Filomat 34:4 (2020), 1329–1346
https://doi.org/10.2298/FIL2004329S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Modified CGLS Iterative Algorithm for Solving the Generalized Sylvester-Conjugate Matrix Equation

Caiqin Song[a,b], Qing-Wen Wang[b]

[a] *School of Mathematical Sciences, University of Jinan, Jinan 250022, China*
[b] *Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China*

Abstract. By introducing the real inner product, this paper offers an modified conjugate gradient least squares iterative algorithm (MCGLS)for solving the generalized Sylvester-conjugate matrix equation. The properties of this algorithm are discussed and the finite convergence of this algorithm is proven. This new iterative method can obtain the symmetric least squares Frobenius norm solution within finite iteration steps in the absence of roundoff errors. Finally, two numerical examples are offered to illustrate the effectiveness of the proposed algorithm.

1. Introduction

Matrix equations often arise from system theory, control theory, neural network, and stability analysis [1-6]. How to solve these matrix equations become an important topic which have received much attention [7-15]. Some different methods have been implemented for solving various linear matrix equations.

Recently, by applying the hierarchical identification principle, Ding etal introduced gradient based iterative algorithms to solve (coupled) generalized Sylvester matrix equations, nonlinear systems [16-25]. For more references, one can refer to [26-32]. Furthermore, the least squares solution to the matrix equations have been focused by many researchers [33-40]. For instance, the LSQR iterative algorithms are used to solve the reflexive least squares solutions of the generalized Sylvester matrix equation $A_1X_1B_1 + A_2X_2B_2 + \cdots + A_lX_lB_l = C$ [34], reflexive least squares solutions of the general coupled matrix equations [37],(R,S)-symmetric least squares solutions of the general coupled matrix equations [38], symmetric least squares solutions of the matrix equation $AXB + CYD = E$ [39]. Very recently, Hajarian [33, 35] extends the conjugate gradient least squares iterative algorithm to solve the general coupled matrix equations and the generalized Sylvester-transpose matrix equations.

Motivated by the above work, in this paper we denote a real inner product over complex field and propose an modified CGLS iterative algorithm (MCGLS) for solving the generalized Sylvester-conjugate matrix equation

---

2010 *Mathematics Subject Classification*. Primaryn 15A09; Secondary 15A21, 16U99
Keywords. Modified CGLS iterative algorithm; Least Frobenius norm symmetric solution; Real inner product
Received: 01 August 2017; Accepted: 12 October 2017
Communicated by Dragana Cvetković Ilić
Corresponding author: Qing-Wen Wang
Research supported by the Natural Science Foundation of China (No. 11971294, No.11501246)
Email addresses: songcaiqin0635@163.com (Caiqin Song), wqw@shu.edu.cn (Qing-Wen Wang)

$$\sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j = E, \tag{1.1}$$

This paper focuses on the following two problems.

Problem 1. For given $A_i \in C^{p\times m}, B_i \in C^{n\times q}, C_j \in C^{p\times m}, D_j \in C^{n\times q}, E \in C^{p\times q},\ i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, t$, find symmetric matrix $X^* \in C^{m\times n}$ such that

$$\|\sum_{i=1}^{p} A_i X^* B_i + \sum_{j=1}^{t} C_j \overline{X^*} D_j - E\| = \min_{X\in C^{m\times n}} \|\sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{t} C_j \overline{X} D_j - E\|. \tag{1.2}$$

Problem 2. Let S_r denote the solution set of Problem 1, for given symmetric matrix $X_0 \in C^{m\times n}$, find the matrix $\check{X} \in S_r$, such that

$$\|\check{X} - X_0\| = \min_{X\in S_r} \|X - X_0\|. \tag{1.3}$$

The remainder of the paper is organized as follows. Section 2 offers the symbols and the preliminaries. Section 3 brings the modified CGLS iterative algorithm. It is shown that the proposed algorithm can obtain the solution of Problem 1 for any initial matrix within a finite number of iterations in the absence of roundoff errors. Two numerical examples are offered in Section 4 to illustrate the effectiveness of the proposed algorithm. Finally, some concluding remarks are given in Section 5.

2. The symbols and preliminaries

Before starting this section, we first introduce the following notations which will be used in the rest of this paper. $C^{m\times n}$ denotes the set of $m \times n$ complex matrices. For a matrix $A \in C^{m\times n}$, we denote its transpose, conjugate, conjugate transpose, trace, Frobenius norm and column space by $A^T, \overline{A}, A^H, \mathrm{tr}(A)$, $\|A\|$ and $R(A)$, respectively. Let I_n and S_n denote the $n \times n$ unit matrix and reverse unit matrix respectively. The symbol $\mathrm{vec}(\cdot)$ stands for the vec operator, i.e., for $A = (a_1, a_2, \cdots, a_n) \in C^{m\times n}$, where $a_i (i = 1, 2 \cdots, n)$ denotes the ith column of A, $\mathrm{vec}(A) = (a_1^T, a_2^T, \cdots, a_n^T)^T$.

For X and Y two matrices in $C^{m\times n}$, we define real inner product as

$$\langle X, Y\rangle = Re[\mathrm{tr}(Y^H X)].$$

According this definition, the real inner product has the following properties.
(1)$\langle X, Y\rangle = \langle Y, X\rangle$,
(2)$\langle kX, Y\rangle = k\langle X, Y\rangle, k \in R$,
(3)$\langle X + Y, Z\rangle = \langle X, Z\rangle + \langle Y, Z\rangle$,
(4)$\langle X, X\rangle > 0$, for all $X \neq 0$,
(5)For matrices R, A, B and X with appropriate dimension, $\langle R, AXB\rangle = \langle A^H R B^H, X\rangle$,
(6)For matrices R, A, B and X with appropriate dimension, $\langle R, A\overline{X}B\rangle = \langle \overline{A}^H R \overline{B}^H, X\rangle$,
(7)Two matrices X and Y are said to be orthogonal if $\langle X, Y\rangle = 0$.

The induced-norm of a matrix A is defined by the following formula:

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{Re(tr[A^H A])} = \sqrt{tr[A^H A]}.$$

The associated norm is the well-known Frobenius norm.

Lemma 1([41]) Let U be an inner product space, V be a subspace of U, and $V^\perp$ be the orthogonal complement subspace of V. For a given $u \in U$, if there exists a $v_0 \in V$ such that $\|u - v_0\| \leq \|u - v\|$ holds for any $v \in V$, then v_0 is unique and $v_0 \in V$ is the unique minimization vector in V if and only if $(u - v_0) \perp V$, i.e. $(u - v_0) \in V^\perp$.

Lemma 2 Supposed that $\hat{R}$ is the residual of Eq.(1.1) corresponding to the symmetric matrix $\hat{X} \in C^{m\times n}$, that is, $\hat{R} = E - \sum_{i=1}^{p} \frac{1}{2}(A_i\hat{X}B_i + A_i\hat{X}^T B_i) - \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{\hat{X}}D_j + C_j\hat{X}^H D_j)$. If

$$\sum_{i=1}^{p} \frac{1}{2}(A_i^H\hat{R}B_i^H + \overline{B_i}\hat{R}^T\overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H\overline{\hat{R}}\overline{D_j}^H + D_j\hat{R}^H C_j) = 0.$$

holds, then the symmetric matrix $\hat{X}$ is the symmetric least squares solution of Eq.(1.1).

Proof. Firstly, we define the linear subspace $W = \{F|F = \sum_{i=1}^{p} \frac{1}{2}(A_iXB_i + A_iX^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{X}D_j + C_jX^HD_j)$ with $A_i \in C^{p\times m}, B_i \in C^{n\times q}, C_j \in C^{p\times m}, D_j \in C^{n\times q}, E \in C^{p\times q}$, and $X \in C^{m\times n}, i = 1, 2, \cdots, p, j = 1, 2, \cdots, t$. Now if we let $\hat{F} = \sum_{i=1}^{p} \frac{1}{2}(A_i\hat{X}B_i + A_i\hat{X}^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{\hat{X}}D_j + C_j\hat{X}^HD_j)$, then $\hat{F} \in W$. Hence, for any $F \in W$,one can obtain

$$\begin{aligned}\langle E - \hat{F}, F\rangle &= \langle E - \sum_{i=1}^{p} \frac{1}{2}(A_i\hat{X}B_i + A_i\hat{X}^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{\hat{X}}D_j + C_j\hat{X}^HD_j),\\ &\quad \sum_{i=1}^{p} \frac{1}{2}(A_iXB_i + A_iX^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{X}D_j + C_jX^HD_j)\rangle\\ &= \langle \hat{R}, \sum_{i=1}^{p} \frac{1}{2}(A_iXB_i + A_iX^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{X}D_j + C_jX^HD_j)\rangle\\ &= \langle \sum_{i=1}^{p} \frac{1}{2}(A_i^H\hat{R}B_i^H + \overline{B_i}\hat{R}^T\overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H\overline{\hat{R}}\overline{D_j}^H + D_j\hat{R}^HC_j), X\rangle\end{aligned}$$

Therefore, if we let $\sum_{i=1}^{p} \frac{1}{2}(A_i^H\hat{R}B_i^H + \overline{B_i}\hat{R}^T\overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H\overline{\hat{R}}\overline{D_j}^H + D_j\hat{R}^HC_j) = 0$, it is can be proved that the above equation $\langle E - \hat{F}, F\rangle = 0$. It follows from Lemma 1 that$(E - \hat{F}) \in W^\perp$. Therefore, the matrix $\hat{X}$ is the symmetric least squares solution of Eq(1.1). □

Lemma 3 Let $\hat{X}$ be a symmetric solution of Problem 1, then any symmetric solution $\tilde{X}$ of Problem 1 can be expressed as $\hat{X} + Z$, where the matrix $Z \in C^{m\times n}$ satisfies

$$\sum_{i=1}^{p} \frac{1}{2}(A_iZB_i + A_iZ^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{Z}D_j + C_jZ^HD_j) = 0. \tag{2.1}$$

Proof. Let $\tilde{X}$ be any solution of Problem 1, if we define the matrix $Z = \tilde{X} - \hat{X}$, then we have $\tilde{X} = \hat{X} + Z$. Now it is showed that Eq.(2.1) holds. By applying Lemma 2, we can obtain

$$\begin{aligned}&\|\sum_{i=1}^{p} \frac{1}{2}(A_i\hat{X}B_i + A_i\hat{X}^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{\hat{X}}D_j + C_j\hat{X}^HD_j) - E\|^2\\ &= \|\sum_{i=1}^{p} \frac{1}{2}(A_i\tilde{X}B_i + A_i\tilde{X}^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{\tilde{X}}D_j + C_j\tilde{X}^HD_j) - E\|^2\end{aligned}$$

$$= \|\sum_{i=1}^{p}\frac{1}{2}[A_i(\hat{X}+Z)B_i + A_i(\hat{X}+Z)^T B_i] + \sum_{j=1}^{t}\frac{1}{2}[C_j\overline{(\hat{X}+Z)}D_j + C_j(\hat{X}+Z)^H D_j] - E\|^2$$
$$= \|\sum_{i=1}^{p}\frac{1}{2}(A_iZB_i + A_iZ^TB_i) + \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{Z}D_j + C_jZ^HD_j) - \hat{R}\|^2$$
$$= \|\sum_{i=1}^{p}\frac{1}{2}(A_iZB_i + A_iZ^TB_i) + \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{Z}D_j + C_jZ^HD_j)\|^2 + \|\hat{R}\|^2$$
$$- 2\langle Z, \sum_{i=1}^{p}\frac{1}{2}(A_i^H\hat{R}B_i^H + \overline{B}_i\hat{R}^T\overline{A}_i) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C}_j^H\overline{\hat{R}}\overline{D}_j^H + D_j\hat{R}^HC_j)\rangle$$
$$= \|\sum_{i=1}^{p}\frac{1}{2}(A_iZB_i + A_iZ^TB_i) + \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{Z}D_j + C_jZ^HD_j)\|^2 + \|\hat{R}\|^2$$

This shows that the Eq.(2.1) holds. □

3. Main results

In this section, the modified CGLS iterative algorithm (MCGLS) for solving the symmetric least squares problem of Eq. (1.1) will be offered and some properties of this algorithm will be established.

Algorithm 1. Step 1. Input matrices $A_i \in C^{p\times m}, B_i \in C^{n\times q}, C_j \in C^{p\times m}, D_j \in C^{n\times q}, E \in C^{p\times q}$ and $X(1) \in C^{m\times n}$ for $i = 1, 2, \cdots, p, j = 1, 2, \cdots, t$;

Step 2. Compute

$$R(1) = E - \sum_{i=1}^{p}\frac{1}{2}(A_iX(1)B_i + A_iX(1)^TB_i) - \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{X(1)}D_j + C_jX(1)^HD_j);$$
$$S(1) = \sum_{i=1}^{p}\frac{1}{2}\left(A_i^HR(1)B_i^H + \overline{B}_iR(1)^T\overline{A}_i\right) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C}_j^H\overline{R(1)}\overline{D}_j^H + D_jR(1)^HC_j);$$

$P(1) = S(1), \ \gamma(1) = \|S(1)\|^2;$

For $k = 1, 2, 3, \cdots$ repeat the following:

Step 3. If $\|R(k)\| = 0$, then stop and $X(k)$ is the solution of Eq.(1.6), break;
else if $\|R(k)\| \neq 0$ but $\|S(k)\| = 0$, then stop and $X(k)$ is the solution of Problem 1, break;
else $k := k + 1$;

Step 4. Compute

$$Q(k) = \sum_{i=1}^{p}\frac{1}{2}(A_iP(k)B_i + A_iP(k)^TB_i) + \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{P(k)}D_j + C_jP(k)^HD_j);$$

$\delta(k) = \gamma(k)/\|Q(k)\|^2;$

$X(k+1) = X(k) + \delta(k)P(k);$

$R(k+1) = R(k) - \delta(k)Q(k);$

$$S(k+1) = \sum_{i=1}^{p}\frac{1}{2}\left(A_i^HR(k+1)B_i^H + \overline{B}_iR(k+1)^T\overline{A}_i\right) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C}_j^H\overline{R(k+1)}\overline{D}_j^H + D_jR(k+1)^HC_j);$$
$$= S(k) - \delta(k)[\sum_{i=1}^{p}\frac{1}{2}\left(A_i^HQ(k)B_i^H + \overline{B}_iQ(k)^T\overline{A}_i\right) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C}_j^H\overline{Q(k)}\overline{D}_j^H + D_jQ(k)^HC_j)];$$

$\gamma(k+1) = \|S(k+1)\|^2;$

$\lambda(k) = \gamma(k+1)/\gamma(k);$

$P(k+1) = S(k+1) + \lambda(k)P(k).$

Step 5. Go to Step 3.

Some basic properties of Algorithm 1 are listed in the following lemmas.

Lemma 4. For the sequences $S(k)$, $P(k)$ and $Q(k)$ which are generated by Algorithm 1, if there exists a positive number r, such that $\|S(u)\| \neq 0$ and $\|Q(u)\| \neq 0, \forall u = 1, 2, \cdots, r$, then the following statements hold for $u, v = 1, 2, \cdots, r$ and $u \neq v$.
(1)$\langle S(u), S(v)\rangle = 0$,
(2)$\langle Q(u), Q(v)\rangle = 0$,
(3)$\langle P(u), S(v)\rangle = 0$.

Proof. Step 1: We prove the conclusion by induction. Because the real inner product is commutative, it is enough to prove three statements for $1 \leq u < v \leq r$.

For $u = 1, v = 2$, by Algorithm 1, one can obtain

$$\begin{aligned}
&\langle S(1), S(2)\rangle \\
&= \langle S(1), S(1) - \delta(1)[\sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(1) B_i^H + \overline{B}_i Q(1)^T \overline{A}_i) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(1)} \overline{D_j}^H + D_j Q(1)^H C_j)]\rangle \\
&= \|S(1)\|^2 - \delta(1)\langle S(1), \sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(1) B_i^H + \overline{B}_i Q(1)^T \overline{A}_i) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(1)} \overline{D_j}^H + D_j Q(1)^H C_j)\rangle \\
&= \|S(1)\|^2 - \delta(1)\langle \sum_{i=1}^{p} \frac{1}{2}(A_i S(1) B_i + A_i S(1)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{S(1)} D_j + C_j S(1)^H D_j), Q(1)\rangle \\
&= \|S(1)\|^2 - \delta(1)\langle Q(1), Q(1)\rangle = 0,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\langle Q(1), Q(2)\rangle = \langle Q(1), \sum_{i=1}^{p} \frac{1}{2}(A_i P(2) B_i + A_i P(2)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{P(2)} D_j + C_j P(2)^H D_j)\rangle \\
&= \langle Q(1), \sum_{i=1}^{p} \frac{1}{2}[A_i(S(2) + \lambda(1)P(1))B_i + A_i(S(2) + \lambda(1)P(1))^T B_i) \\
&+ \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{(S(2) + \lambda(1)P(1))} D_j + C_j(S(2) + \lambda(1)P(1))^H D_j]\rangle \\
&= \lambda(1)\|Q(1)\|^2 + \langle Q(1), \sum_{i=1}^{p} \frac{1}{2}(A_i S(2) B_i + A_i S(2)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{S(2)} D_j + C_j S(2)^H D_j)\rangle \\
&= \lambda(1)\|Q(1)\|^2 + \langle \sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(1) B_i^H + \overline{B}_i Q(1)^T \overline{A}_i) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(1)} \overline{D_j}^H + D_j Q(1)^H C_j), S(2)\rangle \\
&= \lambda(1)\|Q(1)\|^2 - \tfrac{1}{\delta(1)}\langle S(2) - S(1), S(2)\rangle = 0,
\end{aligned} \tag{3.2}$$

and

$$\langle P(1), S(2)\rangle = \langle S(1), S(2)\rangle = 0. \tag{3.3}$$

Step 2: In this case, for $u < w < r$ we assume that

$$\langle S(u), S(w)\rangle = 0, \quad \langle Q(u), Q(w)\rangle = 0, \quad \langle P(u), S(w)\rangle = 0.$$

Therefore, one can obtain

$$\begin{aligned}
&\langle S(u), S(w+1)\rangle \\
&= \langle S(u), S(w) - \delta(w)[\sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(w) B_i^H + \overline{B}_i Q(w)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H + D_j Q(w)^H C_j)]\rangle \\
&= -\delta(w)\langle S(u), \sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(w) B_i^H + \overline{B}_i Q(w)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H + D_j Q(w)^H C_j)\rangle \\
&= -\delta(w)\langle \sum_{i=1}^{p} \frac{1}{2}(A_i S(u) B_i + A_i S(u)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{S(u)} D_j + C_j S(u)^H D_j), Q(w)\rangle \qquad (3.4) \\
&= -\delta(w)\langle \sum_{i=1}^{p} \frac{1}{2}[A_i(P(u) - \lambda(u-1)P(u-1))B_i + A_i S(u)^T B_i] \\
&+ \sum_{j=1}^{t} \frac{1}{2}[C_j \overline{(P(u) - \lambda(u-1)P(u-1))} D_j + C_j (P(u) - \lambda(u-1)P(u-1))^H D_j], Q(w)\rangle \\
&= -\delta(w)\langle Q(u) - \lambda(u-1)Q(u-1), Q(w)\rangle = 0,
\end{aligned}$$

$$\begin{aligned}
&\langle Q(u), Q(w+1)\rangle \\
&= \langle Q(u), \sum_{i=1}^{p} \frac{1}{2}(A_i P(w+1) B_i + A_i P(w+1)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{P(w+1)} D_j + C_j P(w+1)^H D_j)\rangle \\
&= \langle Q(u), \sum_{i=1}^{p} \frac{1}{2}[A_i(S(w+1) + \lambda(w)P(w))B_i + A_i(S(w+1) + \lambda(w)P(w))^T B_i] \\
&+ \sum_{j=1}^{t} \frac{1}{2}[C_j \overline{(S(w+1) + \lambda(w)P(w))} D_j + C_j (S(w+1) + \lambda(w)P(w))^H D_j]\rangle \\
&= \langle Q(u), \sum_{i=1}^{p} \frac{1}{2}(A_i S(w+1) B_i + A_i S(w+1)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{S(w+1)} D_j \\
&\quad + C_j S(w+1)^H D_j) + \lambda(w) Q(w)\rangle \qquad (3.5) \\
&= \langle Q(u), \sum_{i=1}^{p} \frac{1}{2}(A_i S(w+1) B_i + A_i S(w+1)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{S(w+1)} D_j + C_j S(w+1)^H D_j)\rangle \\
&= \langle \sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(u) B_i^H + \overline{B_i} Q(u)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(u)}\,\overline{D_j}^H + D_j Q(u)^H C_j), S(w+1)\rangle \\
&= -\tfrac{1}{\delta(u)}\langle S(u+1) - S(u), S(w+1)\rangle = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\langle P(u), S(w+1)\rangle \\
&= \langle P(u), S(w) - \delta(w)[\sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(w) B_i^H + \overline{B}_i Q(w)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H + D_j Q(w)^H C_j)]\rangle \\
&= -\delta(w)\langle P(u), \sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(w) B_i^H + \overline{B}_i Q(w)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(w)}\,\overline{D_j}^H + D_j Q(w)^H C_j)\rangle \qquad (3.6) \\
&= -\delta(w)\langle \sum_{i=1}^{p} \frac{1}{2}(A_i P(u) B_i + A_i P(u)^T B_i + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j P(u) D_j} + C_j P(u)^H D_j), Q(w)\rangle \\
&= -\delta(w)\langle Q(u), Q(w)\rangle = 0,
\end{aligned}$$

For $u = w$, one has

$$
\begin{aligned}
&\langle S(w), S(w+1)\rangle \\
&= \langle S(w), S(w) - \delta(w)(\sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(w) B_i^H + \overline{B_i} Q(w)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(w) D_j}^H + D_j Q(w)^H C_j))\rangle \\
&= \|S(w)\|^2 - \delta(w)\langle S(w), \sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(w) B_i^H + \overline{B_i} Q(w)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(w) D_j}^H + D_j Q(w)^H C_j)\rangle \\
&= \|S(w)\|^2 - \delta(w)\langle \sum_{i=1}^{p} A_i S(w) B_i + \sum_{j=1}^{t} C_j \overline{S(w)} D_j, Q(w)\rangle \\
&= \|S(w)\|^2 - \delta(w)\langle \textstyle\sum_{i=1}^{p} \frac{1}{2}(A_i S(w) B_i + A_i S(w)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{S(w)} D_j + CjS(w)^H D_j), Q(w)\rangle \\
&= \|S(w)\|^2 - \delta(w)\langle \sum_{i=1}^{p} \frac{1}{2}[A_i(P(w) - \lambda(w-1)P(w-1))B_i + A_i(P(w) - \lambda(w-1)P(w-1))^T B_i) \\
&+ \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{(P(w) - \lambda(w-1)P(w-1))} D_j + Cj(P(w) - \lambda(w-1)P(w-1))^H D_j), Q(w)\rangle \\
&= \|S(w)\|^2 - \delta(w)\langle Q(w) - \lambda(w-1)Q(w-1), Q(w)\rangle = 0,
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&\langle Q(w), Q(w+1)\rangle \\
&= \langle Q(w), \sum_{i=1}^{p} \frac{1}{2}(A_i P(w+1) B_i + A_i P(w)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{P(w)} D_j + CjP(w)^H D_j)\rangle \\
&= \langle Q(w), \sum_{i=1}^{p} \frac{1}{2}[A_i(S(w+1) + \lambda(w)P(w))B_i + A_i(S(w+1) + \lambda(w)P(w))^T B_i] \\
&+ \sum_{j=1}^{t} \frac{1}{2}[C_j \overline{(S(w+1) + \lambda(w)P(w))} D_j + Cj(S(w+1) + \lambda(w)P(w))^H D_j)\rangle \\
&= \langle Q(w), \sum_{i=1}^{p} \frac{1}{2}(A_i S(w+1) B_i + A_i S(w+1)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{S(w+1)} D_j + CjS(w+1)^H D_j) + \lambda(w) Q(w))\rangle \\
&= \lambda(w)\|Q(w)\|^2 + \langle Q(w), \sum_{i=1}^{p} \frac{1}{2}(A_i S(w+1) B_i + A_i S(w+1)^T B_i) \\
&\quad + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{S(w+1)} D_j + CjS(w+1)^H D_j)\rangle \\
&= \lambda(w)\|Q(w)\|^2 + \langle \sum_{i=1}^{p} \frac{1}{2}(A_i^H Q(w) B_i^H + \overline{B_i} Q(w)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{Q(w) D_j}_H + DjQ(w)^H C_j), S(w+1)\rangle \\
&= \lambda(w)\|Q(w)\|^2 - \tfrac{1}{\delta(w)}\langle S(w+1) - S(w), S(w+1)\rangle \\
&= \lambda(w)\|Q(w)\|^2 - \tfrac{1}{\delta(w)}\|S(w+1)\|^2 = 0,
\end{aligned}
\tag{3.8}
$$

and

$$\begin{aligned}
&\langle P(w), S(w+1)\rangle \\
&= \langle P(w), S(w) - \delta(w)[\sum_{i=1}^{p}\frac{1}{2}(A_i^H Q(w)B_i^H + \overline{B}_i Q(w)^T\overline{A_i}) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C_j}^H\overline{Q(w)D_j}_H + DjQ(w)^H C_j)]\rangle \\
&= \langle S(w) + \lambda(w-1)S(w-1) + \lambda(w-1)\lambda(w-2)S(w-2) + \cdots + \lambda(w-1)\ldots\lambda(1)S(1), S(w)\rangle \\
&\quad - \delta(w)\langle P(w), \sum_{i=1}^{p}\frac{1}{2}(A_i^H Q(w)B_i^H + \overline{B}_i Q(w)^T\overline{A_i}) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C_j}^H\overline{Q(w)D_j}^H + DjQ(w)^H C_j)\rangle \\
&= \|S(w)\|^2 - \delta(w)\langle\sum_{i=1}^{p}\frac{1}{2}(A_iP(w)B_i + A_iP(w)^T B_i) + \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{P(w)}D_j + C_jP(w)^H D_j), Q(w)\rangle \\
&= \|S(w)\|^2 - \delta(w)\langle Q(w), Q(w)\rangle = 0.
\end{aligned} \tag{3.9}$$

By the principle of induction, we draw the conclusion. □

Theorem 1. For algorithm 1, if there exists a positive number l such that $\delta(l) = 0$ or $\delta(l) = \infty$, then $X(l)$ is a solution of Problem 1.

Proof. If $\delta(l) = 0$, we have $\|S(l)\|^2 = 0$; If $\delta(l) = \infty$, we have $\|Q(l)\|^2 = 0$. Hence, one can obtain

$$\begin{aligned}
&\|S(l)\|^2 = \langle S(l) + \lambda(l-1)S(l-1) + \lambda(l-1)\lambda(l-2)S(l-2) + \cdots + \lambda(l-1)\ldots\lambda(1)S(1), S(l)\rangle \\
&= \langle P(l), S(l)\rangle \\
&= \langle P(l), \sum_{i=1}^{p}\frac{1}{2}(A_i^H R(l)B_i^H + \overline{B}_i R(l)^T\overline{A_i}) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C_j}^H\overline{R(l)D_j}_H + DjR(l)^H C_j)\rangle \\
&= \langle\sum_{i=1}^{p}\frac{1}{2}(A_iP(l)B_i + A_iP(l)^T B_i) + \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{P(l)}D_j + C_jP(l)^H D_j), R(l)\rangle \\
&= 0.
\end{aligned}$$

Thus, for $\delta(l) = 0$ or $\delta(l) = \infty$ one has

$$S(l) = \sum_{i=1}^{p}\frac{1}{2}(A_i^H R(l)B_i^H + \overline{B}_i R(l)^T\overline{A_i}) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C_j}^H\overline{R(l)D_j}^H + DjR(l)^H C_j) = 0.$$

So by Lemma 2 we can conclude that $X(l)$ is the solution of Problem 1. □

Theorem 2. If the matrix equation (1.1) is consistent, then, for any arbitrary initial matrix $X(l)$, the solution X^* of the Problem 1 can be obtained by using Algorithm 1 within a finite number of iterations in the absence of roundoff errors.

Proof. By Lemma 4, the set $S(i), i = 1, 2, 3, \cdots, m \times n$ is an orthogonal basis of the real inner product space $C^{m\times n}$ with dimension $m \times n$. Therefore, we can obtain $\|S(m \times n + 1)\| = 0$. It is also showed that $X(m \times n + 1)$ is the solution of Problem 1 in the absence of roundoff errors. □

Theorem 3. Let the initial matrix be $\sum_{i=1}^{p}\frac{1}{2}(A_i^H F(l)B_i^H + \overline{B}_i F(l)^T\overline{A_i}) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C_j}^H\overline{F(l)D_j}^H + DjF(l)^H C_j)$, in which $F(1) \in C^{m\times n}$ is an arbitrary symmetric matrix, or especially $X(1) = 0$, then the solution X^* generated by Algorithm 1 is the symmetric least norm solution of Problem 1.

Proof. Supposed that the initial matrix is

$$X(1) = \sum_{i=1}^{p}\frac{1}{2}(A_i^H F(l)B_i^H + \overline{B}_i F(l)^T\overline{A_i}) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C_j}^H\overline{F(l)D_j}^H + DjF(l)^H C_j),$$

then by Algorithm 1 the generated matrix $X(k)$ can be easily expressed as

$$X(k) = \sum_{i=1}^{p} \frac{1}{2}(A_i^H F(k) B_i^H + \overline{B_i} F(k)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{F(k)} \overline{D_j}^H + D_j F(k)^H C_j),$$

for certain matrices $F(k) \in C^{m\times n}$ for $k = 2, 3, \ldots$. It is showed that there exists a matrix $F^* \in C^{m\times n}$ such that

$$X^* = \sum_{i=1}^{p} \frac{1}{2}(A_i^H F^* B_i^H + \overline{B_i}(F^*)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{F^*} \overline{D_j}^H + D_j (F^*)^H C_j).$$

Now if let $\tilde{X}^*$ is an arbitrary solution of Problem 1, then it is followed from Lemma 3 that there exists the matrix $Z^* \in C^{m\times n}$ such that

$$\tilde{X}^* = X^* + Z^*,$$

and

$$\sum_{i=1}^{p} \frac{1}{2}(A_i Z^* B_i + A_i (Z^*)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{Z^*} D_j + C_j (Z^*)^H D_j) = 0.$$

Thus, we have

$$\begin{aligned} &\langle X^*, Z^* \rangle \\ &= \langle \sum_{i=1}^{p} \frac{1}{2}(A_i^H F^* B_i^H + \overline{B_i}(F^*)^T \overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H \overline{F^*} \overline{D_j}^H + D_j (F^*)^H C_j), Z^* \rangle \\ &= \langle F^*, \sum_{i=1}^{p} \frac{1}{2}(A_i Z^* B_i + A_i (Z^*)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{Z^*} D_j + C_j (Z^*)^H D_j) \rangle = 0. \end{aligned} \tag{3.10}$$

Therefore, according to the above Eq.(3.10), it is showed that

$$\|\tilde{X}^*\|^2 = \|X^* + Z^*\|^2 = \|X^*\|^2 + \|Z^*\|^2 + 2\langle X^*, Z^* \rangle = \|X^*\|^2 + \|Z^*\|^2 \geq \|X^*\|^2.$$

This can be showed that the solution X^* is the symmetric least Frobenius norm solution of Problem 1. □

Similar to [34], the minimization property of the proposed algorithm is stated as follows. This property shows that Algorithm 1 converges smoothly.

Theorem 3. For any initial symmetric matrix $X(1) \in C^{m\times n}$, we have

$$\begin{aligned} &\| \sum_{i=1}^{p} \frac{1}{2}(A_i X(k+1) B_i + A_i X(k+1)^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{X(k+1)} D_j + C_j X(k+1)^H D_j) - E \|^2 \\ &= \min_{X \in \psi_k} \| \sum_{i=1}^{p} \frac{1}{2}(A_i X B_i + A_i X^T B_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j \overline{X} D_j + C_j X^H D_j) - E \|^2, \end{aligned} \tag{3.11}$$

where $X(k+1)$ is generated by Algorithm 1 at the $k+1$-th iteration and ψ_k presents an affine subspace which has the following form

$$\psi_k = X(1) + span(P(1), P(2), \ldots, P(k)). \tag{3.12}$$

Proof. For any matrix $X \in \psi_k$, it follows from Eq.(3.12) that there exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$X = X(1) + \sum_{l=1}^{k} \alpha_l P(l). \tag{3.13}$$

Now we define the continuous and differentiable function f with respect to the variable $\alpha_1, \alpha_2, \ldots, \alpha_k$ as

$$\begin{aligned}
&f(\alpha_1, \alpha_2, \ldots, \alpha_k)\\
&= \| \sum_{i=1}^{p} \frac{1}{2}[A_i(X(1) + \sum_{l=1}^{k} \alpha_l P(l))B_i + A_i(X(1) + \sum_{l=1}^{k} \alpha_l P(l))^T B_i]\\
&+ \sum_{j=1}^{t} \frac{1}{2}[C_j\overline{(X(1) + \sum_{l=1}^{k} \alpha_l P(l))}D_j + C_j(X(1) + \sum_{l=1}^{k} \alpha_l P(l))^H D_j] - E\|^2.
\end{aligned}$$

According to Lemma 4, we have

$$\begin{aligned}
f(\alpha_1, \alpha_2, \ldots, \alpha_k) &= \| \textstyle\sum_{i=1}^{p} \frac{1}{2}(A_iX(1)B_i + A_iX(1)^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{X(1)}D_j + C_jX(1)^HD_j) - E\\
&+ \textstyle\sum_{l=1}^{k} \alpha_l \left[\sum_{i=1}^{p} \frac{1}{2}(A_iP(l)B_i + A_iP(l)^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{P(l)}D_j + C_jP(l)^HD_j)\right] \|^2\\
&= \|R(1)\|^2 + \textstyle\sum_{l=1}^{k} \alpha_l^2 \|Q(l)\|^2 - 2\alpha_l \langle Q(l), R(1)\rangle.
\end{aligned}$$

Now we consider the problem of minimizing the function $f(\alpha_1, \alpha_2, \ldots, \alpha_k)$. It is obvious that

$$\min_{\alpha_l} f(\alpha_1, \alpha_2, \ldots, \alpha_k) = \min_{X \in \psi_k} \| \sum_{i=1}^{p} \frac{1}{2}(A_iXB_i + A_iX^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{X}D_j + C_jX^HD_j) - E\|^2.$$

For this function, the minimum occurs when

$$\frac{\partial f(\alpha_1, \alpha_2, \ldots, \alpha_k)}{\partial \alpha_l} = 0 \quad for \;\; l = 1, 2, \cdots, k.$$

Thus, we can get

$$\alpha_l = \frac{\langle Q(l), R(1)\rangle}{\|Q(l)\|^2}.$$

By Algorithm 1, one can obtain

$$R(1) = R(l) + \delta(l-1)Q(l-1) + \delta(l-2)Q(l-2) + \cdots + \delta(1)Q(1).$$

Therefore, it follows from Lemma 4 that

$$\begin{aligned}
\alpha_l &= \tfrac{\langle Q(l),R(l)\rangle}{\|Q(l)\|^2} = \tfrac{\langle P(l), \sum_{i=1}^{p} \frac{1}{2}(A_i^HR(l)B_i^H + \overline{B_i}R(l)^T\overline{A_i}) + \sum_{j=1}^{t} \frac{1}{2}(\overline{C_j}^H\overline{R(l)}\overline{D_j}^H + DjR(l)^HC_j)\rangle}{\|Q(l)\|^2}\\
&= \tfrac{\langle P(l),S(l)\rangle}{\|Q(l)\|^2} = \tfrac{\langle S(l)+\lambda(l-1)P(l-1),S(l)\rangle}{\|Q(l)\|^2}\\
&= \tfrac{\|S(l)\|^2}{\|Q(l)\|^2} = \delta(l).
\end{aligned}$$

Thus, the proof has been completed. □

By Theorem 1, the solution generalized by Algorithm 1 at the $k+1-$th iteration for any initial matrix minimizes the residual norm in the affine subspace ψ_k. Also one has

$$\begin{aligned}
&\| \sum_{i=1}^{p} \frac{1}{2}(A_iX(k+l)B_i + A_iX(k+l)^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{X(k+l)}D_j + C_jX(k+l)^HD_j) - E\|^2\\
&\leq \| \sum_{i=1}^{p} \frac{1}{2}(A_iX(k)B_i + A_iX(k)^TB_i) + \sum_{j=1}^{t} \frac{1}{2}(C_j\overline{X(k)}D_j + C_jX(k)^HD_j) - E\|^2,
\end{aligned} \tag{3.14}$$

which shows that the sequence of the norm of residuals $\|R(1)\|, \|R(2)\|, \cdots$ is monotonically decreasing. This decent property of the norm of the residuals shows that the Algorithm 1 processes fast and smoothly convergence.

In the following, we will solve Problem 2. For a given matrix X_0, one has

$$\begin{aligned}
&\min_{X\in C^{m\times n}} \|\sum_{i=1}^{p}\frac{1}{2}(A_iXB_i + A_iX^TB_i) + \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{X}D_j + C_jX^HD_j) - E\| \\
&= \min_{X\in C^{m\times n}} \|\sum_{i=1}^{p}\frac{1}{2}[A_i(X-X_0)B_i + A_i(X-X_0)^TB_i] + \sum_{j=1}^{t}\frac{1}{2}[C_j\overline{(X-X_0)}D_j + C_j(X-X_0)^HD_j] \\
&\quad - (E - \sum_{i=1}^{p}\frac{1}{2}(A_iX_0B_i + A_iX_0^TB_i) - \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{X_0}D_j + C_jX_0^HD_j)\|.
\end{aligned} \tag{3.15}$$

If we denote the set

$$E_1 = E - \sum_{i=1}^{p}\frac{1}{2}(A_iX_0B_i + A_iX_0^TB_i) - \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{X_0}D_j + C_jX_0^HD_j)$$

and let

$$X_1 = X - X_0,$$

then Problem 2 is equivalent to find the least Frobenius norm solution X_1^* of

$$\min_{X_1\in C^{m\times n}} \|\sum_{i=1}^{p}\frac{1}{2}(A_iX_1B_i + A_iX_1^TB_i) + \sum_{j=1}^{t}\frac{1}{2}(C_j\overline{X_1}D_j + C_jX_1^HD_j) - E\|, \tag{3.16}$$

which can be computed by using Algorithm 1 with the initial matrix

$$X_1(1) = \sum_{i=1}^{p}\frac{1}{2}(A_i^HFB_i^H + \overline{B_i}F^T\overline{A_i}) + \sum_{j=1}^{t}\frac{1}{2}(\overline{C_j}^H\overline{FD_j}^H + D_jF^HC_j),$$

where $F \in C^{m\times n}$ is an arbitrary matrix, or especially $X_1(1) = 0$. Thus the solution of Problem 2 can be stated as

$$\check{X} = X_1^* + X_0. \tag{3.17}$$

4. Numerical examples

In this section, two numerical examples are presented to illustrate the efficiency of Algorithm 1.

Example 4.1. Find the symmetric least Frobenius norm solution of the following generalized sylvester-conjugate matrix equation

$$AXB + C\overline{X}D = M, \tag{4.1}$$

where

$$A = \begin{pmatrix} 1+2i & 13-i & 6+i & 4+3i \\ 2+i & 0 & 12 & 10 \\ 5+6i & 2-3i & 11-2i & i \\ 1 & 12 & 0 & 9i \end{pmatrix}, B = \begin{pmatrix} 2 & 3-12i & 4+6i & 9+8i \\ 46i & 11 & 12 & 9+18i \\ 0 & 12 & 15 & 18 \\ 2i & -9i & 12 & 11 \end{pmatrix},$$

$$C = \begin{pmatrix} i & 9 & 1-5i & 2i \\ 11 & 2-13i & 12 & 11 \\ 1+8i & 11-2i & 21 & i \\ 9+8i & 11 & i & 8+i \end{pmatrix}, D = \begin{pmatrix} 9+2i & 2-i & 2i & 6+8i \\ 4i & 19 & 20 & 11 \\ 23 & 26+3i & 9i & 9 \\ 23 & 0 & 16 & 8i \end{pmatrix},$$

$$M=\begin{pmatrix} 104+7572i & 9059-1126i & 7465+3237i & 6522+7607i \\ 6409-816i & 11016-10231i & 11011-833i & 9606+5155i \\ 7700+9199i & 12752-4698i & 13886+1860i & 9900+7479i \\ 7484+6793i & 13561+3984i & 5700+4557i & 6483+7741i \end{pmatrix}.$$

It can be verified that the generalized Sylvester-conjugate matrix equation (4.1) are consistent and have the solution

$$X=\begin{pmatrix} 4+3i & 2+i & 11 & 6 \\ 2+i & 11+2i & 9+6i & 0 \\ 11 & 9+6i & 2i & 7 \\ 6 & 0 & 7 & 12i \end{pmatrix}.$$

Applying Algorithm 1, if we let the initial matrix $X(0)=zeros(4,4)$,$X(0)=10\times I_4$ and $X(0)=10\times ones(4,4)$ respectively, we have the solution to the equation (4.1)

$$X(35)=\begin{pmatrix} 4.0000+3.0000i & 2.0000+1.0000i & 11.0000 & 5.9999 \\ 2.0000+1.0000i & 11.0000+2.0000i & 9.0000+6.0000i & 0.0000 \\ 11.0000 & 9.0000+6.0000i & 2.0000i & 7.0000 \\ 5.9999 & 0.0000 & 7.0000 & 12.0000i \end{pmatrix}.$$

with corresponding residual

$$r(35)=1.3684\times 10^{-12}, \quad err(35)=4.7075\times 10^{-16}.$$

The residual of solution are presented in figure 1 and figure 2, where

$$r(k)=log_{10}\|M-AX(k)B-C\overline{X(k)}D\|.$$

The relative errors of solution are presented in figure 3 and figure 4, where

$$err(k)=\frac{\|X(k)-X\|}{\|X\|}.$$

Example 4.2. Find the symmetric least Frobenius norm solution of the following matrix equation

$$AXB+CXD+E\overline{X}F=M, \tag{4.2}$$

where

$$A=\begin{pmatrix} 112i & 13-79i & 6+34i & 43+31i \\ 21+2i & 110 & 12 & 10 \\ 5+6i & 2-23i & 113-2i & i \\ 13 & 152 & 0 & 59i \end{pmatrix}, B=\begin{pmatrix} 255 & 35-152i & 45+65i & 95+85i \\ 46i & 151 & 152 & 95+58i \\ 0 & 512 & 155 & 185 \\ 2i & -95i & 125 & 118 \end{pmatrix},$$

$$C=\begin{pmatrix} 16i & 4+9i & 2+123i & 69i \\ -28+9i & 2+8i & 6 & 178 \\ 31 & 228 & 1-i & 6+9i \\ 2 & 3 & 15 & 29i \end{pmatrix}, D=\begin{pmatrix} 2-9i & 111 & 3 & 1-78i \\ 4 & 3+2i & 2 & 6+90i \\ 4 & 2 & 390i & 1 \\ 1 & 22 & 130i & 0 \end{pmatrix},$$

$$E=\begin{pmatrix} 34i & 9-9i & 13-54i & 21i \\ 118 & 27-13i & 572 & 11-78i \\ 188i & 110-2i & 21-96i & i \\ 96+8i & 11i & i & 86+i \end{pmatrix}, F=\begin{pmatrix} 96+2i & 26-i & 26i & 66+84i \\ 446i & 169 & 260 & 116 \\ 23 & 26+63i & 97i & 97 \\ 237 & 0 & 17-9i & 8-56i \end{pmatrix},$$

$$M=\begin{pmatrix} -42399+44932i & 509238+263601i & -932917-126204i & -504532+106117i \\ 8055258+5.601465i & 46181711.463378i & 5333871-931642i & 3762681-1447052i \\ 54811+1167795i & 1932883+684785i & -549837+3173627i & 839701+914561i \\ 1207735+1390280i & 2929188+1517478i & 1866528+1291000i & 1926580+1468563i \end{pmatrix}.$$

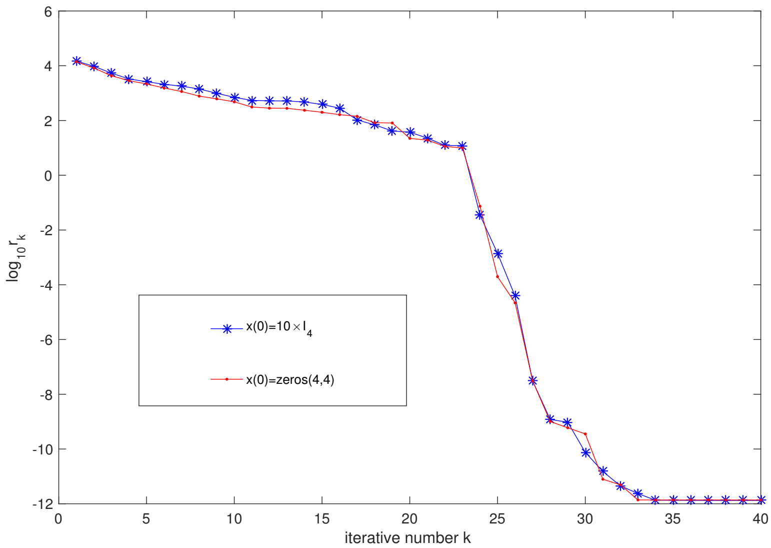

Figure 1: The residual of solution for Example 4.1

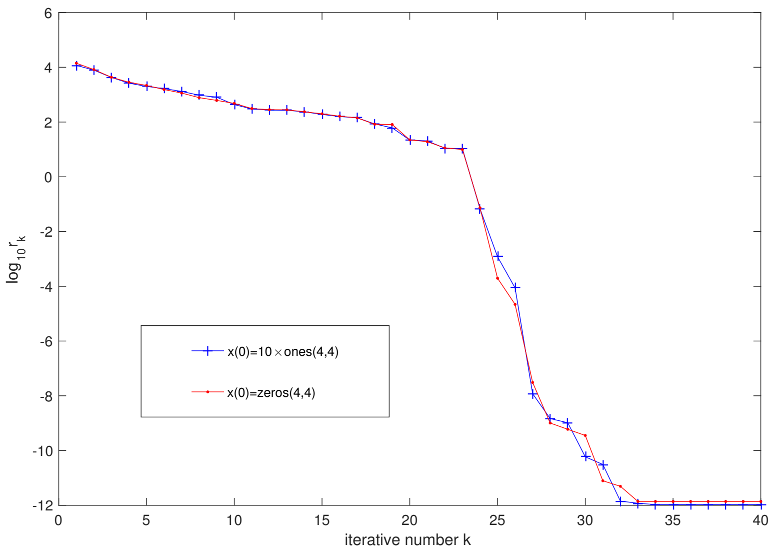

Figure 2: The residual of solution for Example 4.1

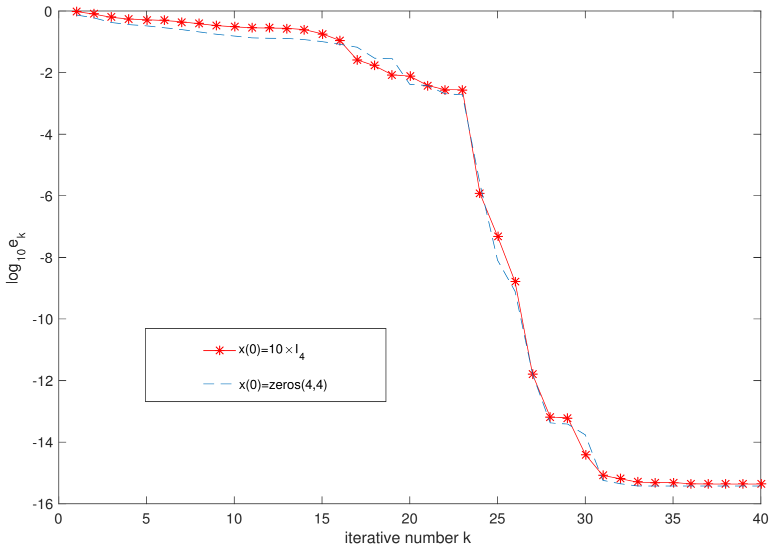

Figure 3: The relative error of solution for Example 4.1

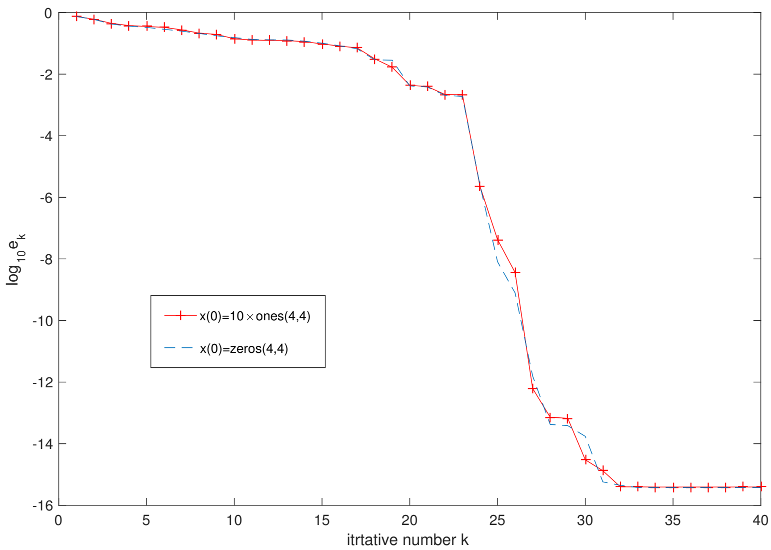

Figure 4: The relative error of solution for Example 4.1

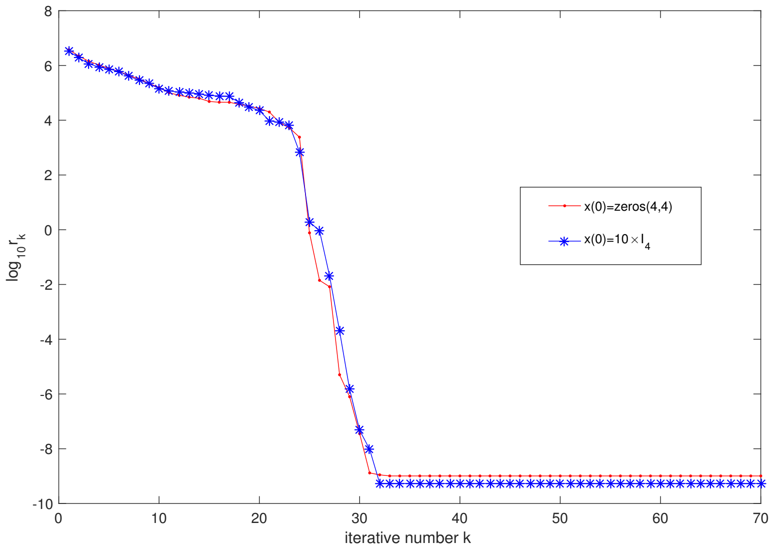

Figure 5: The residual of solution for Example 4.2

It can be verified that the generalized Sylvester-conjugate matrix equation (4.2) are consistent and have the solution

$$X = \begin{pmatrix} 24+13i & 22+i & 11i & 6 \\ 22+i & 21+2i & 19+16i & 10 \\ 11i & 19+16i & 2i & 17 \\ 6 & 10 & 17 & 12i \end{pmatrix}.$$

Let the initial matrix $X(0) = zeros(4,4)$,$X(0) = 10 \times I_4$ and $X(0) = 10 \times ones(4,4)$ respectively, by applying Algorithm 1 we obtain the solution.

The residual of solution are presented in figure 5 and figure 6, where

$$r(k) = log_{10}||M - AX(k)B - C\overline{X(k)}D||.$$

The relative errors of solution are presented in figure 7 and figure 8, where

$$err(k) = \frac{||X(k) - X||}{||X||}.$$

$$X(32) = \begin{pmatrix} 24.0000+12.9999i & 22.0000+0.9999i & 11.0000i & 5.9999 \\ 22.0000+1.0000i & 21.0000+1.9999i & 19.0000+16.0000i & 10.0000 \\ 11.0000i & 19.0000+16.0000i & 2.0000i & 17.0000 \\ 5.9999 & 9.9999 & 17.0000 & 12.0000i \end{pmatrix}.$$

with corresponding residual

$$r(32) = 1.3075 \times 10^{-9}.$$

$$err(32) = 8.1374 \times 10^{-16}.$$

Remark 1. From the two examples above, it is showed that the Algorithm 1 is very efficient for any initial symmetric matrices. Our proposed iterative method can obtain the symmetric least squares Frobenius norm solution within finite iteration steps in the absence of roundoff errors for any initial symmetric matrices.

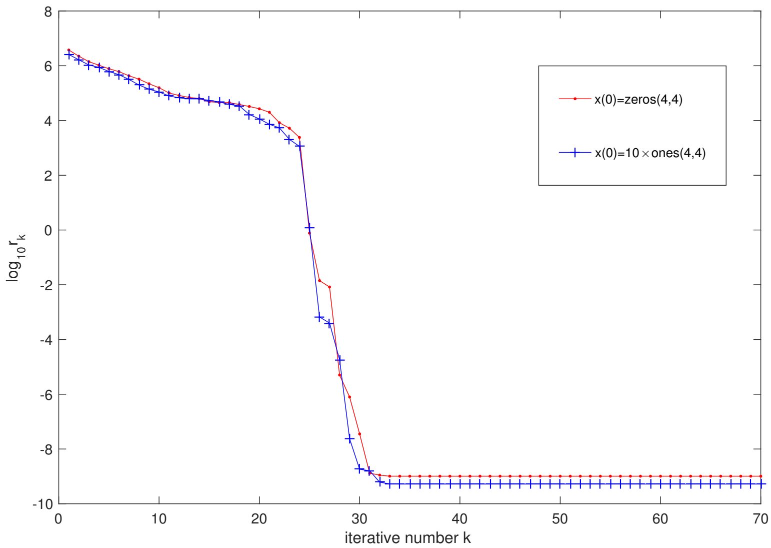

Figure 6: The residual of solution for Example 4.2

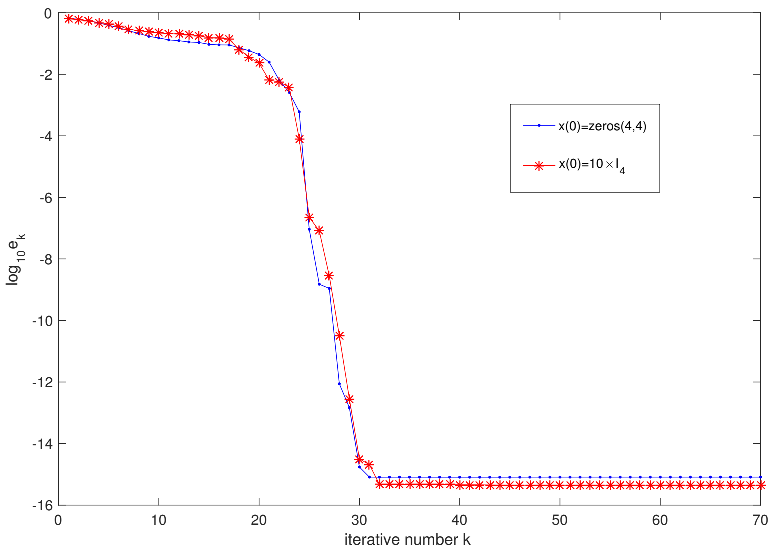

Figure 7: The relative error of solution for Example 4.2

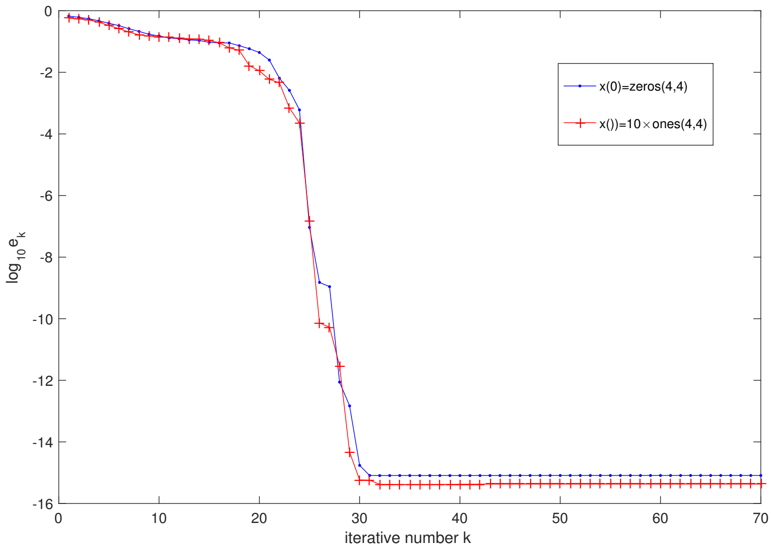

Figure 8: The relative error of solution for Example 4.2

5. Conclusions

In this paper, by extending the CGLS iterative algorithm, we propose the modified CGLS iterative algorithm (MCGLS) to solve the symmetric least squares solution of generalized Sylvester-conjugate matrix equation (1.1). For any initial symmetric matrix $X(1)$, by the proposed MCGLS iterative algorithm, symmetric least Frobenius norm solution X^* can be obtained in finite iteration steps in the absence of roundoff errors. Moreover, by using this MCGLS iterative method, the optimal approximation solution $\check{X}$ to a given matrix X_0 can be derived by first finding the symmetric least Frobenius norm solution of a new corresponding matrix equation. This iterative algorithm can be adapted to solve different classes linear matrix equations. Two numerical examples are offered to illustrate the effectiveness of the proposed algorithm.

Acknowledgements
The authors are very grateful to the anonymous reviewers and the editor for their helpful comments and suggestions which have helped us in improving the quality of this paper.

References

[1] A. Gerheim, Numerical solution of the Lyapunov equation for narrow-band digital filters, IEEE Transaction on Circuits and Systems 31(1984) 991–992.
[2] A.Varga, Robust pole assignment via Sylvester equation based state feedback parametrization, Proceedings of the 2000 IEEE International Symposium on Computer-Aided Control System Design, Alsaka, USA, 2000.
[3] Y.N.Zhang,D.C.Jiang,J.Wang, A recurrent neural network for solving Sylvester equation with time-varying coefficients, IEEE Transaction Neural Network 13(2002)1053–1063.
[4] A.Shahzad, B.L.Jones, E.C. Kerrigan, G.A. Constantinides, An efficient algorithm for the solution of a coupled Sylvester equation appearing in descriptor systems, Automatica 47(2011)244–248.
[5] G.R.Duan, Parametric eigenstructure assignment in high-order linear systems, International Journal of Control Automation and Systems 3(2005)419–429.
[6] B. Zhou, G.R.Duan, On the generalized Sylvester mapping and matrix equationa, Systems Control Letters 57(2008)200–208.
[7] B. Zhou, J. Lam, G.R. Duan, Convergence of gradient-based iterative solution of the coupled Markovian jump Lyapunov equations, Computation Mathematica with Application 56(2008)3070–3078.

[8] Q. W. Wang, C.K. Li, Ranks and the least-norm of the general solution to a system of quaternion matrix equations, Linear Algebra and its Applications 430(2009)1626–1640.

[9] Q. W. Wang, Z.H.He, Systems of coupled generalized Sylvester matrix equations, Automatica 50(2014)2840–2844.

[10] Q. W. Wang, Z. H. He, Solvability conditions and general solution for the mixed Sylvester equations. Automatica 49(2013)2713–2719.

[11] X. P. Sheng, G.L.Chen, A finite iterative method for solving a pair of linear matrix equations $(AXB, CXD) = (E, F)$, Applied Mathematics and Compututation 189(2007)1350–1358.

[12] M. H. Wang, X. H. Cheng, M. S. Wei, Iterative algorithms for solving the matrix equation $AXB+CX^TD = E$, Applied Mathematics and Compututation 187(2007)622–629.

[13] X.P. Shen, G.L. Chen, An iterative method for the symmetric and skew symmetric solutions of a linear matrix equation $AXB + CYD = E$, Journal of Computatational and Applied Mathematics 233(2010)3030–3040.

[14] Q. W. Wang, Bisymmetric and centrosymmetric solutions to systems of real quaternion matrix equations, Computation Mathematica with Application 49(2005)641–650.

[15] Q. W. Wang, The general solution to a system of real quaternion matrix equations, Computation Mathematica with Application 49(2005)665–675.

[16] H. M. Zhang, F. Ding, A property of the eigenvalues of the symmetric positive definite matrix and the iterative algorithm for coupled Sylvester matrix equations, Journal of Franklin Institute 351 (2014)340–357.

[17] H. M. Zhang, Reduced-rank gradient-based algorithms for generalized coupled Sylvester matrix equations and its applications, Computation Mathematica with Application 70 (2015)2049–2062.

[18] F. Ding, T. Chen, Gradient based iterative algorithms for solving a class of matrix equations, IEEE Trans. Autom. Control 50 (2005) 1216–1221.

[19] F. Ding, T. Chen, On iterative solutions of general coupled matrix equations, SIAM Journal Control Optimation 44 (2006) 2269–2284.

[20] F. Ding, T. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, Systerm Control Letters 54 (2005) 95–107.

[21] F. Ding, X.P. Liu, J. Ding, Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle, Applied Mathematics and Compututation 197 (2008)41–50.

[22] L. Xie, J. Ding, F. Ding, Gradient based iterative solutions for general linear matrix equations, Computation Mathematica with Application 58(2009)1441–1448.

[23] F. Ding, T. Chen, On iterative solutions of general coupled matrix equations, SIAM Journal Control Optimation 44(2006) 2269–2284.

[24] F. Ding, T. Chen, Gradient based iterative algorithms for solving a class of matrix equations, IEEE Transaction Automation Control 50(2015)1216–1221.

[25] F. Ding, Y. Wang, J. Ding, Recursive least squares parameter identification algorithms for systems with colored noise using the filtering technique and the auxilary model, Digital Signal Processing 37(2015) 100–108.

[26] M. Hajarian, Developing BiCOR and CORS methods for coupled Sylvester-transpose and periodic Sylvester matrix equations, Applied Mathematical Modeling 39(2015)6073–6084.

[27] M. Hajarian, Matrix iterative methods for solving the Sylvester-transpose and periodic Sylvester matrix equations, Journal of Franklin Insttitute 350(2013)3328–3341.

[28] M. Hajarian, Matrix form of the CGS method for solving general coupled matrix equations, Applied Mathematics Letters 34(2014)37–42.

[29] A. Dmytryshyn, B.Kastrom, Coupled Sylvester-type matrix equations and block diagonalization, SIAM Journal Matrix Analysis Application 38(2015)580–593.

[30] F. L. Li, X. Y. Hu, L. Zhang, The generalized anti-reflexive solutions for a class of matrix equations $BX = C, XD = E$, Computation Mathematica with Application 27(2008)31–46.

[31] Y. X. Yuan, H. Dai, Generalized reflexive solutions of the matrix equation $AXB = D$ and associated optimal approximation problem,Computation Mathematica with Application 56(2008)1643–1649.

[32] J. C. Zhang, S. Z. Zhou, X. Y. Hu, The (P,Q)generalized reflexive and anti-reflexive solutions of the matrix equation $AX = B$, Applied Mathematics and Compututation 209(2009)254–258.

[33] M. Hajarian, Developing the CGLS algorithm for the least squares solutions of the general coupled matrix equations, Mathematical Methods Applied Science 37(2014)2782–2798.

[34] Z. Peng, The reflexive least squares solutions of the matrix equation $A_1X_1B_1 + A_2X_2B_2 + \cdots + A_lX_lB_l = C$ with a submatrix constraint, Numerical Algorithms 64(2013)455–480.

[35] M. Hajarian, Extending the CGLS algorithm for least squares solutions of the generalized Sylvester-transpose matrix equations, Journal of Franklin Institute 353: 11688–1185, 2016.

[36] Å.Björck, Numerical methods for least squares problems, SIAM, Philadelphia, 1996.

[37] Z. Peng, H. Xin, The reflexive least squares solutions of the general coupled matrix equations with a submatrix constraint, Applied Mathematics and Compututation 225(2013)425–445.

[38] Z. Peng, The (R,S)-symmetric least squares solutions of the general coupled matrix equations. Linear and Multilinear Algebra 63(2015)1086–1105.

[39] H. Li, Z. Gao, D. Zhao, Least squares solutions of the matrix equation $AXB + CYD = E$ with the least norm for symmetric arrowhead matrices, Applied Mathematics and Compututation 226(2014)719–724.

[40] F. Ding, T. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, System Control Letters 54(2005)95–107.

[41] R. C. Wang, Functional analysis and optimization theory, Beijing University of Aeronautics and Astronautics Press, Beijing, 2003.