Convolution of two Harmonic Mappings in the Right-half Plane

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Abstract. This paper is to give a univalent criterion and a geometric property of the convolution of two right half-plane harmonic mappings \( f_0(z) \) and \( f(z) \), where \( f_0(z) \) is canonical and the second complex dilatation \( w(z) \) of \( f(z) \) is of the form \( w(z) = -z^2/bz^2 \).

1. Introduction

Let \( \mathcal{H} \) be the class of complex-valued harmonic mappings \( f = u + iv \) defined in the unit disk \( \mathbb{D} = \{z : |z| < 1\} \) and normalized by \( f(0) = 0, f_z(0) = 1 \), where \( u \) and \( v \) are real harmonic in \( \mathbb{D} \). Such functions can be expressed as \( f = h + \overline{g} \), where

\[
h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.
\]

A harmonic mapping \( f \) is locally univalent and sense-preserving in \( \mathbb{D} \) if and only if \( |f'|^2 - |g'|^2 > 0 \) in \( \mathbb{D} \); or equivalently if \( h' \neq 0 \) in \( \mathbb{D} \) and the dilatation \( w = g'/h' \) has the property that \( |w| < 1 \) in \( \mathbb{D} \) [10]. Let \( S_\mathcal{H} \) be the subclass of \( \mathcal{H} \) consisting of univalent and sense-preserving functions. Let \( S^0_\mathcal{H} \) be the subclass of all \( f \in S_\mathcal{H} \) with \( f_z(0) = 0 \) [2]. Furthermore, \( K^0_\mathcal{H} \) be the subclass of \( S^0_\mathcal{H} \) mapping \( \mathbb{D} \) onto convex domains. A domain \( \Omega \) is said to be convex in the direction \( \gamma \) if for all \( a \in \mathbb{C} \), the set \( \Omega \cap \{a + re^{it} : \gamma, t \in \mathbb{R}\} \) is either connected or empty. In particular, a domain is convex in the direction of the real (or imaginary) axis if every line parallel to the real (or imaginary) axis has a connected intersection with the domain.

For two analytic functions \( f = z + \sum_{n=2}^{\infty} a_n z^n \) in \( \mathbb{D} \) and \( F = z + \sum_{n=2}^{\infty} A_n z^n \) in \( \mathbb{D} \), their convolution is defined as \( f \ast F = z + \sum_{n=2}^{\infty} a_n A_n z^n \) in \( \mathbb{D} \). For two harmonic functions \( f = h + \overline{g} = z + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n \) in \( \mathbb{D} \) and \( F = H + \overline{G} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \overline{z}^n \) in \( \mathbb{D} \), we define their harmonic convolution as

\[
f \ast F = h \ast H + g \ast G = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \overline{z}^n, \quad z \in \mathbb{D}.
\]
A harmonic function \( f = h + g \in S_0^H \) is called a right half-plane harmonic mapping, if \( f \) maps the unit disk \( \mathbb{D} \) onto \( R = \{ w : \Re(w) > -1/2, \Re(w), \Im(w) \in \mathbb{R} \} \) such a mapping satisfies \( h + g = \frac{z}{1+z} \in \mathbb{D} \). We denote the class of right half-plane harmonic mappings by \( R_0^H \), which satisfies \( R_0^H \subseteq K_0^H \).

The canonical harmonic right half-plane mapping, \( f_0 = h_0 + \overline{g_0} \in R_0^H \) has a second complex dilatation \( w_0 = -z \). The technique of shear construction [5] yields

\[
h_0(z) = \frac{z - z^2/2}{(1-z)^2} = z + \sum_{n=2}^{\infty} \frac{1+n}{2}z^n = \frac{1}{2}\left( \frac{z}{1-z} + \frac{z}{(1-z)^2} \right), \quad z \in \mathbb{D}
\]

and

\[
g_0(z) = -\frac{z^2/2}{(1-z)^2} = \sum_{n=2}^{\infty} \frac{1}{2}z^n = \frac{1}{2}\left( \frac{z}{1-z} - \frac{z}{(1-z)^2} \right), \quad z \in \mathbb{D}.
\]

If \( f = h + g \in \mathcal{H} \), then

\[
f_0 \ast f = h_0 \ast h + \overline{g_0} \ast g = \frac{h + z\ell'}{2} + \frac{g - zg'}{2}, \quad z \in \mathbb{D}.
\]

Univalent criterions and geometric characterizations of the convolution of two harmonic mappings have attracted one’s much attention [3,4,8,10-14]. Especially, when their dilations are specified, there are some recent results. Among them, Dorff [3] proved

**Theorem A.** Let \( f_1, f_2 \in R_0^H \). If \( f_1 \ast f_2 \) is locally univalent and sense-preserving, then \( f_1 \ast f_2 \in S_0^H \) and is convex in the direction of the real axis.

There were lots of works on concrete conditions determined locally univalence of their convolutions of two harmonic mappings. Dorff, Nowak and Wołoszkiewicz [4] proved the following lemma and theorems.

**Lemma A.** Let \( f = h + \overline{g} \in R_0^H \) with the dilatation \( w(z) \). Then the dilatation \( \overline{w} \) of \( f_0 \ast f \) is

\[
\overline{w}(z) = -\frac{w^2(z) + w(z) - \frac{1}{2}zw'(z) + \frac{1}{2}w'(z)}{1 + w(z) - \frac{1}{2}zw''(z) + \frac{1}{2}z^2w'(z)}.
\]

**Theorem B.** Let \( f = h + g \in R_0^H \) with \( h(z) + g(z) = \frac{z}{1-z} \) and \( w(z) = \frac{g(z)}{h(z)} = \text{e}^{i\theta}z^n \) \( n \in \mathbb{Z}^+ \) and \( \theta \in \mathbb{R}^+ \). If \( n = 1, 2 \), then \( f_0 \ast f \in S_0^H \) and is convex in the direction of the real axis.

**Theorem C.** Let \( f = h + g \in R_0^H \) with \( h(z) + g(z) = \frac{z}{1-z} \) and \( w(z) = \frac{z^n}{1-z} \) with \( a \in (-1, 1) \). Then \( f_0 \ast f \in S_0^H \) and is convex in the direction of the real axis.

Recently, Li and Ponnusamy [12] generalized Theorem C, as follows

**Theorem D.** Let \( f = h + g \in R_0^H \) with the dilatation \( w(z) = \frac{z^n}{1+z^n} \), \( |a| < 1 \). Then \( f_0 \ast f \in S_0^H \) and is convex in the direction of the real axis if and only if

\[
(\Re(a))^2 + 9(\Im(a))^2 \leq 1 \quad \text{and} \quad \Re(a) \neq \pm 1.
\]

Jiang, Rasila and Sun [8] considered Theorem D in a more general setting by allowing a rotation parameter \( \theta \) in the second complex dilation \( w(z) \) of \( f(z) \).

**Theorem E.** Let \( f = h + g \in R_0^H \) with the dilatation \( w(z) = \text{e}^{i\theta}z^n \), where \( a = |a|\text{e}^{i\alpha}, \alpha = \arg a, |a| < 1 \) and \( \theta \in \mathbb{R} \). If

\[
|9 \sin^2(\alpha + \frac{\theta}{2}) + \cos^2(\alpha + \frac{\theta}{2})|a|^2 \leq 1 \quad \text{and} \quad |a| \cos(\alpha + \frac{\theta}{2}) \neq -\cos(\frac{\theta}{2}),
\]

then \( f_0 \ast f \in S_0^H \) and is convex in the direction of the real axis.
A Blaschke product is a product of the type
\[ B(z) = z^{1} \prod_{n=1}^{\infty} \frac{(a_n - z)b_n}{1 - a_n z}, \]
where \( k \) is a nonnegative integer, the sequence \( a_n \) satisfies that \( 0 < |a_n| < 1 \) for each \( n \), and the series \( \sum_{n=1}^{\infty} (1 - |a_n|) \) converges. The function \( B(z) \) is bounded and analytic in \( D \). The zeros of \( B(z) \) are just the numbers \( a_n \) and 0 (if \( k > 0 \)). Specially, if a Blaschke product is of the form
\[ B(z) = c \prod_{n=1}^{m} \frac{a_n - z}{1 - a_n z}, \]
where \( c \in \partial D = \{ z : |z| = 1 \} \), \( m \) is a nonnegative integer, then we say that \( B(z) \) is a finite Blaschke product of degree \( m \).

Blaschke products have been applied to many research fields, for instance, Kraus and Gorkin[1], Dalkayan[6], Hamada[7], Akeroyd and Roth[9].

In [8,12,13], the authors studied the convolution of a classical half-plane harmonic mapping and a right-plane harmonic mapping, with a second complex dilation of a finite Blaschke product of degree one or a Blaschke product of degree two with an \( a_n = 0 \). In this paper, we will consider the case of Blaschke products of degree two with all \( |a_n| \neq 0 \), that is, \( B(z) = -\frac{z-a}{z-b} \), and obtain our main result.

**Theorem 1.1.** Let \( f = h + g \in R_{H}^{0} \) with the dilatation \( w(z) = -\frac{\bar{h} - \bar{g}}{1 - \bar{z} - \bar{b}} \), where \( |a| < 1, |b| \leq 1, (a, b \in \mathbb{R}) \). If
\[ 0 < a + b - 2ab < 2, \]
then \( f_{0} * f \in S_{H}^{0} \) and is convex in the direction of the real axis.

2. Preliminary lemmas and their proofs

**Lemma 2.1.** Let \( f_{0} \) be the canonical harmonic right half-plane mapping. If \( f = h + g \in R_{H}^{0} \) with the dilatation \( w(z) = -\frac{\bar{h} - \bar{g}}{1 - \bar{z} - \bar{b}} \), then the dilatation \( \bar{w} \) of the convolution \( f_{0} * f \) is
\[ \bar{w}(z) = \frac{(z + A)(z + B)(z + C)}{(1 + Az)(1 + Bz)(1 + Cz)} = \bar{z} t(z), \]
where
\[ t(z) = z^{3} + \frac{1}{2}(2 - 3a - 3b)z^{2} + (1 - a - b + 2ab)z - \frac{1}{2}(a + b + 2ab), \]
and
\[ t'(z) = 1 + \frac{1}{2}(2 - 3a - 3b)z + (1 - a - b + 2ab)z^{2} - \frac{1}{2}(a + b + 2ab)z^{3}, \]
Here \(-A,-B,-C\) are the three roots of the equation \( t(z) = 0 \), and \( A,B,C \) may be equal.

**Proof.** Let \( w(z) = -\frac{\bar{h} - \bar{g}}{1 - \bar{z} - \bar{b}} \), then \( \bar{w}'(z) = \frac{-1 + a^{2} + b^{2} - z}{(1 - az)(1 - bz)} + \frac{4a - (1 + b^{2})}{(1 - az)(1 - bz)^{2}} \). We obtain from Lemma A that
\[ \bar{w}(z) = -z \frac{P(z)}{Q(z)}, \]
where
\[ P(z) = (a + b - 2ab - a^2b - ab^2 + 2a^2b^2) - (2 - a - b + a^2b + ab^2 - 2a^2b^2)z + (a + b + 4ab - a^2b - ab^2 - 4a^2b^2)z^2 - (3a + 3b - 3a^2b - 3ab^2)z^3 + (2 - 2ab)z^4 \]
\[ = -(ab - 1)(z - 1)[-a + b - 2ab + 2(1 - a - b + 2ab)z + (2 - 3a - 3b)z^2 + 2z^3] \]

and
\[ Q(z) = (a + b - 2ab - a^2b - ab^2 + 2a^2b^2)z^4 - (2 - a - b + a^2b + ab^2 - 2a^2b^2)z^3 \\
+ (a + b + 4ab - a^2b - ab^2 - 4a^2b^2)z^2 - (3a + 3b - 3a^2b - 3ab^2)z + (2 - 2ab) \\
= (ab - 1)(z - 1)[-a + b - 2ab)z^3 + 2(1 - a - b + 2ab)z^2 + (2 - 3a - 3b)z + 2]. \]

Thus,
\[ \bar{w}(z) = \frac{-zP(z)}{Q(z)} = z^3 + \frac{1}{2}(2 - 3a - 3b)z^2 + (1 - a - b + 2ab)z - \frac{1}{2}(a + b - 2ab) \]
\[ = z \frac{t(z)}{t'(z)}. \]

Suppose that \(-A, -B, -C\) are the three roots of \(t(z) = 0\). Then it follows
\[ t(z) = (z + A)(z + B)(z + C) \]

and
\[ t' = z^3(t'(1/z) = z^3(1/z + A)(1/z + B)(1/z + C) = (1 + Az)(1 + Bz)(1 + Cz). \]

\[ \square \]

**Lemma 2.2.** (1) Let \(|a| \leq 1, |b| \leq 1, \text{ and } a, b \in \mathbb{R}\). Then \(-4 \leq a + b - 2ab \leq 2\). (2) Let \(\Psi = \{(a, b) | -1 < a < 0, -1 < b < 0\}, \Omega = \{(a, b) | 0 \leq a + b - 2ab \leq 2, |a| \leq 1, |b| \leq 1\}, a, b \in \mathbb{R}\). Then \(\Psi \cap \Omega = \phi\).

**Proof.** (1) Let \(f(a, b) = a + b - 2ab\), we have a stable point \((\frac{1}{2}, \frac{1}{2})\), and \(f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}\). Furthermore, the values of \(f(a, b)\) on the boundary as following
\[ f(1, b) = 1 - b, f(-1, b) = -1 + 3b, f(a, 1) = 1 - a, f(a, -1) = -1 + 3a. \]

Using the assumption that \(|a| \leq 1\) and \(|b| \leq 1\), we get \(f(a, b)_{\text{max}} = 2, f(a, b)_{\text{min}} = -4\).

(2) Suppose \((a, b) \in \Psi\), then \(-1 < a < 0, -1 < b < 0\). Since \(f(a, b) = a + b - 2ab = a(1 - b) + b(1 - a)\), it shows that \(f(a, b) < 0\). If \((a, b) \in \Omega\) then \(0 \leq f(a, b) \leq 2\), a contradiction with \(f(a, b) < 0\). Hence \(\Psi \cap \Omega = \phi\). \(\square\)

**Lemma 2.3.** Let \(0 < a + b - 2ab < 2, |a| < 1, |b| < 1, a, b \in \mathbb{R}\). Then
\[ -4 + a + 2a^2 + b + 2ab - 5a^2b + 2b^2 - 5ab^2 + 2a^2b^2 < 0. \] (8)

**Proof.** Let \(\Omega = \{(a, b) | 0 < a + b - 2ab < 2, |a| < 1, |b| < 1\}, \text{ and } \partial \Omega = \{(a, b) | a + b - 2ab = 0, a + b - 2ab = 2, |a| = 1, |b| = 1\}, \)
\[ f(a, b) = -4 + a + 2a^2 + b + 2ab - 5a^2b + 2b^2 - 5ab^2 + 2a^2b^2, \]
then \(f(a, b)\) has a maximum and a minimum on the \(\Omega \cup \partial \Omega\). The system
\[
\begin{align*}
\{ f_a = 1 + 4a + 2b - 10ab - 5b^2 + 4ab^2 = 0 \\
\{ f_b = 1 + 2a - 5a^2 + 4b - 10ab + 4a^2b = 0
\end{align*}
\]
yields stable points \( P_1(-1, \frac{1}{2}), P_2(\frac{1}{2}, -1) \) and the value of \( f(a, b) \) at stable points \( f(-1, \frac{1}{2}) = f(\frac{1}{2}, -1) = -4 \). We will discuss below cases on the boundary

1. If \( a = 1 \), then \(-1 \leq b \leq 1\). Hence,

\[
f(1, b) = -1 - 2b - b^2 = -(1 + b)^2 \leq 0,
\]
equality holds if and only if \( b = -1 \).

2. If \( a = -1 \), then \( \frac{1}{2} \leq b \leq 1 \). Thus,

\[
f(-1, b) = -3 - 6b + 9b^2 = 3(3b(1) - 1) \leq 0,
\]
equality holds if and only if \( b = 1 \).

3. If \( b = 1 \), then \(-1 \leq a \leq 1\). So

\[
f(a, 1) = -1 - 2a - a^2 = -(1 + a)^2 \leq 0,
\]
equality holds if and only if \( a = -1 \).

4. If \( b = -1 \), then \( \frac{1}{2} \leq a \leq 1\). Therefore,

\[
f(a, -1) = -3 - 6a + 9a^2 = 3(3a(1) - 1) \leq 0,
\]
equality holds if and only if \( a = 1 \).

5. If \( a + b - 2ab = 0 \), then

\[
f(a, b) = -4 + a + 2a^2 + b + 2ab - 5a^2b + 2b^2 - 5ab^2 + 2a^2b^2
= -4 + (1 + 2a + 2b - ab)(a + b - 2ab) < 0.
\]

6. If \( a + b - 2ab = 2 \), then \( a = \frac{2 - b}{1 - 2b} \). Since \(|a| = |\frac{2 - b}{1 - 2b}| \leq 1\), it shows that

\[
(2 - b)^2 - (1 - 2b)^2 = 3(1 - b)(1 + b) \leq 0,
\]
yields \( b = 1, a = -1 \) or \( b = -1, a = 1 \). Thus \( f(a, b) = f(-1, 1) = f(1, -1) = 0 \).

In summary, \( f_{\max} = 0, f_{\min} = -4 \). It's easy to see that the function \( f(a, b) \) takes maximum only at points \((1, -1), (-1, 1)\). Thus \( f(a, b) < 0 \) on the \( \Omega \).

In order to establish our main results, we need the following Cohn’s Rule[15,p.375].

**Lemma B.** For a given polynomial

\[
t(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n
\]
of degree \( n \), let

\[
t'(z) = z^n t'(1/z) = \bar{a}_n + \bar{a}_{n-1}z + \bar{a}_{n-2}z^2 + \cdots + \bar{a}_0z^n.
\]
Denote by \( r \) and \( s \) the number of zeros of \( t(z) \) inside and on the unit circle \(|z| = 1\), respectively. If \(|a_0| < |a_n|\), then

\[
t_1(z) = \frac{\bar{a}_n t(z) - a_0 t'(z)}{z}
\]
is of degree \( n - 1 \) and has \( r_1 = r - 1 \) and \( s_1 = s \) number of zeros inside and on the unit circle \(|z| = 1\), respectively.

We also need the following Schur-Cohn algorithm [15,p.383].
Lemma C. For a given polynomial
\[ r(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \]
of degree \( n \), let
\[ M_k = \det \begin{pmatrix} \frac{B_k^T}{A_k} & A_k \\ \frac{A_k^T}{B_k} & B_k \end{pmatrix} \quad (k = 1, 2, \ldots, n), \]
where \( A_k \) and \( B_k \) are the triangular matrices
\[
A_k = \begin{pmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ a_0 & a_1 & \cdots & a_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & \cdots & \cdots & a_{0} \end{pmatrix}, \quad B_k = \begin{pmatrix} \tilde{a}_n & \tilde{a}_{n-1} & \cdots & \tilde{a}_{n-k+1} \\ \tilde{a}_n & \tilde{a}_{n-1} & \cdots & \tilde{a}_{n-k+2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_n & \cdots & \cdots & \tilde{a}_0 \end{pmatrix}.
\]
Then the polynomial \( r(z) \) has all its zeros inside the unit circle \( |z| = 1 \) if and only if the determinants \( M_1, M_2, \ldots, M_n \) are all positive.

3. Proof of Theorem 1.1 and a corollary

Proof. Let us consider two special cases of the second complex dilation \( w(z) \). The case that \( b = 1 \) has already been proved by Dorff et al [5]. If \( b = -1 \), then \( w(z) = -\frac{z^3}{1-z} \), \( -\frac{1}{3} < a < 1 \). Thus, Lemma 2.1 yields
\[
|w(z)| = \left| z + \frac{1-3a}{1-3z} \right| \leq |z| < 1.
\]
Suppose that \(-A, -B, -C\) are the roots of \( t(z) = 0 \). We have form (2.2) that
\[
t(z) = z^3 + \frac{1}{2} (2 - 3a - 3b)z^2 + (1 - a - b + 2ab)z - \frac{1}{2} (a + b - 2ab)
\]
\[= (z + A)(z + B)(z + C).\]
Thus \(|ABC| = | - \frac{1}{2} (a + b - 2ab) | < 1\). Therefore, at least one of \(-A, -B, -C\) lies in \( \mathbb{D} \). Applying the formula (2.2) and (2.3), we can establish the following formula
\[
t_1(z) = \frac{t(z) + \frac{1}{2} (a + b - 2ab) t'(z)}{z}
\]
\[= -\frac{1}{4} \left[ (-4 + a^2 + 2ab - 4a^2b + b^2 - 4ab^2 + 4a^2b^2)z^3 - 2(2 - 2a - a^2 - 2b - 4ab \\
+ 4a^2b - b^2 + 4ab^2 - 4a^2b^2)z - (4 - 2a - 3a^2 - 2b - 2ab + 6a^2b - 3b^2 + 6ab^2) \right].\]
Denote by
\[
f(a, b) = -\frac{1}{4} (-4 + a^2 + 2ab - 4a^2b + b^2 - 4ab^2 + 4a^2b^2)
\]
and
\[
K(a, b) = \frac{1}{4} (-4 + 2a + 3a^2 + 2b + 2ab - 6a^2b + 3b^2 - 6ab^2).\]
After direct calculation, we have

\[ K^2(a, b) - J^2(a, b) = \frac{1}{4} (1 + a)(1 + b)(a + b - 2ab)(-4 + a + 2a^2 + b + 2ab - 5a^2b + 2b^2 - 5ab^2). \]  

(9)

Since \(0 < a + b - 2ab < 2, |a| < 1, |b| < 1\), then Lemma 2.3 shows that the formula (3.1) is negative.

We can use \(t_1(z)\) to construct

\[ t_2(z) = \frac{f(a, b)t_1(z) - K(a, b)t_1'(z)}{z} \]

\[ = -\frac{1}{4} (1 + a)(1 + b)(a + b - 2ab)[(-4 + a + 2a^2 + b + 2ab - 5a^2b + 2b^2 - 5ab^2 + 2a^2b^2)] \]

So \(t_2(z) = 0\) has one zero at

\[ z_0 = \frac{-2 + 2a + a^2 + 2b + 4ab - 4a^2b + b^2 - 4ab^2 + 4a^2b^2}{-4 + a + 2a^2 + b + 2ab - 5a^2b + 2b^2 - 5ab^2 + 2a^2b^2}. \]  

(10)

Denote by

\[ M(a, b) = -2 + 2a + a^2 + 2b + 4ab - 4a^2b + b^2 - 4ab^2 + 4a^2b^2 \]

and

\[ N(a, b) = -4 + a + 2a^2 + b + 2ab - 5a^2b + 2b^2 - 5ab^2 + 2a^2b^2. \]

Some calculation gives

\[ M^2(a, b) - N^2(a, b) = 3(-1 + a^2)(-1 + b^2)(2 - a - b + 2ab). \]  

(11)

Furthermore, the fact that \(0 < a + b - 2ab < 2, |a| < 1, |b| < 1\) implies that the formula (3.3) is negative and \(|z_0| < 1\).

Therefore, by Lemma B, the roots of \(t(z)\) are all in \(D\), that is \(A, B, C \in D\) and \(|w(z)| < 1\) for all \(z \in D\). \(\Box\)

We note that if the case \(b = a\) in Theorem 1.1 is the following corollary.

**Corollary 3.1.** Let \(f_0\) be the canonical harmonic right half-plane mapping. If \(f = h + \bar{g} \in R^0_H\) with the dilatation \(w(z) = -(\frac{z - a}{1 - az})^2\), where \(0 \leq a < 1\). Then \(fo \ast f\) is convex in the direction of the real axis.

**Proof.** By the second result of Lemma 2.2 we have that \(a, b\) are not negative at the same time, that is, if we take \(a = b\), then the condition (1.3) degenerates to the inequality \(0 \leq a < 1\). Then Theorem 1.1 implies Corollary 3.1.

Here, we will give another method to prove this corollary.

By the formula (2.4), we have

\[ \bar{w}(z) = z\frac{z^3 + (1 - 3a)z^2 + (1 - 2a + 2a^2)z + (-a + a^2)}{1 + (1 - 3a)z + (1 - 2a + 2a^2)z^2 + (-a + a^2)z^3} \]

\[ = -z\frac{(a - z)[1 - a + (1 - 2a)z + z^2]}{(1 - az)[(1 - a)z^2 + (1 - 2a)z + 1]} \]

\[ = -z\frac{(a - z)\bar{r}(z)}{(1 - az)(z + D)(z + E)} \]

\[ = \frac{z(a - z)\bar{r}(z)}{(1 - az)(1 + Dz)(1 + Ez)}. \]
where \(-D, -E\) is the two roots of the \(r(z) = 1 - a + (1 - 2a)z + z^2, r'(z) = (1 - a)z^2 + (1 - 2a)z + 1\). Furthermore,

\[
t' = z^2 i (1/z) = z^2 (1/z + D)(1/z + E) = (1 + Dz)(1 + Ez).
\]

We will show that \(D, E \in \mathbb{D}\). By Lemma C, we only need to show that the determinants \(M_1, M_2\) are all positive for \(a \in (0, 1)\). In fact,

\[
M_1 = \det \begin{pmatrix} 1 & 1 - a \\ 1 - a & 1 \end{pmatrix} = a(2 - a) > 0,
\]

\[
M_2 = \det \begin{pmatrix} 1 & 0 & 1 - a & 1 - 2a \\ 1 - 2a & 1 & 0 & 1 - a \\ 1 - a & 0 & 1 & 1 - 2a \\ 1 - 2a & 1 - a & 0 & 1 \end{pmatrix} = 3a^2(1 - a^2) > 0.
\]

Therefore, \(D, E \in \mathbb{D}\) and \(|\frac{w'}{w}| < 1\). As we all know \(|\frac{w}{\bar{w}}| < 1\), for \(0 < a < 1\). Thus \(|\bar{w}(z)| < 1\) for all \(z \in \mathbb{D}\). In addition, when \(a = 0\), the dilatation \(w(z) = -z^2\), which is a special case of \(n = 2, \theta = \pi\) at Theorem B.

4. Auxiliary examples

Example 4.1. Let \(f = h + g \in R_n^0\) with the dilatation \(w(z) = -\left(\frac{z - a}{1 - az}\right)^2\), where \(0 \leq a < 1\). Then

\[
h + g = \frac{z}{1 - z}, \quad \frac{g'}{h'} = -\left(\frac{a - z}{1 - az}\right)^2.
\]

We obtain

\[
h' = \frac{(1 - az)^2}{(1 - a^2)(1 + z)(1 - z)^3} = \frac{1}{8(1 - a^2)} \left(\frac{1 + az}{1 + z} + \frac{(1 + a)^2}{(1 - z)^2} + \frac{2(1 + 2a - 3a^2)}{(1 - z)^2} + \frac{4(1 - a)^2}{(1 - z)^3}\right),
\]

\[
g' = \frac{(a - z)^2}{(1 - a^2)(1 + z)(1 - z)^3} = -\frac{1}{8(1 - a^2)} \left(\frac{1 + az}{1 + z} + \frac{(1 + a)^2}{(1 - z)^2} + \frac{2a^2 + 2a - 3}{(1 - z)^2} + \frac{4(1 - a)^2}{(1 - z)^3}\right).
\]

Integration from 0 to \(z\) gives

\[
h(z) = \frac{1}{8(1 - a^2)} \left(\frac{(1 + a)^2 \ln(1 + z)}{1 - z} + \frac{2(1 + 2a - 3a^2)}{1 - z} + \frac{2(1 - a)^2}{(1 - z)^2}\right) - \frac{1}{2}z' + \frac{1}{2},
\]

\[
g(z) = -\frac{1}{8(1 - a^2)} \left(\frac{(1 + a)^2 \ln(1 + z)}{1 - z} + \frac{2a^2 + 2a - 3}{1 - z} + \frac{2(1 - a)^2}{(1 - z)^2}\right) - \frac{1}{2}.
\]

Let \(F = f_0 * f = h_0 * h + g_0 * g = H + G\). By the definition of convolution and the formula (1.1), we have

\[
H = h_0 * h = \frac{h + zh'}{2}, \quad G = g_0 * g = \frac{g - zg'}{2}.
\]
So

\[
H = \frac{1}{2} \left[ \frac{1}{8(1-a^2)} \left( (1 + a)^2 \ln \left( \frac{1 + z}{1 - z} \right) + \frac{2(1 + 2a - 3a^2)}{1 - z} + \frac{2(1 - a)^2}{(1 - z)^2} \right) \\
+ \frac{z(1-a)^2}{(1-a^2)(1+z)(1-z)^3 - \frac{1}{2}} \right],
\]

\[
G = \frac{1}{2} \left[ -\frac{1}{8(1-a^2)} \left( (1 + a)^2 \ln \left( \frac{1 + z}{1 - z} \right) + \frac{2(a^2 + 2a - 3)}{1 - z} + \frac{2(1 - a)^2}{(1 - z)^2} \right) \\
+ \frac{z(z-a)^2}{(1-a^2)(1+z)(1-z)^3 - \frac{1}{2}} \right].
\]

The real and imaginary parts of \(F(z) = H + \bar{G}\) can be written as follows

\[
\Re(F(z)) = \Re(H + G) = \Re \left[ \frac{(a^2 + 1)z^2 - 4az + a^2 + 1}{2(1-a^2)(1+z)(1-z)^3} z + \frac{1}{2(1-z)} - \frac{1}{2} \right],
\]

\[
\Im(F(z)) = \Im(H - G) = \Im \left[ \frac{1}{8(1-a^2)} \left((1 + a)^2 \ln \left( \frac{1 + z}{1 - z} \right) - \frac{2(1-a)^2}{1 - z} + \frac{2(1-a)^2}{(1 - z)^2} \right) \\
+ \frac{z}{2(1-z)^3} \right].
\]

Images of the unit disk \(\mathbb{D}\) under \(f\) and \(f_0 \ast f\) (right) for \(a = 0.3\) are drawn in Figure 1.

![Figure 1](image1.png)

**FIGURE 1.** Images of \(f\) and \(f_0 \ast f\) (right) for \(a = 0.3\)

Images of the unit disk \(\mathbb{D}\) under \(f\) and \(f_0 \ast f\) (right) for \(a = 0.7\) are drawn in Figure 2.

![Figure 2](image2.png)

**FIGURE 2.** Images of \(f\) and \(f_0 \ast f\) (right) for \(a = 0.7\)
Example 4.2. Let \( f = h + \tilde{g} \in H^0_\mathcal{M} \) with the dilatation \( w(z) = \frac{1 + z^3}{1 + z^2} \). Then
\[
h + g = \frac{z}{1 - z}, \quad g' = -\left(\frac{1 + z}{1 + \frac{1}{2}z^3}\right).
\]

We obtain
\[
h' = \frac{(2 + z)^3}{(1 - z)^3(7 + 13z + 7z^2)} = \frac{1}{(1 - z)^3} + \frac{2}{27(1 - z)} + \frac{14z + 13}{27(7z^2 + 13z + 7)}.
\]
\[
g' = \frac{-(1 + 2z)^3}{(1 - z)^3(7 + 13z + 7z^2)} = \frac{-1}{(1 - z)^3} + \frac{1}{(1 - z)^2} - \frac{2}{27(1 - z)} - \frac{14z + 13}{27(7z^2 + 13z + 7)}.
\]

Integration from 0 to \( z \) gives
\[
h(z) = \frac{1}{2(1 - z)^2} - \frac{2}{27} \ln(z - 1) + \frac{1}{27} \ln(7 + 13z + 7z^2) - \frac{1}{2} - \frac{\ln7}{27},
\]
\[
g(z) = \frac{-1}{2(1 - z)^2} + \frac{1}{1 - z} + \frac{2}{27} \ln(z - 1) - \frac{1}{27} \ln(7 + 13z + 7z^2) - \frac{1}{2} + \frac{\ln7}{27}.
\]

Let \( F = f_0 \ast f = h_0 \ast h + \overline{g_0} \ast \overline{g} = H + \overline{G} \). By the definition of convolution and the formula (1.1), we have
\[
H = h_0 \ast h = \frac{h + z h'}{2}, \quad G = g_0 \ast g = \frac{g - zg'}{2}.
\]

Thus,
\[
H = \frac{z}{2(1 - z)^2} + \frac{1}{4(1 - z)^2} + \frac{1}{54} \left[ \frac{2z}{1 - z} + \frac{14z^2 + 13z}{7 + 13z + 7z^2} + \ln \frac{7 + 13z + 7z^2}{(z - 1)^3} - \ln7 \right] - \frac{1}{4},
\]
\[
G = \frac{z}{2(1 - z)^3} - \frac{z + 1}{2(1 - z)^2} + \frac{1}{54} \left[ \frac{2z + 27}{1 - z} + \frac{14z^2 + 13z}{7 + 13z + 7z^2} - \ln \frac{7 + 13z + 7z^2}{(z - 1)^2} + \ln7 \right] - \frac{1}{4},
\]

and
\[
H' = \frac{(2 + z)^3(28 + 75z + 90z^2 + 50z^3)}{(z - 1)^4(7 + 13z + 7z^2)^2}, \quad G' = \frac{z(1 + 2z)^3(50 + 90z + 75z^2 + 28z^3)}{(z - 1)^4(7 + 13z + 7z^2)^2}.
\]

Therefore, the dilatation \( \overline{w} \) of the convolution \( F = f_0 \ast f \) is
\[
\frac{\overline{w}}{H'} = \frac{1}{2} \left( \frac{1 + 2z^3 (50 + 90z + 75z^2 + 28z^3)}{(2 + z)^2 (50z^3 + 90z^2 + 75z + 28)} \right)
\]

When \( z = -0.8 \), the two formulas \( 50z^3 + 90z^2 + 75z + 28 = 0, 50 + 90z + 75z^2 + 28z^3 = 11.664 \). Thus, the dilatation \( |\overline{w}| \) of the convolution \( F = f_0 \ast f \) tends to infinity. Hence, \( F \) is not sense-preserving.

The real and imaginary parts of \( F(z) = H + \overline{G} \) can be written as follows
\[
\Re(F(z)) = \Re(H + G) = \frac{1}{2} \Re \left\{ \frac{2z}{(1 - z)^2} - \frac{z}{(1 - z)^2} + \frac{4z + 27}{27(1 - z)} + \frac{2(13z + 14z^2)}{27(7 + 13z + 7z^2)} - 1 \right\},
\]
\[
\Im(F(z)) = \Im(H - G) = \frac{1}{2} \Im \left\{ \frac{z + 1}{(1 - z)^2} - \frac{1}{1 - z} + \frac{2}{27} \ln \frac{7 + 13z + 7z^2}{(z - 1)^2} \right\}.
and
\[
\frac{z}{(1-z)^3} = \frac{z(1-3z + 3z^2 - z^3)}{|1-z|^3} = \frac{z - 3|z|^3 + 3|z|^2 + |z|^3}{|1-z|^3},
\]
\[
\frac{z}{(1-z)^2} = \frac{z(1-2z + z^2)}{|1-z|^2} = \frac{z - 2|z|^2 + |z|^2 z}{|1-z|^2},
\]
\[
\frac{13z + 14z^2}{7 + 13z + 7z^2} = \frac{91z + 169|z|^2 + 91|z|^3 + 98|z|^3 + 182|z|^3 z + 98|z|^4}{|7 + 13z + 7z^2|},
\]
\[
\frac{z + 1}{(1-z)^2} = \frac{(z + 1)(1-2z + z^2)}{|1-z|^2} = \frac{z - 2|z|^2 + |z|^2 z + 1 - 2z + z^2}{|1-z|^2}.
\]
Let \(z = e^{i\theta}, \ 0 \leq \theta \leq 2\pi\) then
\[
\Re\left[\frac{z}{(1-z)^3}\right] = \frac{\cos \theta - 3 \cos \theta - \cos 2\theta}{(1 - \cos \theta)^2 + \sin^2 \theta)^3} = \frac{\cos \theta - 2 \cos^2 \theta - 2}{(2 - 2 \cos \theta)^3} = \frac{1}{4(1 - \cos \theta)}
\]
\[
\Re\left[\frac{z}{(1-z)^2}\right] = \frac{\cos \theta - 2 \cos \theta}{(1 - \cos \theta)^2 + \sin^2 \theta)^2} = \frac{2 \cos \theta - 2}{(2 - 2 \cos \theta)^2} = \frac{1}{2(1 - \cos \theta)}
\]
\[
\Re\left[\frac{z}{1-z}\right] = \frac{\cos \theta - 1}{(1 - \cos \theta)^2 + \sin^2 \theta)^2} = \frac{\cos \theta - 1}{2 - 2 \cos \theta} = \frac{1}{2}
\]
\[
\Re\left[\frac{1}{1-z}\right] = \frac{1 - \cos \theta}{(1 - \cos \theta)^2 + \sin^2 \theta)^2} = \frac{1 - \cos \theta}{2 - 2 \cos \theta} = \frac{1}{2}
\]
\[
\Re\left[\frac{13z + 14z^2}{7 + 13z + 7z^2}\right] = \frac{91 \cos \theta + 169 + 91 \cos \theta + 98 \cos 2\theta + 182 \cos \theta + 98}{(7 + 13 \cos \theta + 7 \cos 2\theta)^2 + (13 \sin \theta + 7 \sin 2\theta)^2}
\]
\[
= \frac{364 \cos \theta + 196 \cos^2 \theta + 169}{(13 \cos \theta + 14 \cos^2 \theta)^2 + (13 \sin \theta + 14 \sin \theta \cos \theta)^2}
\]
\[
= \frac{364 \cos \theta + 196 \cos^2 \theta + 169}{(13 + 14 \cos \theta)^2}
\]
\[
= 1.
\]
Thus
\[
\Re(F) = \frac{1}{2} \times [2 \times \frac{1}{4(\cos \theta - 1)} - \frac{1}{2(\cos \theta - 1)} - \frac{1}{2} \times \frac{4}{27} + \frac{1}{2} + \frac{2}{27} \times 1 - 1]
\]
\[
= -\frac{1}{4}.
\]
Furthermore

\[
\mathfrak{I}\left[\frac{z+1}{(1-z)^2}\right] = \frac{2 \sin \theta - \sin 2\theta}{[(1 - \cos \theta)^2 + \sin^2 \theta]^2} = \frac{\sin \theta}{2(1 - \cos \theta)}.
\]

\[
\mathfrak{I}\left[\frac{1}{1-z}\right] = \text{Im}\left[\frac{1-z}{|1-z|^2}\right] = \frac{\sin \theta}{(1 - \cos \theta)^2 + \sin^2 \theta} = \frac{\sin \theta}{2(1 - \cos \theta)}.
\]

\[
\mathfrak{I}[\ln(z-1)] = \arg(z-1), \quad \mathfrak{I}[\ln(7+13z+7z^2)] = \arg(7+13z+7z^2)
\]

and

\[
z - 1 = \cos \theta - 1 + i \sin \theta
= -2 \sin^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2}
= 2 \sin \frac{\theta}{2} \left(-\sin \frac{\theta}{2} + i \cos \frac{\theta}{2}\right)
= 2 \sin \frac{\theta}{2} e^{i\left(\frac{\pi}{2} + \frac{\theta}{2}\right)}.
\]

\[
7 + 13z + 7z^2 = (7 + 13 \cos \theta + 7 \cos 2\theta) + i(13 \sin \theta + 7 \sin 2\theta)
= \cos \theta(13 + 14 \cos \theta) + i \sin \theta(13 + 14 \cos \theta)
= (13 + 14 \cos \theta)(\cos \theta + i \sin \theta)
= (13 + 14 \cos \theta)e^{i\theta},
\]

\[
\arg(z-1) = \left\{ \begin{array}{ll}
\frac{\theta}{2} + \frac{\pi}{2}, & (0 \leq \theta < \pi) \\
\frac{\theta}{2} - \frac{3\pi}{2}, & (\pi \leq \theta \leq 2\pi)
\end{array} \right.
\]

\[
\arg(7 + 13z + 7z^2) = \left\{ \begin{array}{ll}
\theta, & (0 \leq \theta < \pi) \\
\theta - 2\pi, & (\pi \leq \theta \leq 2\pi)
\end{array} \right.
\]

Thus

\[
\mathfrak{I}(F) = \frac{1}{2\pi} \arg(7 + 13z + 7z^2) - \frac{2}{2\pi} \arg(z-1)
= \left\{ \begin{array}{ll}
-\frac{\pi}{2\pi}, & (0 \leq \theta < \pi) \\
\frac{\pi}{2\pi}, & (\pi \leq \theta \leq 2\pi)
\end{array} \right.
\]

Images of the $T^0 = \{ |z| = 1 \} \setminus \{ 1 \}$, $z = 1$ under $f_0 \ast f$ is drawn two points $(-\frac{1}{4}, \pm \frac{\sqrt{3}}{2})$, $\partial(f_0 \ast f) = \{ \text{the boundary of the image domain of function } f_0 \ast f \}$ \{\{(-\frac{1}{4}, \pm \frac{\sqrt{3}}{2})\}\} respectively.

Images of the unit disk $D$ under $f$ and $f_0 \ast f$ (right) are drawn as follows in Figure 3.
Remark 4.3. In Example 4.1, the convolution for a right plane harmonic mapping \( f(z) \) with the second complex dilation \( w(z) = -\frac{z-a}{1-az} \) and the the canonical harmonic right half-plane mapping belongs to \( S_{0}^{H} \) and is convex in the direction of the real axis. Example 4.2 says that when the second complex dilation of \( f(z) \) is \( w(z) = -(\frac{z+1}{z-1})^{3} \), it is a different result. Thus, it is interesting to determine which condition to make the convolution of \( f_{0} \) and a right half-plane mapping with a Blaschke product of degree three as its second complex dilation. Furthermore, which condition should be made is satisfy for the general case \( w(z) = \prod_{n=1}^{m} \frac{a_{n}z}{1-a_{n}z} \) \((m \geq 3)\).

References