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Convolution of two Harmonic Mappings in the Right-half Plane

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Abstract. This paper is to give a univalent criterion and a geometric property of the convolution of two right half-plane harmonic mappings $f_0(z)$ and $f(z)$, where $f_0(z)$ is canonical and the second complex dilatation $w(z)$ of $f(z)$ is of the form $w(z) = -\frac{z-a}{1-az}\frac{z-b}{1-bz}$.

1. Introduction

Let $\mathcal{H}$ be the class of complex-valued harmonic mappings $f = u + iv$ defined in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and normalized by $f(0) = 0$, $f_z(0) = 1$, where u and v are real harmonic in $\mathbb{D}$. Such functions can be expressed as $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.$$

A harmonic mapping f is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $J_f = |h'|^2 - |g'|^2 > 0$ in $\mathbb{D}$; or equivalently if $h' \neq 0$ in $\mathbb{D}$ and the dilatation $w = g'/h'$ has the property that $|w| < 1$ in $\mathbb{D}$ [10]. Let S_H be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving functions. Let S_H^0 be the subclass of all $f \in S_H$ with $f_{\bar{z}}(0) = 0$ [2]. Furthermore, K_H^0 be the subclass of S_H^0 mapping $\mathbb{D}$ onto convex domains. A domain Ω is said to be convex in the direction γ if for all $a \in \mathbb{C}$, the set $\Omega \cap \{a + te^{i\gamma} : \gamma, t \in \mathbf{R}\}$ is either connected or empty. In particular, a domain is convex in the direction of the real (or imaginary) axis if every line parallel to the real (or imaginary) axis has a connected intersection with the domain.

For two analytic functions $f = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathbb{D}$ and $F = z + \sum_{n=2}^{\infty} A_n z^n$ in $\mathbb{D}$, their convolution is defined as $f * F = z + \sum_{n=2}^{\infty} a_n A_n z^n$ in $\mathbb{D}$. For two harmonic functions $f = h + \bar{g} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\overline{z^n}$ in $\mathbb{D}$ and $F = H + \overline{G} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n}\overline{z^n}$ in $\mathbb{D}$, we define their harmonic convolution as

$$f * F = h * H + \overline{g * G} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n}\overline{z^n}, \quad z \in \mathbb{D}.$$

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A harmonic function $f = h + \bar{g} \in S_H^0$ is called a right half-plane harmonic mapping, if f maps the unit disk $\mathbb{D}$ onto $R = \{w : \Re(w) > -1/2\}$, such a mapping satisfies $h + g = \frac{z}{1-z}$ in $\mathbb{D}$. We denote the class of right half-plane harmonic mappings by R_H^0 which satisfies $R_H^0 \subseteq K_H^0$.

The canonical harmonic right half-plane mapping, $f_0 = h_0 + \overline{g_0} \in R_H^0$ has a second complex dilatation $w_0 = -z$. The technique of shear construction [5] yields

$$h_0(z) = \frac{z - z^2/2}{(1-z)^2} = z + \sum_{n=2}^{\infty} \frac{1+n}{2} z^n = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right), \quad z \in \mathbb{D}$$

and

$$g_0(z) = \frac{-z^2/2}{(1-z)^2} = \sum_{n=2}^{\infty} \frac{1-n}{2} z^n = \frac{1}{2}\left(\frac{z}{1-z} - \frac{z}{(1-z)^2}\right), \quad z \in \mathbb{D}.$$

If $f = h + \bar{g} \in \mathcal{H}$, then

$$f_0 * f = h_0 * h + \overline{g_0 * g} = \frac{h + zh'}{2} + \overline{\frac{g - zg'}{2}}, \quad z \in \mathbb{D}. \tag{1}$$

Univalent criterions and geometric characterizations of the convolution of two harmonic mappings have attracted one's much attention [3,4,8,10-14]. Especially, when their dilations are specified, there are some recent results. Among them, Dorff [3] proved

Theorem A. *Let $f_1, f_2 \in R_H^0$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in S_H^0$ and is convex in the direction of the real axis.*

There were lots of works on concrete conditions determined locally univalence of their convolutions of two harmonic mappings. Dorff, Nowak and Woloszkiewicz [4] proved the following lemma and theorems.

Lemma A. *Let $f = h + \bar{g} \in R_H^0$ with the dilatation $w(z)$. Then the dilatation $\tilde{w}$ of $f_0 * f$ is*

$$\tilde{w}(z) = -z \frac{w^2(z) + w(z) - \frac{1}{2} z w'(z) + \frac{1}{2} w'(z)}{1 + w(z) - \frac{1}{2} z w'(z) + \frac{1}{2} z^2 w'(z)}. \tag{2}$$

Theorem B. *Let $f = h + \bar{g} \in R_H^0$ with $h(z) + g(z) = \frac{z}{1-z}$ and $w(z) = \frac{g'(z)}{h'(z)} = e^{i\theta} z^n$ $n \in \mathbf{Z}^+$ and $\theta \in \mathbf{R}^+$. If $n = 1, 2$, then $f_0 * f \in S_H^0$ and is convex in the direction of the real axis.*

Theorem C. *Let $f = h + \bar{g} \in R_H^0$ with $h(z) + g(z) = \frac{z}{1-z}$ and $w(z) = \frac{z+a}{1+az}$ with $a \in (-1, 1)$. Then $f_0 * f \in S_H^0$ and is convex in the direction of the real axis.*

Recently, Li and Ponnusamy [12] generalized Theorem C, as follows

Theorem D. *Let $f = h + \bar{g} \in R_H^0$ with the dilatation $w(z) = \frac{z+a}{1+\bar{a}z}$, $|a| < 1$. Then $f_0 * f \in S_H^0$ and is convex in the direction of the real axis if and only if*

$$(\Re(a))^2 + 9(\Im(a))^2 \leq 1 \text{ and } \Re(a) \neq \pm 1.$$

Jiang, Rasila and Sun[8] considered Theorem D in a more general setting by allowing a rotation parameter θ in the second complex dilation $w(z)$ of $f(z)$.

Theorem E. *Let $f = h + \bar{g} \in R_H^0$ with the dilatation $w(z) = e^{i\theta} \frac{z+a}{1+\bar{a}z}$, where $a = |a| e^{i\alpha}$, $\alpha = \arg a$, $|a| < 1$ and $\theta \in \mathbf{R}$. If*

$$[9\sin^2(\alpha + \frac{\theta}{2}) + \cos^2(\alpha + \frac{\theta}{2})]|a|^2 \leq 1 \quad \text{and} \quad |a|\cos(\alpha + \frac{\theta}{2}) \neq -\cos(\frac{\theta}{2}),$$

then $f_0 * f \in S_H^0$ and is convex in the direction of the real axis.

A Blaschke product is a product of the type

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{(a_n - z)|a_n|}{(1 - \bar{a}_n z)a_n},$$

where k is a nonnegative integer, the sequence a_n satisfies that $0 < |a_n| < 1$ for each n, and the series $\sum_{n=1}^{\infty}(1 - |a_n|)$ converges. The function $B(z)$ is bounded and analytic in $\mathbb{D}$. The zeros of $B(z)$ are just the numbers a_n and 0 (if $k > 0$). Specially, if a Blaschke product is of the form

$$B(z) = c \prod_{n=1}^{m} \frac{a_n - z}{1 - \bar{a}_n z},$$

where $c \in \partial\mathbb{D} = \{z : |z| = 1\}$, m is a nonnegative integers, then we say that $B(z)$ is a finite Blaschke product of degree m.

Blaschke products have been applied to many research fields, for instance, Kraus and Gorkin[1], Dallakyan[6], Hamada[7], Akeroyd and Roth[9].

In [8,12,13], the authors studied the convolution of a canonical right half-plane harmonic mapping and a right-plane harmonic mapping with a second complex dilation of a finite Blaschke product of degree one or a Blaschke product of degree two with an $a_n = 0$. In this paper, we will consider the case of Blaschke products of degree two with all $|a_n| \neq 0$, that is, $B(z) = -\frac{z-a}{1-az}\frac{z-b}{1-bz}$, and obtain our main result.

Theorem 1.1. *Let $f = h + \bar{g} \in R_H^0$ with the dilatation $w(z) = -\frac{z-a}{1-az}\frac{z-b}{1-bz}$, where $|a| < 1$, $|b| \leq 1$, $(a, b \in \mathbf{R})$. If*

$$0 < a + b - 2ab < 2, \tag{3}$$

then $f_0 * f \in S_H^0$ and is convex in the direction of the real axis.

2. Preliminary lemmas and their proofs

Lemma 2.1. *Let f_0 be the canonical harmonic right half-plane mapping. If $f = h + \bar{g} \in R_H^0$ with the dilatation $w(z) = -\frac{z-a}{1-az}\frac{z-b}{1-bz}$, then the dilatation $\tilde{w}$ of the convolution $f_0 * f$ is*

$$\tilde{w}(z) = z\frac{(z + A)(z + B)(z + C)}{(1 + \bar{A}z)(1 + \bar{B}z)(1 + \bar{C}z)} = z\frac{t(z)}{t^*(z)}, \tag{4}$$

where

$$t(z) = z^3 + \frac{1}{2}(2 - 3a - 3b)z^2 + (1 - a - b + 2ab)z - \frac{1}{2}(a + b - 2ab), \tag{5}$$

and

$$t^*(z) = 1 + \frac{1}{2}(2 - 3a - 3b)z + (1 - a - b + 2ab)z^2 - \frac{1}{2}(a + b - 2ab)z^3. \tag{6}$$

Here $-A$,$-B$,$-C$ are the three roots of the equation $t(z) = 0$, and A,B,C may be equal.

Proof. Let $w(z) = -\frac{z-a}{1-az}\frac{z-b}{1-bz}$, then $w'(z) = \frac{(-1+a^2)(b-z)}{(1-az)^2(1-bz)} + \frac{(a-z)(-1+b^2)}{(1-az)(1-bz)^2}$. We obtain from Lemma A that

$$\tilde{w}(z) = -z\frac{P(z)}{Q(z)},$$

where

$$\begin{aligned}P(z) =&(a+b-2ab-a^2b-ab^2+2a^2b^2)-(2-a-b+a^2b+ab^2-2a^2b^2)z+(a\\&+b+4ab-a^2b-ab^2-4a^2b^2)z^2-(3a+3b-3a^2b-3ab^2)z^3+(2-2ab)z^4]\\=&-(ab-1)(z-1)[-(a+b-2ab)+2(1-a-b+2ab)z+(2-3a-3b)z^2\\&+2z^3]\end{aligned}$$

and

$$\begin{aligned}Q(z) =&(a+b-2ab-a^2b-ab^2+2a^2b^2)z^4-(2-a-b+a^2b+ab^2-2a^2b^2)z^3\\&+(a+b+4ab-a^2b-ab^2-4a^2b^2)z^2-(3a+3b-3a^2b-3ab^2)z+(2-2ab)\\=&(ab-1)(z-1)[-(a+b-2ab)z^3+2(1-a-b+2ab)z^2+(2-3a-3b)z+2].\end{aligned}$$

Thus,

$$\begin{aligned}\tilde{w}(z)=-z\frac{P(z)}{Q(z)}&=z\frac{z^3+\frac{1}{2}(2-3a-3b)z^2+(1-a-b+2ab)z-\frac{1}{2}(a+b-2ab)}{1+\frac{1}{2}(2-3a-3b)z+(1-a-b+2ab)z^2-\frac{1}{2}(a+b-2ab)z^3}\\&=z\frac{t(z)}{t^*(z)}.\end{aligned} \tag{7}$$

Suppose that $-A$,$-B$,$-C$ are the three roots of $t(z)=0$. Then it follows

$$t(z)=(z+A)(z+B)(z+C)$$

and

$$t^*=z^3\overline{t(1/\bar{z})}=z^3\overline{(1/\bar{z}+A)(1/\bar{z}+B)(1/\bar{z}+C)}=(1+\bar{A}z)(1+\bar{B}z)(1+\bar{C}z).$$

□

Lemma 2.2. (1) *Let* $|a|\le 1$, $|b|\le 1$, *and* $a, b\in\mathbf{R}$. *Then* $-4\le a+b-2ab\le 2$.

(2) *Let* $\Psi=\{(a,b)|-1<a<0,-1<b<0\}$, $\Omega=\{(a,b)|0\le a+b-2ab\le 2,|a|\le 1,|b|\le 1\}$, $a, b\in\mathbf{R}$. *Then* $\Psi\bigcap\Omega=\phi$.

Proof. (1) Let $f(a,b)=a+b-2ab$, we have a stable point $(\frac{1}{2},\frac{1}{2})$, and $f(\frac{1}{2},\frac{1}{2})=\frac{1}{2}$. Furthermore, the values of $f(a,b)$ on the boundary as following

$$f(1,b)=1-b, f(-1,b)=-1+3b, f(a,1)=1-a, f(a,-1)=-1+3a.$$

Using the assumption that $|a|\le 1$ and $|b|\le 1$, we get $f(a,b)_{max}=2$, $f(a,b)_{min}=-4$.

(2) Suppose $(a,b)\in\Psi$, then $-1<a<0$, $-1<b<0$. Since $f(a,b)=a+b-2ab=a(1-b)+b(1-a)$, it shows that $f(a,b)<0$. If $(a,b)\in\Omega$ then $0\le f(a,b)\le 2$, a contradiction with $f(a,b)<0$. Hence $\Psi\bigcap\Omega=\phi$. □

Lemma 2.3. *Let* $0<a+b-2ab<2, |a|<1, |b|<1$, $a, b\in\mathbf{R}$. *Then*

$$-4+a+2a^2+b+2ab-5a^2b+2b^2-5ab^2+2a^2b^2<0. \tag{8}$$

Proof. Let $\Omega=\{(a,b)|0<a+b-2ab<2,|a|<1,|b|<1\}$, and
$\partial\Omega=\{(a,b)|a+b-2ab=0,a+b-2ab=2,|a|=1,|b|=1\}$,

$$f(a,b)=-4+a+2a^2+b+2ab-5a^2b+2b^2-5ab^2+2a^2b^2,$$

then $f(a,b)$ has a maximum and a minimum on the $\Omega\bigcup\partial\Omega$. The system

$$\begin{cases}f_a=1+4a+2b-10ab-5b^2+4ab^2=0\\f_b=1+2a-5a^2+4b-10ab+4a^2b=0\end{cases}$$

yields stable points $P_1(-1, \frac{1}{3})$, $P_2(\frac{1}{3}, -1)$ and the value of $f(a, b)$ at stable points $f(-1, \frac{1}{3}) = f(\frac{1}{3}, -1) = -4$. We will discuss below cases on the boundary

(1) If $a = 1$, then $-1 \leq b \leq 1$. Hence,

$$f(1, b) = -1 - 2b - b^2 = -(1 + b)^2 \leq 0,$$

equality holds if and only if $b = -1$.

(2) If $a = -1$, then $\frac{1}{3} \leq b \leq 1$. Thus,

$$f(-1, b) = -3 - 6b + 9b^2 = 3(3b + 1)(b - 1) \leq 0,$$

equality holds if and only if $b = 1$.

(3) If $b = 1$, then $-1 \leq a \leq 1$. So

$$f(a, 1) = -1 - 2a - a^2 = -(1 + a)^2 \leq 0,$$

equality holds if and only if $a = -1$.

(4) If $b = -1$, then $\frac{1}{3} \leq a \leq 1$. Therefore,

$$f(a, -1) = -3 - 6a + 9a^2 = 3(3a + 1)(a - 1) \leq 0,$$

equality holds if and only if $a = 1$.

(5) If $a + b - 2ab = 0$, then

$$\begin{aligned} f(a, b) &= -4 + a + 2a^2 + b + 2ab - 5a^2 b + 2b^2 - 5ab^2 + 2a^2 b^2 \\ &= -4 + (1 + 2a + 2b - ab)(a + b - 2ab) < 0. \end{aligned}$$

(6) If $a + b - 2ab = 2$, then $a = \frac{2-b}{1-2b}$. Since $|a| = |\frac{2-b}{1-2b}| \leq 1$, it shows that

$$(2 - b)^2 - (1 - 2b)^2 = 3(1 - b)(1 + b) \leq 0,$$

yields $b = 1, a = -1$ or $b = -1, a = 1$. Thus $f(a, b) = f(-1, 1) = f(1, -1) = 0$.

In summary, $f_{max} = 0$, $f_{min} = -4$. It's easy to see that the function $f(a, b)$ takes maximum only at points $(1, -1)$, $(-1, 1)$. Thus $f(a, b) < 0$ on the Ω. $\square$

In order to establish our main results, we need the following Cohn's Rule[15,p.375].

Lemma B. *For a given polynomial*

$$t(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

of degree n, let

$$t^*(z) = z^n \overline{t(1/\overline{z})} = \overline{a_n} + \overline{a_{n-1}} z + \overline{a_{n-2}} z^2 + \cdots + \overline{a_0} z^n.$$

Denote by r and s the number of zeros of $t(z)$ inside and on the unit circle $|z| = 1$, respectively. If $|a_0| < |a_n|$, then

$$t_1(z) = \frac{\overline{a_n} t(z) - a_0 t^*(z)}{z}$$

is of degree $n - 1$ and has $r_1 = r - 1$ and $s_1 = s$ number of zeros inside and on the unit circle $|z| = 1$, respectively.

We also need the following Schur-Cohn algorithm [15,p.383].

Lemma C. *For a given polynomial*

$$r(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

of degree n, let

$$M_k = det\begin{pmatrix} \overline{B_k}^T & A_k \\ \overline{A_k}^T & B_k \end{pmatrix}(k = 1, 2\cdots, n),$$

where A_k and B_k are the triangular matrices

$$A_k = \begin{pmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ & a_0 & \cdots & a_{k-2} \\ & & \ddots & \vdots \\ & & & a_0 \end{pmatrix}, \quad B_k = \begin{pmatrix} \bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_{n-k+1} \\ & \bar{a}_n & \cdots & \bar{a}_{n-k+2} \\ & & \ddots & \vdots \\ & & & \bar{a}_n \end{pmatrix}.$$

Then the polynomial $r(z)$ has all its zeros inside the unit circle $|z| = 1$ if and only if the determinants $M_1, M_2 \cdots, M_n$ are all positive.

3. Proof of Theorem 1.1 and a corollary

Proof. Let us consider two special cases of the second complex dilation $w(z)$. The case that $b = 1$ has already been proved by Dorff et al [5]. If $b = -1$, then $w(z) = -\frac{z-a}{1-az}$, $\frac{1}{3} < a < 1$. Thus, Lemma 2.1 yields

$$|\widetilde{w}(z)| = |z\frac{z + \frac{1-3a}{2}}{1 + \frac{1-3a}{2}z}| \leq |z| < 1.$$

Suppose that $-A$,$-B$,$-C$ are the three roots of $t(z) = 0$. We have form (2.2) that

$$\begin{aligned} t(z) &= z^3 + \frac{1}{2}(2 - 3a - 3b)z^2 + (1 - a - b + 2ab)z - \frac{1}{2}(a + b - 2ab) \\ &= (z + A)(z + B)(z + C). \end{aligned}$$

Thus $|ABC| = | - \frac{1}{2}(a + b - 2ab)| < 1$. Therefore, at lease one of $-A$, $-B$, $-C$ lies in $\mathbb{D}$. Applying the formula (2.2) and (2.3), we can establish the following formula

$$\begin{aligned} t_1(z) =& \frac{t(z) + \frac{1}{2}(a + b - 2ab)t^*(z)}{z} \\ =& -\frac{1}{4}[(-4 + a^2 + 2ab - 4a^2b + b^2 - 4ab^2 + 4a^2b^2)z^2 - 2(2 - 2a - a^2 - 2b - 4ab \\ &+ 4a^2b - b^2 + 4ab^2 - 4a^2b^2)z - (4 - 2a - 3a^2 - 2b - 2ab + 6a^2b - 3b^2 + 6ab^2)]. \end{aligned}$$

Denote by

$$J(a, b) = -\frac{1}{4}(-4 + a^2 + 2ab - 4a^2b + b^2 - 4ab^2 + 4a^2b^2)$$

and

$$K(a, b) = \frac{1}{4}(-4 + 2a + 3a^2 + 2b + 2ab - 6a^2b + 3b^2 - 6ab^2).$$

After direct calculation, we have

$$
\begin{aligned}
K^2(a,b) - J^2(a,b) =& \frac{1}{4}(1+a)(1+b)(a+b-2ab)(-4+a+2a^2+b+2ab \\
& -5a^2b+2b^2-5ab^2+2a^2b^2).
\end{aligned} \tag{9}
$$

Since $0 < a+b-2ab < 2, |a| < 1, |b| < 1$, then Lemma 2.3 shows that the formula (3.1) is negative.

We can use $t_1(z)$ to construct

$$
\begin{aligned}
t_2(z) =& \frac{\overline{J(a,b)}t_1(z) - K(a,b)t_1^*(z)}{z} \\
=& -\frac{1}{4}(1+a)(1+b)(a+b-2ab)[(-4+a+2a^2+b+2ab-5a^2b+2b^2-5ab^2 \\
& +2a^2b^2)z + (-2+2a+a^2+2b+4ab-4a^2b+b^2-4ab^2+4a^2b^2)].
\end{aligned}
$$

So $t_2(z) = 0$ has one zero at

$$
z_0 = \frac{-2+2a+a^2+2b+4ab-4a^2b+b^2-4ab^2+4a^2b^2}{-4+a+2a^2+b+2ab-5a^2b+2b^2-5ab^2+2a^2b^2}. \tag{10}
$$

Denote by

$$
M(a,b) = -2+2a+a^2+2b+4ab-4a^2b+b^2-4ab^2+4a^2b^2
$$

and

$$
N(a,b) = -4+a+2a^2+b+2ab-5a^2b+2b^2-5ab^2+2a^2b^2.
$$

Some calculation gives

$$
M^2(a,b) - N^2(a,b) = 3(-1+a^2)(-1+b^2)(-2-a-b+2ab)(2-a-b+2ab). \tag{11}
$$

Furthermore, the fact that $0 < a+b-2ab < 2$, $|a| < 1$, $|b| < 1$ implies that the formula (3.3) is negative and $|z_0| < 1$.

Therefore, by Lemma B, the roots of $t(z)$ are all in $\mathbb{D}$, that is $A, B, C \in \mathbb{D}$ and $|\widetilde{w}(z)| < 1$ for all $z \in \mathbb{D}$. □

We note that if the case $b = a$ in Theorem 1.1 is the following corollary.

Corollary 3.1. *Let f_0 be the canonical harmonic right half-plane mapping. If $f = h + \bar{g} \in R_H^0$ with the dilatation $w(z) = -(\frac{z-a}{1-az})^2$, where $0 \le a < 1$. Then $f_0 * f$ is convex in the direction of the real axis.*

Proof. By the second result of Lemma 2.2 we have that a, b are not negative at the same time, that is, if we take $a = b$, then the condition (1.3) degenerates to the inequality $0 \le a < 1$. Then Theorem 1.1 implies Corollary 3.1.

Here, we will give another method to prove this corollary.

By the formula (2.4), we have

$$
\begin{aligned}
\tilde{w}(z) =& z\frac{z^3+(1-3a)z^2+(1-2a+2a^2)z+(-a+a^2)}{1+(1-3a)z+(1-2a+2a^2)z^2+(-a+a^2)z^3} \\
=& -z\frac{(a-z)[1-a+(1-2a)z+z^2]}{(1-az)[(1-a)z^2+(1-2a)z+1]} \\
=& -z\frac{(a-z)}{(1-az)}\frac{r(z)}{r^*(z)} \\
=& -z\frac{(a-z)}{(1-az)}\frac{(z+D)(z+E)}{(1+\bar{D}z)(1+\bar{E}z)},
\end{aligned}
$$

where $-D$, $-E$ is the two roots of the $r(z) = 1 - a + (1-2a)z + z^2$, $r^*(z) = (1-a)z^2 + (1-2a)z + 1$. Furthermore,

$$t^* = z^2\overline{t(1/\bar{z})} = z^2\overline{(1/\bar{z} + D)(1/\bar{z} + E)} = (1 + \bar{D}z)(1 + \bar{E}z).$$

We will show that D, $E \in \mathbb{D}$. By Lemma C, we only need to show that the determinants M_1, M_2 are all positive for $a \in (0,1)$. In fact,

$$M_1 = det\begin{pmatrix} 1 & 1-a \\ 1-a & 1 \end{pmatrix} = a(2-a) > 0,$$

$$\begin{aligned} M_2 &= det\begin{pmatrix} 1 & 0 & 1-a & 1-2a \\ 1-2a & 1 & 0 & 1-a \\ 1-a & 0 & 1 & 1-2a \\ 1-2a & 1-a & 0 & 1 \end{pmatrix} \\ &= 3a^2(1-a^2) > 0. \end{aligned}$$

Therefore, $D, E \in \mathbb{D}$ and $|\frac{r(z)}{r^*(z)}| < 1$. As we all know $|\frac{a-z}{1-az}| < 1$, for $0 < a < 1$. Thus $|\widetilde{w}(z)| < 1$ for all $z \in \mathbb{D}$. In addition, when a=0, the dilatation $w(z) = -z^2$, which is a special case of $n = 2$, $\theta = \pi$ at Theorem B. □

4. Auxiliary examples

Example 4.1. *Let $f = h + \bar{g} \in R_H^0$ with the dilatation $w(z) = -(\frac{z-a}{1-az})^2$, where $0 \le a < 1$. Then*

$$h + g = \frac{z}{1-z}, \ \frac{g'}{h'} = -(\frac{a-z}{1-az})^2.$$

We obtain

$$\begin{aligned} h' &= \frac{(1-az)^2}{(1-a^2)(1+z)(1-z)^3} \\ &= \frac{1}{8(1-a^2)}[\frac{(1+a)^2}{1+z} + \frac{(1+a)^2}{1-z} + \frac{2(1+2a-3a^2)}{(1-z)^2} + \frac{4(1-a)^2}{(1-z)^3}], \end{aligned}$$

$$\begin{aligned} g' &= \frac{(a-z)^2}{(1-a^2)(1+z)(1-z)^3} \\ &= -\frac{1}{8(1-a^2)}[\frac{(1+a)^2}{1+z} + \frac{(1+a)^2}{1-z} + \frac{2(a^2+2a-3)}{(1-z)^2} + \frac{4(1-a)^2}{(1-z)^3}]. \end{aligned}$$

Integration from 0 to z gives

$$h(z) = \frac{1}{8(1-a^2)}[(1+a)^2\ln(\frac{1+z}{1-z}) + \frac{2(1+2a-3a^2)}{1-z} + \frac{2(1-a)^2}{(1-z)^2}] - \frac{1}{2},$$

$$g(z) = -\frac{1}{8(1-a^2)}[(1+a)^2\ln(\frac{1+z}{1-z}) + \frac{2(a^2+2a-3)}{1-z} + \frac{2(1-a)^2}{(1-z)^2}] - \frac{1}{2}.$$

Let $F = f_0 * f = h_0 * h + \overline{g_0 * g} = H + \overline{G}$. By the definition of convolution and the formula (1.1), we have

$$H = h_0 * h = \frac{h + zh'}{2}, \ G = g_0 * g = \frac{g - zg'}{2}.$$

So

$$H = \frac{1}{2}\{\frac{1}{8(1-a^2)}[(1+a)^2\ln(\frac{1+z}{1-z}) + \frac{2(1+2a-3a^2)}{1-z} + \frac{2(1-a)^2}{(1-z)^2}] + \frac{z(1-az)^2}{(1-a^2)(1+z)(1-z)^3} - \frac{1}{2}\},$$

$$G = \frac{1}{2}\{-\frac{1}{8(1-a^2)}[(1+a)^2\ln(\frac{1+z}{1-z}) + \frac{2(a^2+2a-3)}{1-z} + \frac{2(1-a)^2}{(1-z)^2}] + \frac{z(z-a)^2}{(1-a^2)(1+z)(1-z)^3} - \frac{1}{2}\}.$$

The real and imaginary parts of $F(z) = H + \bar{G}$ *can be written as follows*

$$\Re(F(z)) = \Re(H+G) = \Re[\frac{(a^2+1)z^2 - 4az + a^2 + 1}{2(1-a^2)(1+z)(1-z)^3}z + \frac{1}{2(1-z)} - \frac{1}{2}],$$

$$\Im(F(z)) = \Im(H-G) = \Im[\frac{1}{8(1-a^2)}((1+a)^2\ln(\frac{1+z}{1-z}) - \frac{2(1-a)^2}{1-z} + \frac{2(1-a)^2}{(1-z)^2} + \frac{z}{2(1-z)^2})].$$

Images of the unit disk $\mathbb{D}$ *under* f *and* $f_0 * f$ *(right) for* $a = 0.3$ *are drawn in Figure*1.

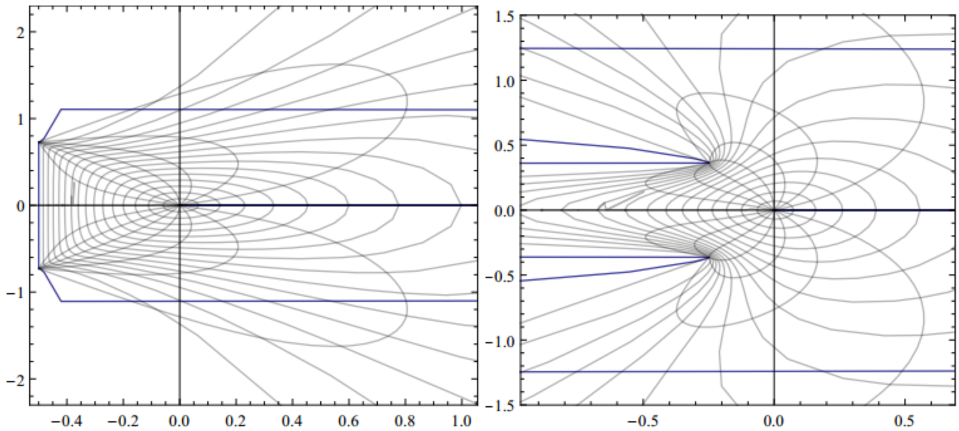

FIGURE 1. *Images of* f *and* $f_0 * f$ *(right) for* $a = 0.3$

Images of the unit disk $\mathbb{D}$ *under* f *and* $f_0 * f$ *(right) for* $a = 0.7$ *are drawn in Figure*2.

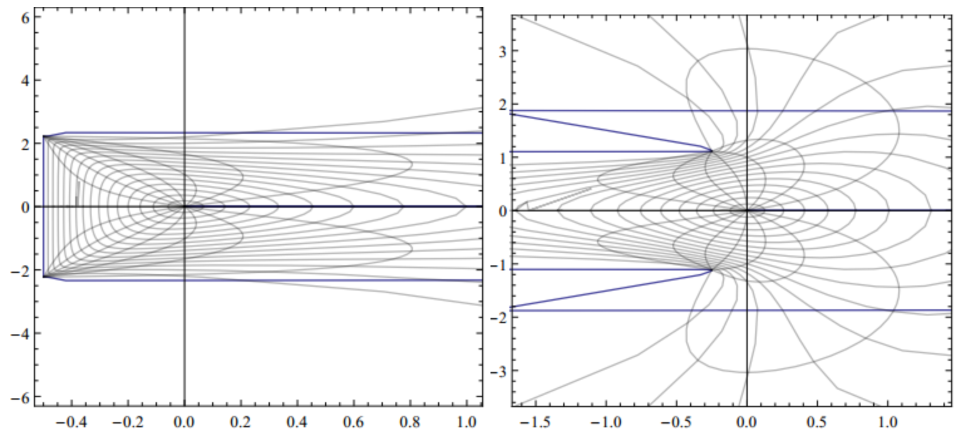

FIGURE 2. *Images of* f *and* $f_0 * f$ *(right) for* $a = 0.7$

Example 4.2. *Let $f = h + \bar{g} \in R_H^0$ with the dilatation $w(z) = -(\frac{\frac{1}{2}+z}{1+\frac{1}{2}z})^3$. Then*

$$h + g = \frac{z}{1-z}, \quad \frac{g'}{h'} = -(\frac{\frac{1}{2}+z}{1+\frac{1}{2}z})^3.$$

We obtain

$$h' = \frac{(2+z)^3}{(1-z)^3(7+13z+7z^2)} = \frac{1}{(1-z)^3} + \frac{2}{27(1-z)} + \frac{14z+13}{27(7z^2+13z+7)},$$

$$g' = \frac{-(1+2z)^3}{(1-z)^3(7+13z+7z^2)} = \frac{-1}{(1-z)^3} + \frac{1}{(1-z)^2} - \frac{2}{27(1-z)} - \frac{14z+13}{27(7z^2+13z+7)}.$$

Integration from 0 to z gives

$$h(z) = \frac{1}{2(1-z)^2} - \frac{2}{27}\ln(z-1) + \frac{1}{27}\ln(7+13z+7z^2) - \frac{1}{2} - \frac{\ln 7}{27},$$

$$g(z) = \frac{-1}{2(1-z)^2} + \frac{1}{1-z} + \frac{2}{27}\ln(z-1) - \frac{1}{27}\ln(7+13z+7z^2) - \frac{1}{2} + \frac{\ln 7}{27}.$$

Let $F = f_0 * f = h_0 * h + \overline{g_0 * g} = H + \overline{G}$. By the definition of convolution and the formula (1.1), we have

$$H = h_0 * h = \frac{h + zh'}{2}, \quad G = g_0 * g = \frac{g - zg'}{2}.$$

Thus,

$$H = \frac{z}{2(1-z)^3} + \frac{1}{4(1-z)^2} + \frac{1}{54}[\frac{2z}{1-z} + \frac{14z^2+13z}{7+13z+7z^2} + \ln\frac{7+13z+7z^2}{(z-1)^2} - \ln 7] - \frac{1}{4},$$

$$G = \frac{z}{2(1-z)^3} - \frac{z+\frac{1}{2}}{2(1-z)^2} + \frac{1}{54}[\frac{2z+27}{1-z} + \frac{14z^2+13z}{7+13z+7z^2} - \ln\frac{7+13z+7z^2}{(z-1)^2} + \ln 7] - \frac{1}{4},$$

and

$$H' = \frac{(2+z)^2(28+75z+90z^2+50z^3)}{(z-1)^4(7+13z+7z^2)^2}, \quad G' = \frac{z(1+2z)^2(50+90z+75z^2+28z^3)}{(z-1)^4(7+13z+7z^2)^2}.$$

Therefore, the dilatation $\widetilde{w}$ of the convolution $F = f_0 * f$ is

$$\widetilde{w} = \frac{G'}{H'} = z\frac{(1+2z)^2}{(2+z)^2}\frac{(50+90z+75z^2+28z^3)}{(50z^3+90z^2+75z+28)}.$$

When $z = -0.8$, the two formulas $50z^3 + 90z^2 + 75z + 28 = 0$, $50 + 90z + 75z^2 + 28z^3 = 11.664$. Thus, the dilatation $|\widetilde{w}|$ of the convolution $F = f_0 * f$ tends to infinity. Hence, F is not sense-preserving.

The real and imaginary parts of $F(z) = H + \bar{G}$ can be written as follows

$$\Re(F(z)) = \Re(H+G) = \frac{1}{2}\Re[\frac{2z}{(1-z)^3} - \frac{z}{(1-z)^2} + \frac{4z+27}{27(1-z)} + \frac{2(13z+14z^2)}{27(7+13z+7z^2)} - 1],$$

$$\Im(F(z)) = Im(H-G) = \frac{1}{2}\Im[\frac{z+1}{(1-z)^2} - \frac{1}{1-z} + \frac{2}{27}\ln\frac{7+13z+7z^2}{(z-1)^2}]$$

and

$$\frac{z}{(1-z)^3} = \frac{z(1-3\overline{z}+3\overline{z}^2-\overline{z}^3)}{|1-z|^6} = \frac{z-3|z|^2+3|z|^2\overline{z}+|z|^2\overline{z}^2}{|1-z|^6},$$

$$\frac{z}{(1-z)^2} = \frac{z(1-2\overline{z}+\overline{z}^2)}{|1-z|^4} = \frac{z-2|z|^2+|z|^2\overline{z}}{|1-z|^4},$$

$$\frac{13z+14z^2}{7+13z+7z^2} = \frac{91z+169|z|^2+91|z|^2\overline{z}+98|z|^2+182|z|^2z+98|z|^4}{|7+13z+7z^2|^2},$$

$$\frac{z+1}{(1-z)^2} = \frac{(z+1)(1-2\overline{z}+\overline{z}^2)}{|1-z|^4} = \frac{z-2|z|^2+|z|^2\overline{z}+1-2\overline{z}+\overline{z}^2}{|1-z|^4}.$$

Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ *then*

$$\begin{aligned}\Re[\frac{z}{(1-z)^3}] &= \frac{\cos\theta - 3 + 3\cos\theta - \cos 2\theta}{[(1-\cos\theta)^2+\sin^2\theta]^3}\\ &= \frac{4\cos\theta - 2\cos^2\theta - 2}{(2-2\cos\theta)^3}\\ &= \frac{-2(1-\cos\theta)^2}{(2-2\cos\theta)^3}\\ &= -\frac{1}{4(1-\cos\theta)},\end{aligned}$$

$$\Re[\frac{z}{(1-z)^2}] = \frac{\cos\theta-2+\cos\theta}{[(1-\cos\theta)^2+\sin^2\theta]^2} = \frac{2\cos\theta-2}{(2-2\cos\theta)^2} = -\frac{1}{2(1-\cos\theta)},$$

$$\Re[\frac{z}{1-z}] = \frac{\cos\theta-1}{[(1-\cos\theta)^2+\sin^2\theta]^2} = \frac{\cos\theta-1}{2-2\cos\theta} = -\frac{1}{2},$$

$$\Re[\frac{1}{1-z}] = \frac{1-\cos\theta}{[(1-\cos\theta)^2+\sin^2\theta]^2} = \frac{1-\cos\theta}{2-2\cos\theta} = \frac{1}{2},$$

$$\begin{aligned}\Re[\frac{13z+14z^2}{7+13z+7z^2}] &= \frac{91\cos\theta+169+91\cos\theta+98\cos 2\theta+182\cos\theta+98}{(7+13\cos\theta+7\cos 2\theta)^2+(13\sin\theta+7\sin 2\theta)^2}\\ &= \frac{364\cos\theta+196\cos^2\theta+169}{(13\cos\theta+14\cos^2\theta)^2+(13\sin\theta+14\sin\theta\cos\theta)^2}\\ &= \frac{364\cos\theta+196\cos^2\theta+169}{(13+14\cos\theta)^2}\\ &= 1.\end{aligned}$$

Thus

$$\begin{aligned}\Re(F) &= \frac{1}{2}\times[2\times\frac{1}{4(\cos\theta-1)} - \frac{1}{2(\cos\theta-1)} - \frac{1}{2}\times\frac{4}{27}+\frac{1}{2}+\frac{2}{27}\times 1 - 1]\\ &= -\frac{1}{4}.\end{aligned}$$

Furthermore

$$\Im[\frac{z+1}{(1-z)^2}] = \frac{2\sin\theta - \sin 2\theta}{[(1-\cos\theta)^2 + \sin^2\theta]^2} = \frac{\sin\theta}{2(1-\cos\theta)},$$

$$\Im[\frac{1}{1-z}] = Im[\frac{1-\bar{z}}{|1-z|^2}] = \frac{\sin\theta}{(1-\cos\theta)^2 + \sin^2\theta} = \frac{\sin\theta}{2(1-\cos\theta)},$$

$$\Im[\ln(z-1)] = \arg(z-1), \quad \Im[\ln(7+13z+7z^2)] = \arg(7+13z+7z^2)$$

and

$$\begin{aligned} z-1 &= \cos\theta - 1 + i\sin\theta \\ &= -2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= 2\sin\frac{\theta}{2}(-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}) \\ &= 2\sin\frac{\theta}{2}e^{i(\frac{\pi}{2}+\frac{\theta}{2})}, \end{aligned}$$

$$\begin{aligned} 7+13z+7z^2 &= (7+13\cos\theta+7\cos 2\theta) + i(13\sin\theta + 7\sin 2\theta) \\ &= \cos\theta(13+14\cos\theta) + i\sin\theta(13+14\cos\theta) \\ &= (13+14\cos\theta)(\cos\theta + i\sin\theta) \\ &= (13+14\cos\theta)e^{i\theta}, \end{aligned}$$

$$\arg(z-1) = \begin{cases} \frac{\theta}{2} + \frac{\pi}{2}, & (0 \le \theta < \pi) \\ \frac{\theta}{2} - \frac{3\pi}{2}, & (\pi \le \theta \le 2\pi) \end{cases},$$

$$\arg(7+13z+7z^2) = \begin{cases} \theta, & (0 \le \theta < \pi) \\ \theta - 2\pi, & (\pi \le \theta \le 2\pi) \end{cases}.$$

Thus

$$\begin{aligned} \Im(F) &= \frac{1}{27}\arg(7+13z+7z^2) - \frac{2}{27}\arg(z-1) \\ &= \begin{cases} -\frac{\pi}{27}, & (0 \le \theta < \pi) \\ \frac{\pi}{27}, & (\pi \le \theta \le 2\pi) \end{cases}. \end{aligned}$$

Images of the $T^0 = \{|z| = 1\}\backslash\{1\}$, $z = 1$ *under* $f_0 * f$ *is drawn two points* $(-\frac{1}{4}, \pm\frac{\pi}{27})$, $\partial(f_0 * f)$= {*the boundary of the image domain of function* $f_0 * f$ }$\backslash\{(-\frac{1}{4}, \pm\frac{\pi}{27})\}$ *respectively.*

Images of the unit disk $\mathbb{D}$ *under* f *and* $f_0 * f$ *(right) are drawn as follows in Figure*3.

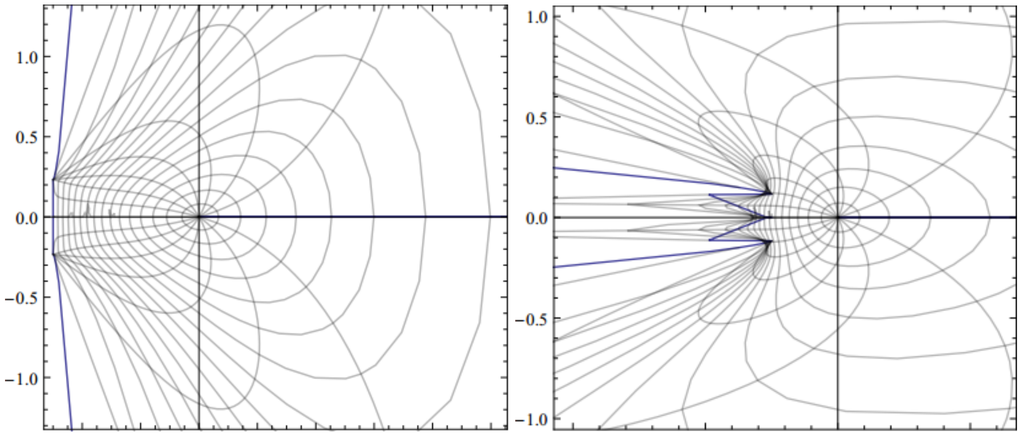

FIGURE 3. Images of f and $f_0 * f$ (right)

Remark 4.3. *In Example 4.1, the convolution for a right plane harmonic mapping $f(z)$ with the second complex dilation $w(z) = -\frac{z-a}{1-az}\frac{z-b}{1-bz}$ and the the canonical harmonic right half-plane mapping belongs to S_H^0 and is convex in the direction of the real axis. Example 4.2 says that when the second complex dilation of $f(z)$ is $w(z) = -(\frac{a-z}{1-az})^3$, it is a different result. Thus, it is interesting to determine which condition to make the convolution of f_0 and a right half-plane mapping with a Blaschke product of degree three as its second complex dilation. Furthermore, which condition should be made is satisfy for the general case $w(z) = c\prod_{n=1}^{m}\frac{a_n-z}{1-\bar{a}_n z}$ $(m \geq 3)$.*

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