta = \Delta_A$. $\square$

We give an example to illustrate the above proposition.

**Example 3.2.** We use the same notation as in Example 2.6. Let

$$A := [1, 1]\,.$$

Then $\Phi = \Delta A^* = [f\ g\ \frac{z}{\sqrt{2}}\ \frac{z}{\sqrt{2}}]^t$ does not admit a DSS factorization. Note that

$$\Omega = \text{left-g.c.d}\left(\widetilde{[\Delta, \Delta_c]},\ \widetilde{[A, 0]}\right) = I_3.$$

It follows from the above proposition that $\ker H_{\Phi^*} = [\Delta, \Delta_c]H^2_{\mathbb{C}^3}$.

Next we extend the above proposition by using the notion of degree of cyclicity due to V.I. Vasyunin and N.K. Nikolskii [18] (or [16]): If $F \subseteq H^2_{\mathbb{C}^n}$, then the *degree of cyclicity*, denoted by $\mathrm{dc}(F)$, of $F$ is defined by the number

$$\mathrm{dc}(F) := n - \max_{\zeta\in\mathbb{D}} \dim\left\{g(\zeta) : g \in H^2_{\mathbb{C}^n} \ominus E^*_F\right\},$$

where $E^*_F$ denotes the smallest $S^*_E$-invariant subspace containing $F$, i.e., $E^*_F = \bigvee\left\{S^{*n}_E F :\ n \geq 0\right\}$. It is known from [4, Lemma 2.13] that if $\Phi \equiv [\Phi_1, \cdots, \Phi_n]$ ($\Phi_j \in L^\infty_{\mathbb{C}^m}$) is an $m \times n$ matrix-valued function then

$$\ker H_{\Phi^*} = \Theta H^2_{\mathbb{C}^r} \iff \mathrm{dc}\{\Phi_+\} = n - r, \tag{16}$$

where $\Theta$ is an $m \times r$ inner matrix function and $\{\Phi_+\} := \{(\Phi_1)_+, \cdots, (\Phi_n)_+\} \subseteq H^\infty_{\mathbb{C}^m}$ (where $(\Phi_j)_+$ denotes the analytic part of $\Phi_j$).

**Remark 3.3.** *Suppose $\Phi \in L^\infty_{M_{n\times m}}$ does not admit a DSS factorization. Let*

$$\Phi = \Delta A^* \quad (\textit{pseudo-DSS factorization}).$$

*Suppose that $[\Delta, \Delta_c] \in H^\infty_{M_{n\times s}}$, $\Omega \equiv$ left-g.c.d$\left(\widetilde{[\Delta, \Delta_c]},\ \widetilde{[A, 0]}\right)$ is two-sided inner and dc$\{\Phi_+\} = n - s$. Then by the same argument as the proof of Proposition 3.1, we have that $\Theta = \Delta_A\Gamma$ for some inner function $\Gamma$. By Theorem 2.1, $\ker H_{\Phi^*} = \Delta_A H^2_{\mathbb{C}^r}$ for some $r \leq n$. By the assumption, dc$\{\Phi_+\} = n - s$ and by (16), $r = s$. Therefore $\Theta = \Delta_A\Gamma$ implies that $\Gamma$ is a $s \times s$ two-sided inner matrix. Thus by the same argument as the proof of Proposition 3.1, we have that*

$$\Phi = \Theta B^* \quad (\textit{canonical pseudo-DSS factorization}),$$

*where $\Theta = [\Delta, \Delta_c]\,\check{\Omega}$.*

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