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On The Representation of Monogenic Functions by The Product Bases of Polynomials

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Abstract.
The main purpose of this paper is to study questions concerning representations of Clifford valued functions by the product bases of Clifford polynomials. By the way we generalize several results from complex analysis to the setting of Clifford analysis.

1. Introduction

Hyper-complex function theory is one of the possible generalizations of the theory of holomorphic functions of one complex variable taking advantage of Clifford algebras and provides the fundamentals of Clifford analysis as a refinement of harmonic analysis in higher dimensions which have many applications in mathematical physics. In the mid of 1980′*s*, it became clear that Clifford analysis provided a natural framework for generalizing a lot of results from complex analysis in the plane to the higher dimensional case (see [11, 12]).

With this in hand, an extension of the theory of bases (basic sets) of polynomials in one complex variable, as introduced by J.M. Whittaker and B. Cannon (see [31]) to the setting of Clifford analysis has been given in (see [2–6, 9, 10, 13, 32, 33]), where an important subclass of the Clifford regular functions were considered, for which several results on their representations in closed ball were obtained.

From this starting point, many results on the polynomial bases in the complex case of one complex variable were refined and generalized to the Clifford setting (see [1–9, 27]). In this line of research in Clifford setting, one of the interesting problem has been investigated by Zayed et al. [32] where the authors explored the effectiveness of the hypercomplex derivative and primitive basic sets associated with the previously mentioned polynomials. Recently, these polynomials were used to prove a counterpart of Hadamard's three-hyperballs theorem within Clifford analysis and to establish an overconvergence property of a special monogenic simple series (see [9]). As an application of this overconvergence property on generalized monogenic Bessel polynomials, we refer to [1] .

The aim of this contribution is to study questions concerning representations of special monogenic functions by the product bases of polynomials. We work with the field $\mathcal{A}_m$ of the real 2^m-dimensional

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Clifford algebra, where it is a real algebra freely generated by the standard basis $e_0, e_1, e_2, \ldots, e_m$ in $\mathbb{R}^{m+1}$ subject to the conditions $e_0 = 1$ and $e_j e_k + e_k e_j = -2\delta_{jk}$ for $1 \leq j, k \leq m$ (we refer to [8], [9] for the basic facts about $\mathcal{A}_m$). Note that e.g. $\mathcal{A}_0$ is the field of real numbers, $\mathcal{A}_1$ is the field of complex numbers and $\mathcal{A}_2 = H$ the quaternionic skew field, respectively.

We embed canonically $\mathbb{R}^{m+1}$ in $\mathcal{A}_m$. For $x \in \mathcal{A}_m$, $\mathbf{Re}x$ the real part of x, will stand for the e_0-component of x and $\mathbf{Im}x = x - (\mathbf{Re}x)e_0$.

We also equip $\mathcal{A}_m$ with the Euclidean norm $|x|^2 = \mathbf{Re}(x\overline{x})$ where the conjugation is the unique linear morphism of $\mathcal{A}_m$ for which $\overline{e}_0 = e_0$, $\overline{e}_j = -e_j$ for $1 \leq j \leq m$ and $\overline{xy} = \overline{y}.\overline{x}$ for all $x, y \in \mathcal{A}_m$.

As $\mathcal{A}_m$ is isomorphic to $\mathbb{R}^{2^m}$ we may provide it with the $\mathbb{R}^{2^m}$-norm $|a|$ and one sees easily that for any $a, b \in \mathcal{A}_m$, $|a.b| \leq 2^{\frac{m}{2}} |a| . |b|$, where $a = \sum_{A \subseteq M} a_A e_A$ and M stands for $\{1, 2, ..., m\}$.

Suggested by the case $m = 1$, call a $\mathcal{A}_m$-valued function f in $\mathbb{R}^{m+1}$ Clifford analytic (monogenic), provided it is annihilated by the generalized Cauchy-Riemann operator

$$D = \sum_{j=0}^{m} e_j \left(\frac{\partial}{\partial x_j}\right), \text{ i.e. } Df = 0.$$

The right $\mathcal{A}_m$-module $\mathcal{A}_m[x]$ defined by $\mathcal{A}_m[x] = \mathrm{span}_{\mathcal{A}_m}\{z_n(x) : n \in \mathbb{N}\}$ is called the space of special monogenic polynomials, where $\mathcal{A}_m$ is the Clifford algebra and x is the Clifford variable. $z_n(x)$ is defined by (see [2])

$$z_n(x) = \sum_{i+j=n} \frac{(\frac{m-1}{2})_i (\frac{m+1}{2})_j}{i!j!} \overline{x}^i x^j$$

where for $b \in \mathbb{R}$, $(b)_\ell$ stands for $b(b+1)...(b+\ell-1)$, $\overline{x}$ is the conjugate of x, $x \in \mathbb{R}^{m+1}$, $\mathbb{R}^{m+1}$ is identified with a subset of $\mathcal{A}_m$.

If $P_n(x)$ is homogeneous special monogenic polynomial of degree n in x, then (see. [2]) $P_n(x) = z_n(x).\alpha$, α is some constant in $\mathcal{A}_m$ and

$$\sup_{|x|=R} |z_n(x)| = \begin{pmatrix} m+n-1 \\ n \end{pmatrix} R^n = \frac{(m)_n}{n!} R^n$$

where

$$\frac{(m)_n}{n!} = (m+n-1)!/n!(m-1)!$$

Definition 1.1. *(Special monogenic function) Let Ω be a connected open subset of $\mathbb{R}^{m+1}$ containing 0, then a monogenic function f in Ω is said to be special monogenic in Ω iff its Taylor series near zero (which is known to exist) has the form $f(x) = \sum_{n=0}^{\infty} z_n(x) c_n$, $c_n \in \mathcal{A}_m$. A function f is said to be special monogenic on $\overline{B}(R)$ if it is special monogenic on some connected open neighborhood Ω_f of $\overline{B}(R)$.*

The fundamental references for special monogenic functions are [17, 28].

Definition 1.2. *A set $\beta = \{P_k(x) : k \in \mathbb{N}\}$ of special monogenic polynomials is called basic if and only if it is a base for the space $\mathcal{A}_m[x]$ of special monogenic polynomials, in the sense of Hamel basis.*

Suppose that $P_n(x) = \sum_{j=0}^{\infty} z_j(x) P_{nj}$, $P_{nj} \in \mathcal{A}_m$. The base $\{P_n(x)\}$ is said to be simple if $P_n(x)$ has degree n, for all $n \in \mathbb{N}$, and a simple base is called simple monic if $P_{nn} = 1$ for every $n \in \mathbb{N}$. The matrix $P = (P_{nj})$ is called the matrix of Clifford coefficients of the base $\{P_n(x)\}$.

From the definition of $P_n(x)$, we shall have the $z_n(x)$ representation in the form

$$z_n(x) = \sum P_i(x) \pi_{ni}, \pi_{ni} \in \mathcal{A}_m \tag{1}$$

where the matrix $\Pi = (\pi_{ni})$ is said to be the matrix of operators of $\{P_n(x)\}$.

Let $f(x)$ be special monogenic function as defined above, then there is formally an associated basic series given by (see [2])

$$\sum_{k=0}^{\infty} P_k(x) \left(\sum_{n=0}^{\infty} \pi_{nk} c_k \right) \tag{2}$$

when this associated basic series (2) converges normally to $f(x)$ in $\overline{B}(R)$, then it is said that the basic series represents $f(x)$ in $\overline{B}(R)$.

A base $\{P_n(x)\}$ is said to be effective if for every special monogenic function f, defined in a closed neighborhood of zero $\overline{B}(R)$ of the radius $R > 0$, the series (2) converges normally to f in $\overline{B}(R)$.

Write

$$\lambda_n(x) = \sum_{k=0}^{\infty} \sup_{|x|=R} |P_k(x)\pi_{nk}| = \left(\sum_k \|P_k(x)\pi_{nk}\|_R \right)$$

and

$$\lambda(R) = \limsup_{n\to\infty} (\lambda_n(R))^{\frac{1}{n}}$$

The base $\{P_n(x)\}$ is called effective in $\overline{B}(R)$ iff $\lambda(R) = R$ (see [2], Theorem 1).

The order ω and type γ of a Cannon base have been adapted to the Clifford case and introduced in [2] as follows:

$$\omega = \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \lambda_n(R)}{n \log n}.$$

If $0 < \omega < \infty$, the type is

$$\gamma = \lim_{R\to\infty} \frac{e}{\omega} \limsup_{n\to\infty} \frac{[\lambda_n(R)]^{\frac{1}{n\omega}}}{n}$$

If $\sum_0^{\infty} z_n(x)c_n$ is special monogenic on all of $\mathbb{R}^{m+1}$, then its "order" is defined to be

$$\limsup_{n\to\infty} (n \log n)/\log(1/|c_n|), \text{ (c.f. [2]).}$$

It has been shown in [2, 3] that a base $\{P_n(x)\}$ of order ω will represent in any closed ball $\overline{B}(R)$ every entire special monogenic function of order less than $\frac{1}{\omega}$.

Definition 1.3. *(The product base) If Q and P are the matrices of Clifford coefficients of the respective bases $\{Q_n(x)\}$ and $\{P_n(x)\}$, then one can show that the matrix QP is the matrix of Clifford coefficients of a base $\{U_n(x)\}$, given by $U_n(x) = \sum_{j=0}^{\infty} Q_j(x)P_{nj}$, $P_{nj} \in \mathcal{A}_m$. The base $\{U_n(x)\}$ is said to be the product base of the bases $\{Q_n(x)\}$ and $\{P_n(x)\}$ in the given order.*

In analogue with the complex case, an important question arises in the theory of bases in Clifford analysis that is: when the product base of special monogenic polynomials is effective?

This question is partially answered in ([5, 6]) for the product base, under some restrictive conditions, that is for the case of simple bases.

It has been shown in [5] that in general the product of two effective bases need not be effective.

Other possibilities, for effectiveness of the product base when each of its factors is effective, that its product base is not effective in some closed ball (see [5]).

Besides, if each of the factors of the product base is not effective in a closed ball $\overline{B}(R)$, is the product base $\{U_n(x)\}$ not effective there either? The answer is negative since we can take $\{P_n(x)\}$ as the inverse base of $\{Q_n(x)\}$ to yield for the product base $\{U_n(x)\}$, the unit base $\{z_n(x)\}$ which is everywhere effective.

Remark 1.1. *Several open questions concerning the convergence properties of the product bases of special monogenic polynomials have not yet been investigated.*

(I) Concerning effectiveness for functions of bounded radii of convergence. The first trial to obtain, the effectiveness of the product base $\{U_n(x)\}$ of special monogenic polynomial in the closed ball $\bar{B}(R)$, is due to Abul-Ez [5] who started with the following special cases:

Result(1): Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be simple monic bases of special monogenic polynomials both effective in $\bar{B}(R)$. Then the product base $\{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}$ is effective in $\bar{B}(R)$.

Result(2): If $\{P_n(x)\}$ and $\{Q_n(x)\}$ are such that $\{P_n(x)\}$ is simple and $\{Q_n(x)\}$ is simple monic and both are effective in $\bar{B}(R)$ then $\{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}$is effective in $\bar{B}(R)$.

In order to obtain the effectiveness property of the product of the two non-monic simple bases, it is necessary to impose some additional conditions on $\{P_n(x)\}$ and $\{Q_n(x)\}$ and in this respect Abul-Ez [5] have obtained the following result.

Result(3): Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be simple bases of special monogenic polynomials and suppose that $\{Q_n(x)\}$ is effective in $\bar{B}(R)$ and satisfying

$$\lim_{n\to\infty} \left|q_{nn}\right|^{\frac{1}{n}} = H, \quad 0 < H < \infty$$

Then the product base $\{U_n(x)\}$ is effective in $\bar{B}(R)$, if and only if, the base $\{P_n(x)\}$ in effective in $\{\bar{B}(HR)\}$.

(II) Concerning effectiveness for entire functions in terms of the mode of increase, we have the following interesting result due to Abul-Ez [6]. For which he had obtained the representation of Clifford valued function by the product base of special monogenic polynomials.

Result(4): Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be simple bases of special monogenic polynomials of order ω_1 and ω_2 respectively, and suppose that $f(x)$ is an entire special monogenic function of order $< \frac{1}{\omega_1+2\omega_2}$. If the base $\{Q_n(x)\}$ is monic, then the product base $\{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}$ represents $f(x)$ in any closed ball $\bar{B}(R)$.

The above result (4) gives us an estimation of an upper bound of the order of the product base of special monogenic polynomials, stated in the following result [6].

Result(5): Let $\{Q_n(x)\}$ and $\{P_n(x)\}$ be simple bases of special monogenic polynomials of orders ω_1 and ω_2 respectively. If $\{Q_n(x)\}$ monic base, then the order ω_U of the product set $\{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}$ does not exceed $\omega_1 + 2\omega_2$.

As it is interesting to know the value of the lower bound of the order ω_U of the product base $\{U_n(x)\}$, it is known that $\omega_U \geq 0$, but the value of the lower bound is not always zero as the following results show [6].

Result(6): If $0 \leq \omega_1 \leq \frac{1}{2}\omega_2$, then the order ω_U of the product base $\{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}$ of the two simple monic bases $\{Q_n(x)\}$ and $\{P_n(x)\}$ is such that,

$$(\frac{1}{2}\omega_2 - \omega_1) \leq \omega_U \leq (\omega_1 + 2\omega_2)$$

Result(7): If $0 \leq \omega_2 \leq \frac{1}{2}\omega_1$, then the order ω_U of the product base $\{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}$, satisfies the relation

$$(\omega_1 - 2\omega_2) \leq \omega_U \leq (\omega_1 + 2\omega_2)$$

provided that $\{Q_n(x)\}$ is a monic base.

Corollary 1.1. *If $\frac{1}{2}\omega_2 \leq \omega_1 \leq 2\omega_2$, then ω_U may be equal to zero.*

2. Aim of the work

Now we are ready to carry out our goal in this paper that is to obtain generalizations of the previous results and by the way to get some extensions of some results in complex case as given in ([14–16, 18–26, 29]), we start with:

3. Effectiveness of the product base for functions with bounded radii of convergence

The main aim of this section is to generalize the original results of Nassif ([21–24]), and Tantawi ([29, 30]). This generalizes also to the Clifford setting the analogue results in the complex case given by Mikhail ([18, 19]). We start with the following result (with Newns condition [26]).

Theorem 3.1. *Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be two bases of special monogenic polynomials such that $\{P_n(x)\}$ be effective in $\bar{B}(aR^+)$ and $\{Q_n(x)\}$ effective in $\bar{B}(R^+)$. If*

$$\mu(R^+) \leq aR, \qquad a \text{ is a positive constant} \tag{3}$$

$$\nu(\rho) > aR\,, \qquad \text{for all } \ \rho > R \tag{4}$$

$$\mu(R) = \limsup_{n\to\infty}\{B_n(R)^{\frac{1}{n}}\} \ \text{ and } \ \nu(R) = \liminf_{n\to\infty}\{B_n(R)^{\frac{1}{n}}\}.$$

Then the product base $\{U_n(x)\}$ is effective in $\bar{B}(R^+)$.

Proof. We shall take R_5 any number $> R$ and then choose the intervening $R's$, $R < R_1 < R_2 < R_3 < R_4 < R_5$ to suit their requirements. We also associate the expressions $B_n(R)$, λ_{ni}, $\theta_n(R)$, $\sigma(R)$ with the base $\{Q_n(x)\}$ and the expressions $C_n(R), \delta_{ni}, \Phi_n(R), \tau(R)$ with the base $\{U_n(x)\}$, then from (4) we have

$$(aR_4)^n < k_1 B_n(R_5) \quad \text{for all } \ n, \ k_1 \text{ is constant.} \tag{5}$$

Since $\{P_n(x)\}$ is effective in $B(aR^+)$, $\kappa(aR^+) = aR$.
Hence, $\kappa(aR_3) < aR_4$. So that

$$F_n(aR_3) < k_2(aR_4)^n, \ \forall n, \quad k_2 \text{ is constant.} \tag{6}$$

Now

$$\begin{aligned}
\Phi_n(R) &= \sup_{|x|=R}\left|U_i(x)\delta_{nj} + \cdots + U_k(x)\right| \\
&= \sup_{|x|=R}\left|\sum_{i=j}^{k} U_i(x)\{\sum_t \pi_{ti}\lambda_{nt}\}\right| \\
&= \sup_{|x|=R}\left|\sum_t \{U_j(x)\pi_{tj} + \cdots + U_k(x)\pi_{tk}\}\lambda_{nt}\right| \\
&= \sup_{|x|=R}\left|\sum_i \{U_j(x)\pi_{ij} + \cdots + U_k(x)\pi_{ik}\}\lambda_{ni}\right|.
\end{aligned}$$

Thus

$$\Phi_n(R) \leq 2^{\frac{m}{2}} \sum_i \sup_{|x|=R}\left|U_j(x)\pi_{ij} + \cdots + U_k(x)\pi_{ik}\right| |\lambda_{ni}|\,. \tag{7}$$

We now write

$$f_i(x) = P_j(x)\pi_{ij} + \cdots + P_k(x)\pi_{ik} = \sum_t z_t(x) f_{it}. \tag{8}$$

Then from the relation $U_n(x) = \sum_i Q_i(x)p_{ni} = \sum_i \sum_j z_j(x)q_{ij}p_{ni}$, we have

$$g_i(x) = U_j(x)\pi_{ij} + \cdots + U_k(x)\pi_{ik} = \sum_t Q_t(x) f_{it}. \tag{9}$$

Hence, writing $L_i(R) = \sup_{|x|=R} \left|f_i(x)\right|$, $N_i(R) = \sup_{|x|=R} \left|g_i(x)\right|$. We have

$$L_i(R) \le F_i(R). \tag{10}$$

Then from (8) and (10) and relying on Cauchy's inequality [3] for the special monogenic polynomials in (8) we get

$$\begin{aligned} \left|f_{it}\right| &\le \sqrt{\frac{n!}{(m)_n}} \left(\frac{\sup\limits_{|x|=aR_3} \left|f_i(x)\right|}{(aR_3)^t} \right) \le \sqrt{\frac{n!}{(m)_n}} \left(\frac{L_i(aR_3)}{(aR_3)^t} \right) \\ &= \sqrt{\frac{n!}{(m)_n}} \left(\frac{F_i(aR_3)}{(aR_3)^t} \right) \end{aligned}$$

Hence from (9) and (3) one can deduce

$$\begin{aligned} N_i(R_1) &= \sup_{|x|=R_1} \left|g_i(x)\right| = \sup_{|x|=R_1} \left| \sum_t Q_t(x) f_{it} \right| \\ &\le 2^{\frac{m}{2}} \sum \sup_{|x|=R_1} |Q_t(x)| \left|f_{it}\right| \\ &\le 2^{\frac{m}{2}} \sum_t B_t(R_1) \sqrt{\frac{n!}{(m)_n}} \left(\frac{F(aR_3)}{(aR_3)^t} \right) \\ &< 2^{\frac{m}{2}} \sum_t k_3 (aR_2)^t \sqrt{\frac{n!}{(m)_n}} \left(\frac{F_i(aR_3)}{(aR_3)^t} \right) \\ &= 2^{\frac{m}{2}} \sqrt{\frac{n!}{(m)_n}} k_3 F_i(aR_3) \sum_t \left(\frac{aR_2}{aR_3} \right)^t. \end{aligned}$$

Thus $N_i(R_1) < k_4 F_i(aR_3)$. Now (5), (6) and (7) lead to

$$\begin{aligned} \Phi_n(R_1) &< 2^{\frac{m}{2}} k_4 \sum_i F_i(aR_3) \, |\lambda_{ni}| \\ &< 2^{\frac{m}{2}} k_4 \sum_i k_2 (aR_4)^i \, |\lambda_{ni}| \\ &< 2^{\frac{m}{2}} k_4 k_2 \sum_i k_1 B_i(R_5) \, |\lambda_{ni}| \\ &< k_5 N_n \max_{i=0}^{\infty} B_i(R_5) \, |\lambda_{ni}| \\ &< k_5 N_n \theta_n(R_5) \end{aligned}$$

Therefore, $\tau(R_1) \le \sigma(R_5)$. Making $R_5 \to R^+$, $R_1 \to R^+$, we get $\tau(R^+) \le \sigma(R^+)$. Since $\{Q_n(x)\}$ is effective in $B(R^+)$, $\sigma(R^+) = R$. Hence $\tau(R^+) = R$. The Theorem is therefore established. □

Theorem 3.2. *Let $\{P_n(x)\}$ be a base effective in the open ball $B(aR)$. Let $\{Q_n(x)\}$ be a base effective in the open ball $B(R)$ and such that*

$$\mu(r) < aR, \quad \forall r < R \tag{11}$$

$$\nu(R^-) \ge aR. \tag{12}$$

Then the product base $\{U_n(x)\}$ is effective in $B(R)$.

Proof. Similarly, the proof of theorem 3.2 will be very parallel to the proof of theorem 3.1. □

4. Order's bounds of the product base

The upper and lower bounds of the order of the product base, in terms of the orders of the factor bases are given by the following theorems.

Theorem 4.1. *Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be two simple bases of special monogenic polynomials, of order ω_P and ω_Q respectively. Also let*

$$\frac{\log\left|q_{nn}\right|}{n\log n} \to 0 \quad as\ n \to \infty.$$

Then the product base $\{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}$ is of order $\omega_U \leq \omega_P + 2\omega_Q$.

Theorem 4.2. *Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be two simple bases of special monogenic polynomials, of order ω_P and ω_Q respectively, where*

$$\omega_P > 2\omega_Q \geq 0.$$

Also, let

$$\frac{\log\left|q_{nn}\right|}{n\log n} \to 0 \qquad as \quad n \to \infty.$$

Then the product base $\{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}$ is of order

$$\omega_U \geq \omega_P - 2\omega_Q$$

Theorem 4.3. *Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be two simple bases of special monogenic polynomials, of orders ω_P and ω_Q respectively where*

$$\omega_Q > 2\omega_P \geq 0.$$

Also, let

$$\frac{\log\left|p_{nn}\right|}{n\log n} \to 0, \quad \frac{\log\left|q_{nn}\right|}{n\log n} \to 0 \quad as \quad n \to \infty.$$

Then the product base $\{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}$ is of order $\omega_U \geq \frac{1}{2}\omega_Q - \omega_P$.

Remark 4.1.

(1) *Theorem 4.1 generalizes a result of Nassif [21] to the Clifford setting.*

(2) *The original forms of theorems 4.2 and 4.3 in complex analysis are due to Ewida [14]. Our results here (theorems 4.2 and 4.3) are not only generalizations of those of Ewida [14], but also are extension to more general classes of bases of special monogenic polynomials.*

(3) *Also in complex case, Mikhail [19] has proved (using his own method) the analogue of theorems 4.1, 4.2, and 4.3 considering the condition $q_{nn} = o(n^{\hbar}), \hbar$ finite, in theorems 4.1, 4.2, and 4.3, and the corresponding condition $p_{nn} = o(n^{\hbar})$ in theorem 4.3.*

(4) *We shall improve these conditions given in the above statements of* (3) *by introducing more general ones. With this in hand, and using the analogue of Mikhail's method in complex case, we shall investigate in the present work, the extent of generalization to Clifford case, of the results in ([14, 19, 21]), as stated in theorems 4.1 4.2, and 4.3.*

Now, In order to prove these results (theorems 4.1, 4.2, and 4.3), an interesting lemma is given first which in fact is the generalization of the one given by Mikhail [18].

Lemma 4.1. *Let $\{P_n(x)\}$ and $\{Q_n(x)\}$be two bases of special monogenic polynomials for which*

$$\limsup_{n\to\infty}\frac{D(n)}{n}=\frac{1}{2^m}D_P,\ \frac{1}{2^m}D_Q \text{ respectively.}$$

Then the product base $\{U_n(x)\}=\{Q_n(x)\}\{P_n(x)\}$ is of order

$$\omega_U\le 2^m\limsup_{n\to\infty}\frac{\log\max_k|\lambda_{nk}|}{n\log n}+\omega_p D_Q+D_pD_Q\limsup_{n\to\infty}\frac{\log\max_j\left|q_{nj}\right|}{n\log n}.$$

Proof. For

$$\begin{aligned}\omega_U &= \frac{\limsup_{n\to\infty}\log\max_{i,j}|\delta_{ni}|\left|u_{ij}\right|}{n\log n}\\ &= \frac{\limsup_{n\to\infty}\max|\sum\pi_{ki}\lambda_{nk}|\left|\sum q_{tj}p_{it}\right|}{n\log n}\end{aligned}$$

That k ranges from 0 to $D_Q(n)$ at most, and t ranges from 0 to $D_p(k)$ at most.

$$\omega_U\le 2^m\limsup_{n\to\infty}\frac{\log\{[D_Q(n)+1][D_P(D_Q(n)+1)+1]\}\max_{i,j}|\pi_{ki}|\left|p_{it}\right|\cdot|\lambda_{nk}|\left|q_{tj}\right|}{n\log n}$$

Since

$$\limsup_{n\to\infty}\frac{k}{n}\le\limsup_{n\to\infty}\frac{D_Q(n)}{n}=\frac{1}{2^m}D_Q$$

and

$$\begin{aligned}\limsup_{n\to\infty}\frac{t}{n}&\le\limsup_{n\to\infty}\frac{D_p(k)}{n}\\ &\le\limsup_{n\to\infty}\left(\frac{D_p(D_Q(n))}{D_Q(n)}\cdot\frac{D_Q(n)}{n}\right)\\ &\le\frac{1}{2^m}D_p\cdot\frac{1}{2^m}D_Q,\end{aligned}$$

it follows that

$$\begin{aligned}\omega_U \le\ & 2^m\limsup_{n\to\infty}\frac{\log[D_Q(n)+1][D_p(D_Q(n)+1)]}{n\log n}\\ &+2^m\limsup_{n\to\infty}\frac{\log\max_k|\lambda_{nk}|}{n\log n} && (1^{st}\text{term})\\ &+2^m\limsup_{n\to\infty}\frac{\log\max_i\left|p_{it}\right||\pi_{ki}|}{n\log n} && (2^{nd}\text{term})\\ &+2^m\limsup_{n\to\infty}\frac{\log\max_j\left|q_{tj}\right|}{n\log n} && (3^{rd}\text{term})\end{aligned}$$

We note that:
The 2^{nd} term

$$2^m \limsup_{n\to\infty} \frac{\log \max_{i,j} |p_{it}| |\pi_{ki}|}{k \log k} \frac{k \log k}{n \log n} \leq 2^m \omega_p \frac{D_Q}{2^m} = \omega_P D_Q$$

The 3^{rd} term

$$\begin{aligned} 2^m \limsup_{n\to\infty} \frac{\log \max_{j} |q_{tj}|}{t \log t} \cdot \frac{t \log t}{n \log n} &\leq 2^m \frac{D_P}{2^m} \cdot \frac{D_Q}{2^m} \limsup_{n\to\infty} \frac{\log \max_{j} |q_{nj}|}{t \log t} \\ &\leq D_P D_Q \limsup_{n\to\infty} \frac{\log \max |q_{nj}|}{n \log n} \end{aligned}$$

Then

$$\omega_U \leq 2^m \limsup_{n\to\infty} \frac{\log \max_{k} |\lambda_{nk}|}{n \log n} + \omega_P D_Q + D_P D_Q \limsup_{n\to\infty} \frac{\log \max_{j} |q_{nj}|}{n \log n}$$

Hence the required result is already proved.

Proof of theorems 4.1, 4.2, and 4.3.

From the above lemma we can easily deduce the above three results (theorems 4.1 4.2, and 4.3) as follows.

Since the bases considered are simple ones then

$$D_P = D_Q = 1$$

Hence by the above lemma we have:

$$\omega_U \leq 2^m \limsup_{n\to\infty} \frac{\log \max_{k} |\lambda_{nk}|}{n \log n} + \omega_P + \limsup_{n\to\infty} \frac{\log \max_{k} |q_{nj}|}{n \log n} \tag{13}$$

But

$$\begin{aligned} 2^m \limsup_{n\to\infty} \frac{\log \max_{k} |\lambda_{nk}|}{n \log n} &\leq 2^m \limsup_{n\to\infty} \frac{\log \max_{k(n)} |\lambda_{n,k(n)}|}{n \log n} \\ &= 2^m \limsup_{n\to\infty} \frac{\log \max |\lambda_{n,D_Q(n)}| |q_{nn}|}{D_Q(n) \log D_Q(n)} \cdot \frac{D_Q(n) \log D_Q(n)}{n \log n} \\ &\leq 2^m \omega_Q \frac{D_Q}{2^m} = \omega_Q D_Q \end{aligned} \tag{14}$$

Since

$$\frac{\log |q_{nn}|}{n \log n} \to 0 \quad \text{as} \quad n \to \infty, \quad k(n) \leq n$$

It follows that

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log \max_{j} |q_{nj}|}{n \log n} &\leq \limsup_{n\to\infty} \frac{\log |q_{n,j(n)}|}{n \log n} \\ &= \limsup_{n\to\infty} \frac{\log |\lambda_{nn}| |q_{n,\, j(n)}|}{n \log n} = \omega_Q. \end{aligned}$$

Since $\log|\lambda_{nn}| = -\log|q_{nn}|$ and the base $\{Q_n(x)\}$ being a simple monic, it follows that

$$\limsup_{n\to\infty} \frac{\log \max |q_{nj}|}{n \log n} \le \omega_Q. \tag{15}$$

Substituting from (14) and (15) into (13) we obtain

$$\omega_U \le \omega_P + 2\omega_Q.$$

Writing $\{P_n(x)\} = \{\bar{Q}_n(x)\}\{U_n(x)\}$, where $\{\bar{Q}_n(x)\}$ is the inverse base of $\{Q_n(x)\}$, we have

$$\begin{aligned}
\omega_P &\le \limsup_{n\to\infty} \frac{\log \max |q_{nk}|}{n \log n} + \omega_U + \limsup_{n\to\infty} \frac{\log \max |\lambda_{nj}|}{n \log n} \\
&= \limsup_{n\to\infty} \frac{\log |\lambda_{nn}| \, |q_{n,k(n)}|}{n \log n} + \omega_U + \limsup_{n\to\infty} \frac{\log |\lambda_{n,j(n)}| \, |q_{j(n),j(n)}|}{n \log n}
\end{aligned}$$

Hence $\omega_U \ge \omega_P - 2\omega_Q$. Writing$\{Q_n(x)\} = \{U_n(x)\}\{\bar{P}_n(x)\}$, theorem 4.1 gives

$$\omega_Q \le \bar{\omega}_Q + 2\omega_U.$$

But by theorem (2) of [8] we have $\bar{\omega}_P \le 2\omega_P$. Therefore $\omega_Q \le 2\omega_P + 2\omega_U$ and then $\omega_U \ge \frac{1}{2}\omega_Q - \omega_P$. □

We note that the upper and lower bounds of the order ω_U, given in the above theorems. are all attainable. These facts are illustrated by the following two examples.

Example 4.1. *Let*

$$P_n(x) = \begin{cases} 1 + n^n z_{n-1}(x) + z_n(x) & \text{if } n \text{ is odd} \\ z_n(x) & \text{if } n \text{ is even} \end{cases}$$

and

$$Q_n(x) = \begin{cases} z_n(x) & \text{if } n \text{ is odd} \\ 1 + n^{3n} z_{n-1}(x) + z_n(x) & \text{if } n \text{ is even} \\ 1 & \text{if } n = 0 \end{cases}$$

It is easy to see that

$$U_n(x) = \begin{cases} 1 + n^n + n^n(n-1)^{3(n-1)} z_{n-2}(x) + n^n z_{n-1}(x) + z_n(x) & \text{if } n \text{ is odd} \\ 1 + n^{3n} z_{n-1}(x) + z_n(x) & \text{if } n \text{ is even} \end{cases}$$

We easily verify that $\omega_P = 1$, $\omega_Q = 3$, $\omega_U = 7$. *Thus* $\omega_U = \omega_P + 2\omega_Q$. *It is also clear that* $\bar{\omega}_Q = 3$. *Writing* $\{P_n(x)\} = \{U_n(x)\}\{\bar{Q}_n(x)\}$, *and noticing that*

$$\omega_P = 1 = \omega_U - 2\bar{\omega}_Q$$

gives the fact that the lower bound in theorem 4.2 is attainable.

Example 4.2. *Suppose that*

$$P_n(x) = \begin{cases} z_n(x) - n^{\omega_1 n} z_{n-1}(x) + n^{n\omega_1} z_{n-2}(x) & \text{if } n \text{ is even} \\ z_n(x) - z_{n-1}(x) & \text{if } n \text{ is odd} \end{cases}$$

and

$$Q_n(x) = \begin{cases} z_n(x) + n^{n\omega_1} z_{n-1}(x) + n^{n\omega_1}(n-2)^{(n-2)(\frac{\omega_2}{2}-\omega_1)} z_{n-1}(x) & \textit{if n is even} \\ z_n(x) + \{(n-1)^{(n-1)(\frac{\omega_2}{2}-\omega_1)} + 1\} z_{n-1}(x) + (n-1)^{(n-1)\omega_1} z_{n-2}(x) & \\ \quad +(n-1)^{(n-1)\omega_1}(n-3)^{(n-3)(\frac{\omega_2}{2}-\omega_1)} z_{n-3}(x) & \textit{if n is odd} \end{cases}$$

Then

$$U_n(x) = \begin{cases} z_n(x) & \textit{if n is even} \\ {}_n(x) + (n-1)^{(n-1)(\frac{\omega_2}{2}-\omega_1)} z_{n-1}(x) & \textit{if n is odd} \end{cases}$$

The bases $\{P_n(x)\}$, $\{Q_n(x)\}$ and $\{U_n(x)\}$ are respectively of orders ω_1, ω_2 and $-\omega_1$.

5. Growth order and type of the product base

The main result is the following:

Theorem 5.1. *Let $\{P_n(x)\}$ be a base of special monogenic polynomials of order F and type Γ. Let $\{Q_n(x)\}$ be a simple base satisfying*

$$0 < a \leq \liminf_{n\to\infty} \left|q_{nn}\right|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left|q_{nn}\right|^{\frac{1}{n}} \leq b < \infty \tag{16}$$

and such that

$$\frac{\theta(R)}{R} \to c, \quad as \quad R \to \infty,$$

where $\theta(R)$ corresponds to $\lambda(R)$. Then the product base $\{U_n(x)\}$ is of increase less than order F, type $\Gamma(\frac{c}{a})^{\frac{1}{F}}$.

Proof. From (16) we get

$$(a_1 R)^n < k_1 B_n(R), \;\; a_1 < a, \;\; \text{for all } n, \;\; \text{and all } R. \tag{17}$$

$$B_n(R) < k_2 (b_1 R)^n, \;\; b_1 > b, \; \text{for all } n, \;\; \text{and all } R. \tag{18}$$

Also from the definition of the order and type, we have for a general base the number $F = \lim_{R\to\infty} F(R)$ is defined as the order of the base $\{P_n(x)\}$, (see [3]), where

$$F(R) = \limsup_{n\to\infty} \frac{\log F_n(R)}{n \log n}$$

is called the order on the ball $\bar{B}(R)$.

When $0 < F < \infty$ the base is also said to be of type Γ, where.

$$\Gamma = \lim_{R\to\infty} \Gamma(R) = \lim_{n\to\infty} \frac{e}{F} \limsup_{n\to\infty} \frac{\{F_n(R)\}^{\frac{1}{nF}}}{n}.$$

Then we get

$$F_n(R) < k_3 (\frac{n\Gamma_1 F}{e})^{nF}, \;\; \Gamma_1 > \Gamma \;\; \forall n \text{ and } R. \tag{19}$$

Hence as in the proof of theorem 3.1 we have

$$\begin{aligned} N_i(R) &\leq 2^{\frac{m}{2}} \sum_t B_t(R) \sqrt{\frac{n!}{(m)_n}} (\frac{F_i(b_2 R)}{(b_2 R)^t}) \\ &< 2^{\frac{m}{2}} \sum_t k_2 (b_1 R)^t \sqrt{\frac{n!}{(m)_n}} (\frac{F_i(b_2 R)}{(b_2 R)^t}) \end{aligned}$$

Taking $b_2 > b_1$ then

$$N_i(R) < k_4 F_i(b_2 R) \tag{20}$$

Hence, from (7), (19), and (20) we obtain

$$\begin{aligned} \Phi_n(R) &< 2^{\frac{m}{2}} k_4 \sum F_i(b_2 R) \, |\lambda_{ni}| \\ &< 2^{\frac{m}{2}} k_4 \sum_i k_3 (\frac{i\Gamma_1 F}{e})^{iF} \, |\lambda_{ni}| \\ &< 2^{\frac{m}{2}} k_3 k_4 \sum_i (\frac{i\Gamma_1 F}{e})^{iF} \frac{1}{(a_1 R)^i} \, |\lambda_{ni}| \, (a_1 R)^i \\ &< k_5 (\frac{n\Gamma_1 F}{e})^{nF} \frac{1}{(a_1 R)^n} \sum_i |\lambda_{ni}| \, (a_1 R)^i, \end{aligned}$$

where $(\frac{i\Gamma_1 F}{e})^{iF} \frac{1}{(a_1 R)^i} \div (\frac{n\Gamma_1 F}{e})^{nF} \frac{1}{(a_1 R)^n} < 1$, however large is R, for sufficiently large n.

Substituting for $(a_1 R)^i$ from (17), we get

$$\begin{aligned} \Phi_n(R) &< k_1 k_5 \{\frac{n\Gamma_1 F}{e}\}^{nF} \frac{1}{(a_1 R)^n} \sum_i |\lambda_{ni}| \, B_i(a_1 R) \\ &< k_6 (\frac{n\Gamma_1 F}{e})^{nF} \frac{1}{(a_1 R)^n} N_n \max_{i=0}^{\infty} B_i(a_1 R) \, |\lambda_{ni}| \\ &< k_6 \{\frac{n\Gamma_1 F}{e}\}^{nF} \frac{1}{(a_1 R)^n} N_n \theta_n(R) \end{aligned}$$

If follows immediately that $\{U_n(x)\}$ is of order$\leq F$, and if the order is F, the type $\leq \{\frac{\theta(R)}{a_1 R}\}^{\frac{1}{F}} \Gamma_1$i.e. the type $\leq ((\frac{e}{a_1})^{\frac{1}{F}} \Gamma_1)$.

Since Γ_1 is arbitrarily $> \Gamma$ and a_1 is arbitrarily$< a$, it follows that the type $< (\frac{e}{a})^{\frac{1}{F}} \Gamma$, which is the required result. □

Remark 5.1.

(1) *It is worthy to mention here that the above result (theorem 5.1)is the extent of generalization of the one given by Nassif [23].*
(2) *The original proof in complex case of Nassif [23] is in fact not easy reading and covers only Cannon bases but our result here (theorem 5.1)covers the case of general base in Clifford setting.*
(3) *Our proof here is much easier to the one given in complex case introduced by Nassif [23].*

From theorem 5.1 we deduce the following important result.

Corollary 5.1. *If $\{P_n(x)\}$ is a base of order F, type Γ, and $A \neq 0$, B are any real constants, then $\{P_n(Ax + B\}$ is a base of order F and type $|A|^{-\frac{1}{F}} \Gamma$.*

Proof. Using the relation (2), it is seen at once that

$$\begin{aligned}\{P_n(Ax+B)\} &= \{U_n(x)\} = \{Q_n(x)\}\{P_n(x)\}\\ \{Q_n(x)\} &= \{z_n(Ax+B)\}.\end{aligned}$$

Thus the base $\{Q_n(x)\}$ is a simple base and hence the product base $\{P_n(Ax+B)\}$ is a base.
Also the base $\{Q_n(x)\}$ is effectively equivalent to $\{z_n(x+\frac{B}{A})\}$. For the later base

$$z_n(x) = z_n\left(\left(x+\frac{B}{A}\right)-\frac{B}{A}\right).$$

Hence $\theta_n(R)$ which is the same for the two bases, is equal to

$$\theta_n(R) = \frac{(m)_n}{n!}2^n\sum_i\left(\begin{array}{c}n\\ i\end{array}\right)\left(R+\left|\frac{B}{A}\right|\right)^i\left|\frac{B}{A}\right|^{n-i} = \frac{(m)_n}{n!}\left(R+2\left|\frac{B}{A}\right|\right)^n ,$$

where $\sup\limits_{|x|=R}|z_n(x)| = \frac{(m)_n}{n!}R^n$ thus $c = 1$. Lastly, $q_{nn} = A^n$ for all n that $q_{nn}^{\frac{1}{n}} = A$ for all n. Now, applying theorem 5.1, the base $\{U_n(x)\}$ is of order $F^* \leq F$. But the base $\{U_n(x)\}$, may be written as $\{P_n(\frac{x}{A}-\frac{B}{A})\}$ and hence $F \leq F^*$. Thus $F^* = F$. Again the type of $\{U_n(x)\}$ by theorem 5.1 is $\Gamma^* \leq A^{-\frac{1}{F}}\Gamma$. Also $\Gamma \leq A^{\frac{1}{F}}\Gamma^*$. Hence

$$\Gamma^* = A^{-\frac{1}{F}}\Gamma.$$

□

Remark 5.2. *The result in the above corollary generalizes the one given by Abul-Ez and Constales in [3], who dealt only the case of Cannon bases.*

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