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Existence and Exponential Stability of Almost Pseudo Automorphic Solution for Neutral Stochastic Evolution Equations Driven by G-Brownian Motion

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Abstract. This paper mainly concerns the quasi sure exponential stability of square mean almost pseudo automorphic mild solution for a class of neutral stochastic evolution equations driven by G-Brownian motion. By means of evolution operator theorem and fixed point theorem, existence and uniqueness of square mean almost pseudo automorphic mild solution is obtained. Also, a series of sufficient conditions on exponential stability and quasi sure exponential stability are established.

1. Introduction

The article aims to the quasi sure exponential stability of square mean almost pseudo automorphic mild solution for neutral stochastic evolution equations driven by G-Brownian motion(G-NSEEs for short)

$$\begin{aligned} \mathrm{d}\left[X(t)-D\left(t,X(t)\right)\right] &= \left[AX(t)+f\left(t,X(t)\right)\right]\mathrm{d}t+g\left(t,X(t)\right)\mathrm{d}\left\langle B\right\rangle(t) \\ &\quad +h\left(t,X(t)\right)\mathrm{d}B(t),\ t\in\mathbb{R} \end{aligned} \tag{1}$$

where $A(t):\mathcal{D}(A(t))\subset L_G^2(\mathcal{F})\to L_G^2(\mathcal{F})$ is densely closed linear operator, and satisfies the well known Acquistapace-Terrani conditions(one can see [1] and [5]). $B(t)$ is a one dimensional G-Brownian motion, the functions $D,\ f,\ g$ and $h:\mathbb{R}\times L_G^2(\mathcal{F})\to L_G^2(\mathcal{F})$ are jointly continuous. Since Bochner [3] firstly introduced the results of automorphy, many authors made further study and improvement(one can see [4], [10], [17]). Because of various applications of almost pseudo automorphy, there have been a wide range of interests on this issue. In particular, under the framework of classical Brownian motion, the stability and existence of pseudo almost automorphic solutions of stochastic differential equations have been considerably discussed. Chen and Lin [6] studied the square mean almost pseudo automorphic solution of SEEs. By means of Weyl fractional derivative, Pardoa and Lizama [19] obtained the existence and uniqueness of weighted pseudo almost automorphic mild solutions for fractional abstract differential equation. In the sense of distribution, Feng and Zong [9] discussed the square mean pseudo almost automorphic solution to stochastic differential equation driven by Lévy process. Most recently, by use of C_0-semigroup, Cui and Rong [7] established exponential stability of μ-pseudo almost automorphic mild solutions for nonlinear SEEs.

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In order to solve some problems in finance, Peng [20] firstly established the basic theory of G-expectation. Moreover, Peng [21] introduced the G-Brownian motion and related G-Itô stochastic calculus. Since then, many scholars made further research on the the G-Brownian motion(one can see [8], [14], [25]). Importantly, based on the G-stochastic analysis theory, stochastic differential equations driven by G-Brownian motion(G-SDEs in short) have been attracting much attention(one can see [2], [16], [22], [23]). Especially, Gu et al. [12] established existence and uniqueness of square mean pseudo almost automorphic mild solutions for G-SEEs.

As we know, stability has been one of the most interesting topics of SDEs since Mao [18] established the stability theorem. As to G-SDEs, there are a lot of interesting works including exponential stability, H_∞ stability and almost sure exponential stability(one can see [24], [27]). By Razumikhin theorm, Li and Yang [15] derived pth moment exponential stability of mild solution of neutral stochastic functional differential equations driven by G-Brownian motion. Recently, Hu et al. [13] considered exponential stability of the square mean almost automorphic for a class of impulsive G-SDEs with the help of Lyapunov function.

However, to the best of our knowledge, there is no result on the existence and stability of square mean pseudo almost automorphic mild solutions for G-NSEEs. To close the gap, we first aim to derive the existence and uniqueness of the system. Moreover, exponentially stability and quasi sure exponential stability of square mean pseudo almost automorphic mild solutions will be discussed with sufficient conditions.

The structure of this article is arranged as follows. In section 2, some basic notions, preliminaries and lemmas are provided. Section 3 is devoted to studying the existence and uniqueness of square mean pseudo almost automorphic mild solutions for G-NSEEs. In section 4, we shall discuss exponential stability and quasi sure exponential stability of square mean pseudo almost automorphic mild solutions.

2. Notations and Preliminaries

Throughout the paper, we will use the following specified notation. Denote $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N} = \{1, 2, \cdots\}$. If A is a vector or matrix, its transpose is denoted by A^T and the norm $|A|^2 = \mathrm{trace}\left(AA^T\right)$.

2.1. Itô integral of G-Brownian motion

In this subsection, we begin with some notations and preliminary results with respect to G-Brownian motion. Ω is the space of all $\mathbb{R}^n$-valued continuous functions with $\omega_0 = 0$, equipped with the distance

$$\rho\left(\omega^1, \omega^2\right) = \sum_{k=1}^{\infty} 2^{-i}\left[\left(\max_{t\in[0,k]} |\omega_t^1 - \omega_t^2|\right) \wedge 1\right],$$

then (Ω, ρ) is a metric space. We suppose that $\mathcal{H}$ satisfies $c \in \mathcal{H}$ for each constant c. If $X \in \mathcal{H}$, then $|X| \in \mathcal{H}$. If $X_1, X_2, \cdots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \cdots X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ is defined as

$$C_{l,Lip}(\mathbb{R}^n) = \{\varphi : \mathbb{R}^n \to \mathbb{R} | \exists C \in \mathbb{R}^+,\ m \in \mathbb{N}\ s.t.\ |\varphi(x) - \varphi(y)| \le C(1 + |x|^m + |y|^m)|x - y|\}.$$

Definition 2.1. $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is called as a sublinear expectation, if for any $X, Y \in \mathcal{H}$,

(1) if $X \ge Y$, then $\mathbb{E}(X) \ge \mathbb{E}(Y)$.

(2) $\mathbb{E}(c) = c$, for any $c \in \mathbb{R}$.

(3) $\mathbb{E}(X + Y) \le \mathbb{E}(X) + \mathbb{E}(Y)$.

(4) $\mathbb{E}(\lambda X) = \lambda\mathbb{E}(X)$, for any $\lambda \ge 0$.

For any $\omega \in \Omega$, the canonical process $B_t(\omega)$ is defined by $B_t(\omega) = \omega_t,\ t \geq 0$. The filtration $\mathcal{F}_t$ generated by $(B_t)_{t\geq 0}$ is defined

$$\mathcal{F}_t = \sigma\left(B_s,\ 0 \leq s \leq t\right).$$

Let $C_{b,Lip}\left(\mathbb{R}^n\right)$ denote the set of all bounded and continuous Lipschitz functions on $\mathbb{R}^n$. For any $t > 0$, let

$$\mathcal{L}_{Lip}\left(\mathcal{F}_t\right) = \left\{\xi(\omega) := \phi\left(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}\right), n \geq 1, t_1, t_2, \cdots, t_n \in [0,t], \phi \in C_{b,Lip}\left(\mathbb{R}^{n\times n}\right)\right\}.$$

Let

$$\mathcal{L}_{Lip}(\mathcal{F}) = \bigcup_{i=1}^{\infty} \mathcal{L}_{Lip}(\mathcal{F}_i).$$

If $\xi(\omega) = \phi\left(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}\right) \in \mathcal{L}_{Lip}(\mathcal{F})$ with $0 < t_1 < t_2 < \cdots < t_n < \infty$, we let

$$\mathbb{E}\left[\phi\left(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}\right)\right] := \phi_n,$$

where ϕ_n is iterative procedure defined as

$$\begin{aligned}
\phi_1\left(x_1, x_2, \cdots, x_{n-1}\right) &= \mathbb{E}\left[\phi\left(x_1, x_2, \cdots, x_{n-1}, B_{t_n} - B_{t_{n-1}}\right)\right], \\
\phi_2\left(x_1, x_2, \cdots, x_{n-2}\right) &= \mathbb{E}\left[\phi_1\left(x_1, x_2, \cdots, x_{n-2}, B_{t_{n-1}} - B_{t_{n-2}}\right)\right], \\
&\ \vdots \\
\phi_{n-1}\left(x_1\right) &= \mathbb{E}\left[\phi_{n-2}\left(x_1, B_{t_2} - B_{t_1}\right)\right], \\
\phi_n\left(x_1\right) &= \mathbb{E}\left[\phi_{n-1}\left(B_{t_1}\right)\right].
\end{aligned}$$

The conditional expectation of $\xi := \phi\left(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}\right)$ is given by

$$\mathbb{E}\left[\xi | \mathcal{F}_{t_j}\right] := \phi_{n-j}\left(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_j} - B_{t_{j-1}}\right).$$

Definition 2.2. (G-normal distribution)Assuming that $\underline{\sigma}$, $\overline{\sigma}$ are given nonnegative numbers satisfying $0 \leq \underline{\sigma} \leq \overline{\sigma}$. A random variable X is subject to G-normal distribution, denoted by $X \sim N\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$, if for each $\phi \in \mathcal{L}_{Lip}(\mathcal{F})$, the operator is defined by

$$\mathbb{E}\left[\phi\left(B(t) + X\right)\right] := u(t, X),$$

$u(t,x)$ is the viscosity solution of the following nonlinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - G\left(\dfrac{\partial^2 u}{\partial t^2}\right) = 0, \\ u(0,x) = \phi(x). \end{cases}$$

where $G(r) = \frac{1}{2}\left(\overline{\sigma}^2 r^+ - \underline{\sigma}^2 r^-\right), r \in \mathbb{R}$.

Definition 2.3. *(G-Brownian motion) The expectation operator $\mathbb{E}$ on $\mathcal{H}$ defined through the above process is called the G-expectation and the canonical process $B(t)$ is called G-Brownian motion.*

Next, we give the definitions of Itô Integral and quadratic variation process with respect to G-Brownian motion.

Definition 2.4. (1) *For $p \geq 1,\ T > 0$, $\mathcal{M}_G^{p,0}([0,T])$ denotes the space of simple processes by*

$$\mathcal{M}_G^{p,0}([0,T]) = \Big\{\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_{t_j}(\omega) I_{[t_j,\, t_{j+1})}(t);\ \xi_{t_j} \in L_G^p(\mathcal{F}_{t_j}), \forall N \geq 1,$$
$$0 = t_0 < t_1 < \cdots < t_N = T,\ j = 0, 1,\ 2, \cdots,\ N-1\Big\}.$$

(2) *For any $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_{t_j}(\omega) I_{[t_j, t_{j+1})} \in \mathcal{M}_G^{p,0}([0,T])$, its Bochner integral is defined as follows*

$$\int_0^t \eta_t(\omega)\mathrm{d}t = \sum_{j=0}^{N-1} \xi_{t_j}(\omega)(t_{j+1} - t_j).$$

(3) *Let*

$$\tilde{\mathbb{E}}_T = \frac{1}{T}\int_0^T \mathbb{E}[\eta_t]\mathrm{d}t = \frac{1}{T}\sum_{j=0}^{N-1} \mathbb{E}\left[\xi_{t_j}(\omega)\right](t_{j+1} - t_j),$$

then, $\tilde{\mathbb{E}}_T : \mathcal{M}_G^{p,0} \to \mathbb{R}$ is also a sublinear expectation.

For each $p \geq 1$, $\mathcal{M}_G^p([0,T])$ is the completion of $\mathcal{M}_G^{p,0}([0,T])$ equipped with the form

$$\|\eta\|_{\mathcal{M}_G^p([0,T])} = \left(\frac{1}{T}\int_0^T \|\eta_s^p\|\mathrm{d}s\right)^{\frac{1}{p}} = \left(\frac{1}{T}\sum_{j=0}^{N-1} \mathbb{E}\left[\xi_{t_j}(\omega)\right]^p (t_{j+1} - t_j)\right)^{\frac{1}{p}}.$$

Definition 2.5. *(Itô Integral)For $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_{t_j}(\omega) I_{[t_j,t_{j+1})} \in \mathcal{M}_G^{p,0}([0,T])$, the Itô integral is defined by*

$$I(\eta) = \int_0^T \eta_s \mathrm{d}B(s) := \sum_{j=0}^{N-1} \xi_{t_j}\left(B(t_{j+1}) - B(t_j)\right).$$

$L_G^p(\mathcal{F}_T)$ $(p \geq 1)$ is the completion of $\mathcal{L}_{Lip}(\mathcal{F}_T)$ with norm of $\|X\| = \{\mathbb{E}|X|^p\}^{\frac{1}{p}}$, as well as, $L_G^p(\mathcal{F})$ the completion of $\mathcal{L}_{Lip}(\mathcal{F})$. It is natural to construct the G-expectation on $\left(\Omega, L_G^p(\mathcal{F}\right)$(one can see [21]).

Remark 2.6. *For any $\eta \in \mathcal{M}_G^{p,0}([0,T])$, we have $\mathbb{E}\left[\int_0^T \eta_s \mathrm{d}B(s)\right] = 0$.*

Remark 2.7. *From [11] and [26], we conclude that the map $I : \mathcal{M}_G^{p,0}([0,T]) \to L_G^p(\mathcal{F}_T)$ is linear and continuous. Moreover, it can be extended as $I : \mathcal{M}_G^p([0,T]) \to L_G^p(\mathcal{F}_T)$.*

Definition 2.8. *When $t > 0$, the sequence π_t^N is partitions of $[0,\ t]$, $\pi_t^N : 0 = t_0^N < t_1^N < \cdots < t_{N-1}^N = t$, with the mesh $\mu(\pi_t^N) \to 0$ as $N \to \infty$. The quadratic variation process of G-Brownian motion $B(t)$ is*

$$\langle B\rangle(t) := \lim_{N\to\infty}\sum_{j=0}^{N-1}\left(B(t_{j+1}^N) - B(t_j^N)\right)^2 = B^2(t) - 2\int_0^t B(s)\mathrm{d}B(s).$$

In addition, the mutual variation process of B and $\bar{B}$ is

$$\langle B,\ \bar{B}\rangle(t) := \frac{1}{4}\left(\langle B + \bar{B}\rangle(t) - \langle B - \bar{B}\rangle(t)\right).$$

Definition 2.9. *(Integral w.r.t $\langle B\rangle$) For any $\eta_t \in \mathcal{M}_G^{1,0}([0,T])$, the map $Q_{0,T}(\eta) : \mathcal{M}_G^{1,0}([0,T]) \to L_G^1(\mathcal{F}_T)$ is defined by*

$$Q_{0,T}(\eta) = \int_0^T \eta_t \mathrm{d}\langle B\rangle(t) := \sum_{j=0}^{N-1} \xi_{t_j}\left[\langle B\rangle(t_{j+1}) - \langle B\rangle(t_j)\right].$$

Remark 2.10. *$Q_{0,T}(\eta)$ is linear and continuous, and can be continuously extended $Q_{0,T}(\eta) : \mathcal{M}_G^1([0,T]) \to L_G^1(\mathcal{F}_T)$.*

In order to get the our main results, we introduce some technical lemmas which can be found in [11], [20].

Lemma 2.11. *For any $0 \le T < \infty$,*

(1) $\mathbb{E}\left[\left|\int_0^T \eta_t \mathrm{d}\langle B\rangle(t)\right|\right] \le \bar{\sigma}^2 \mathbb{E}\left[\int_0^T |\eta_t| \mathrm{d}t\right]$*, for any $\eta_t \in \mathcal{M}_G^1([0,T])$.*

(2) $\mathbb{E}\left[\left(\int_0^T \eta_t \mathrm{d}B(t)\right)^2\right] = \mathbb{E}\left[\int_0^T \eta_t^2 \mathrm{d}\langle B\rangle(t)\right]$*, for any $\eta_t \in \mathcal{M}_G^2([0,T])$.*

(3) $\mathbb{E}\left[\left(\int_0^T |\eta_t|^p \mathrm{d}t\right)\right] \le \int_0^T [\mathbb{E}|\eta_t|^p]\,\mathrm{d}t$*, for any $\eta_t \in \mathcal{M}_G^p([0,T])$, $p \ge 1$.*

Lemma 2.12. *Let $p \ge 2$, $\eta = \{\eta_s\} \in \mathcal{M}_G^p([0,T])$. Then,*

$$\mathbb{E}\left(\sup_{s\le u\le t}\left|\int_s^u \eta_r \mathrm{d}B(r)\right|\right)^p \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\left(\left|\int_s^t \eta_r \mathrm{d}B(r)\right|\right)^p.$$

Lemma 2.13. *For $p \ge 1$, $\eta = \{\eta_s\} \in \mathcal{M}_G^p([0,T])$. Then,*

$$\mathbb{E}\left(\sup_{s\le u\le t}\left|\int_s^u \eta_r \mathrm{d}\langle B\rangle(r)\right|\right)^p \le \bar{\sigma}^p |t-s|^p \int_s^t \mathbb{E}|\eta_r|^p \mathrm{d}r.$$

By Denis et al. [8] and Wei et al. [28], there exists a weakly compact family $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\mathbb{E}[X] = \sup_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[X], \ \forall X \in L_G^1(\mathcal{F}).$$

And the related Choquet capacities is defined by

$$\bar{C}(A) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(A), \ A \in \mathcal{B}(\Omega).$$

A set A is called polar if $\bar{C}(A) = 0$, and a property holds quasi surely(q.s. in short) if it holds outside a polar set.

Lemma 2.14. *Suppose $X \in L_G^1(\mathcal{F}_T)$ satisfies $\mathbb{E}|X|^p < \infty$ for some $p > 0$. Then*

$$\bar{C}(|X| > M) \le \frac{\mathbb{E}|X|^p}{M^p}.$$

2.2. square mean almost automorphic stochastic process

In this subsection, we introduce some concepts of square mean almost automorphic stochastic processes and related properties.

Definition 2.15. *A stochastically continuous process $X(t):\mathbb{R}\to L_G^2(\mathcal{F})$ is square mean almost automorphic if for any real sequence $\{r_n'\}_{n\in\mathbb{N}}$ there exist a subsequence $\{r_n\}_{n\in\mathbb{N}}$ and $Y(t):\mathbb{R}\to L_G^2(\mathcal{F})$ such that*

$$\lim_{n\to\infty}\mathbb{E}\|X(t+r_n)-Y(t)\|^2=0 \quad and \quad \lim_{n\to\infty}\mathbb{E}\|Y(t-r_n)-X(t)\|^2=0$$

hold. The collection of all square mean almost automorphic processes is denoted by $SAA\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$.

$SBC\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$ is served as the collection of all the stochastically bounded and continuous processes.

Remark 2.16. *$SBC\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$ is a Banach space with the norm*

$$\left\|X\right\|_\infty=\sup_{t\in\mathbb{R}}\left(\mathbb{E}\|X(t)\|^2\right)^{\frac{1}{2}}.$$

Definition 2.17. *A stochastic process $X(t)$ belongs to $SBC_0\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$, if it is one of $SBC\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$ and satisfies*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\|X(t)\|^2\mathrm{d}t=0.$$

Remark 2.18. *$SBC_0\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$ is also a Banach space with the norm $\left\|X\right\|_\infty$.*

Remark 2.19. *If $X(t)\in SAA\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$, then $X(t)$ is bounded with the norm $\left\|X\right\|_\infty$. That is, $SAA\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)\subset SBC\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$.*

Definition 2.20. *A continuous stochastic process $f(t):\mathbb{R}\to L_G^2(\mathcal{F})$ is called square mean pseudo almost automorphic if it can be decomposed as $f(t)=g(t)+\varphi(t)$, where $g(t)\in SAA\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$, $\varphi(t)\in SBC_0\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$.*

We denote $SPAA\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$ the collection of square mean pseudo almost automorphic processes.

Remark 2.21. *Under the norm $\left\|X\right\|_\infty$, $SPAA\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$ is a Banach space.*

Definition 2.22. *A jointly continuous function $f(t,x):\mathbb{R}\times L_G^2(\mathcal{F})\to L_G^2(\mathcal{F})$ is square mean pseudo almost automorphic at t for any $x\in L_G^2(\mathcal{F})$ if it can be decomposed as $f=g+\varphi$, where $g\in SAA\Big(\mathbb{R}\times L_G^2(\mathcal{F}),L_G^2(\mathcal{F})\Big)$, $\varphi\in SBC_0\Big(\mathbb{R}\times L_G^2(\mathcal{F}),L_G^2(\mathcal{F})\Big)$. We denote the set of all such stochastically continuous processes by $SPAA\Big(\mathbb{R}\times L_G^2(\mathcal{F}),L_G^2(\mathcal{F})\Big)$.*

Lemma 2.23. ([5]) *If $f(t,x):\mathbb{R}\times L_G^2(\mathcal{F})\to L_G^2(\mathcal{F})$ is square mean almost automorphic and satisfies*

$$\mathbb{E}\left\|f(t,x)-f(t,y)\right\|^2\le C_1\left\|x-y\right\|^2, \text{ for all } x,y\in L_G^2(\mathcal{F}),\ t\in\mathbb{R},$$

where $C_1\ge 0$ is independent of t. Then for each $X(t)\in SPAA\Big(\mathbb{R},L_G^2(\mathcal{F})\Big)$, the stochastic process $F(\cdot)=f(\cdot,X(\cdot))$ is also square mean almost automorphic.

Lemma 2.24. *Suppose that $f(t,x)\in SPAA\left(\mathbb{R}\times L_G^2(\mathcal{F}),L_G^2(\mathcal{F})\right)$, and there exists nonnegative constant C such that,*

$$\mathbb{E}\left\|f(t,x)-f(t,y)\right\|^2\le C\left\|x-y\right\|^2, \text{ for any } x,y\in L_G^2(\mathcal{F}),\ t\in\mathbb{R}.$$

Then, $f(t,X(t))\in SPAA\left(\mathbb{R},L_G^2(\mathcal{F})\right)$ for any $X(t)\in SPAA\left(\mathbb{R},L_G^2(\mathcal{F})\right)$.

3. Existence of square mean pseudo almost automorphic mild solution

In order to investigate the existence and uniqueness of square mean pseudo almost automorphic mild solution for G-NSEEs, we begin with definition of the mild solutions and some assumptions.

Definition 3.1. An $\mathcal{F}_t$−progressively measurable process $\{X(t)\}_{t\in\mathbb{R}}$ is called a mild solution of the (1) if the following stochastic integral equation is satisfied

$$\begin{aligned} X(t) - D(t, X(t)) &= U(t,s)\left[X(s) - D(s, X(s))\right] + \int_s^t U(t,r) f(r, X(r))\,\mathrm{d}r \\ &\quad + \int_s^t U(t,r) g(r, X(r))\,\mathrm{d}\langle B\rangle(r) + \int_s^t U(t,r) h(r, X(r))\,\mathrm{d}B(r) \end{aligned} \tag{2}$$

for any $t \geq s$ and $s \in \mathbb{R}$.

In order to get the main results, we impose the following assumptions on evolution family and coefficients.

(H1) There exist positive constants M and μ such that the evolution family $U(t,s)$ generated by $A(t)$ is exponentially stable,
$$\|U(t,s)\| \leq Me^{-\mu(t-s)}, \quad t \geq s.$$

(H2) The coefficients $D(t,x), f(t,x), g(t,x)$ and $h(t,x)$: $\mathbb{R}\times L_G^2(\mathcal{F}) \to L_G^2(\mathcal{F})$ are functions of $SPAA\left(\mathbb{R}\times L_G^2(\mathcal{F}), L_G^2(\mathcal{F})\right)$. Furthermore, there exist nonnegative constants $L_D, L_f,\ L_g$ and L_h such that
$$\left\|D(t,x) - D(t,y)\right\|^2 \leq L_D\left\|x-y\right\|^2, \ \left\|f(t,x) - f(t,y)\right\|^2 \leq L_f\left\|x-y\right\|^2$$
and
$$\left\|g(t,x) - g(t,y)\right\|^2 \leq L_g\left\|x-y\right\|^2, \ \left\|h(t,x) - h(t,y)\right\|^2 \leq L_h\left\|x-y\right\|^2$$
for $x, y \in L_G^2(\mathcal{F})$ and $t \in \mathbb{R}$.

(H3) $D = D_1 + D_2 \in SPAA\left(\mathbb{R}\times L_G^2(\mathcal{F}), L_G^2(\mathcal{F})\right)$, where $D_1 \in SAA\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$, $D_2 \in SBC_0\left(\mathbb{R}\times L_G^2(\mathcal{F}), L_G^2(\mathcal{F})\right)$. $f = f_1 + f_2 \in SPAA\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$, where $f_1 \in SAA\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$, $f_2 \in SBC_0\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$. $g = g_1 + g_2 \in SPAA\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$, where $g_1 \in SAA\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$, $g_2 \in SBC_0\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$. $h = h_1 + h_2 \in SPAA\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$, where $h_1 \in SAA\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$, $h_2 \in SBC_0\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$.

The following theorem presents the existence and uniqueness of square mean pseudo almost automorphic mild solution.

Theorem 3.2. *Assuming that the conditions* (H1)-(H3) *are satisfied, and*
$$4L_D + \frac{4M^2L_f}{\mu^2} + \frac{4\bar{\sigma}^4M^2L_g}{\mu^2} + \frac{2\bar{\sigma}^2M^2L_h}{\mu} < 1.$$

Then, the system (1) *has a unique mild solution* $X \in SPAA\left(\mathbb{R}, L_G^2(\mathcal{F})\right)$. *Moreover, the solution can be expressed by*

$$\begin{aligned} X(t) &= D(t, X(t)) + \int_{-\infty}^t U(t,r) f(r, X(r))\,\mathrm{d}r + \int_{-\infty}^t U(t,r) g(r, X(r))\,\mathrm{d}\langle B\rangle(r) \\ &\quad + \int_{-\infty}^t U(t,r) h(r, X(r))\,\mathrm{d}B(r). \end{aligned} \tag{3}$$

Proof: Existence Firstly, we claim that (3) satisfies that (2) for all $t \geq s$ at each $s \in \mathbb{R}$. So $X(t)$ given by (3) is a mild solution of (1).

For any $X(t) \in SPAA\Big(\mathbb{R},\ L_G^2(\mathcal{F})\Big)$, we define the operator

$$\begin{aligned}(\Phi X)(t) &= D(t,\ X(t)) + \int_{-\infty}^{t} U(t,r) f(r, X(r))\,\mathrm{d}r + \int_{-\infty}^{t} U(t,r) g(r, X(r))\,\mathrm{d}\langle B\rangle(r) \\ &\quad + \int_{-\infty}^{t} U(t,r) h(r, X(r))\,\mathrm{d}B(r), \end{aligned} \tag{4}$$

which is well defined and satisfies (2). From (H3), we have

$$\begin{aligned}(\Phi X)(t) &= \Bigg(D_1(t, X(t)) + \int_{-\infty}^{t} U(t,r) f_1(r, X(r))\,\mathrm{d}r + \int_{-\infty}^{t} U(t,r) g_1(r, X(r))\,\mathrm{d}\langle B\rangle(r) \\ &\quad + \int_{-\infty}^{t} U(t,r) h_1(r, X(r))\,\mathrm{d}B(r)\Bigg) + \Bigg(D_2(t, X(t)) + \int_{-\infty}^{t} U(t,r) f_2(r, X(r))\,\mathrm{d}r \\ &\quad + \int_{-\infty}^{t} U(t,r) g_2(r, X(r))\,\mathrm{d}\langle B\rangle(r) + \int_{-\infty}^{t} U(t,r) h_2(r, X(r))\,\mathrm{d}B(r)\Bigg). \\ &= (\Phi_1 X)(t) + (\Phi_2 X)(t). \end{aligned} \tag{5}$$

In the following part, we will show that $(\Phi X)(t)$ is in $SPAA\Big(\mathbb{R},\ L_G^2(\mathcal{F})\Big)$. That is to say, it needs to verify that $(\Phi_1 X)(t)$ is in $SAA\Big(\mathbb{R},\ L_G^2(\mathcal{F})\Big)$ and $(\Phi_2 X)(t)$ is in $SBC_0\Big(\mathbb{R}, L_G^2(\mathcal{F})$. We illustrate the facts through three steps.

Step 1. We begin with the continuity of $(\Phi_1 X)(t)$. From the definition of $(\Phi_1 X)(t)$, we have

$$\begin{aligned}&\mathbb{E}\Big\|\,(\Phi_1 X)(t+s) - (\Phi_1 X)(t)\Big\|^2 \\ &= \mathbb{E}\Bigg\|D_1(t+s, X(t+s)) - D_1(t, X(t)) + \int_{-\infty}^{t+s} U(t+s,r) f_1(r, X(r))\,\mathrm{d}r - \int_{-\infty}^{t} U(t,r) f_1(r, X(r))\,\mathrm{d}r \\ &\quad + \int_{-\infty}^{t+s} U(t+s,r) g_1(r, X(r))\,\mathrm{d}\langle B\rangle(r) - \int_{-\infty}^{t} U(t,r) g_1(r, X(r))\,\mathrm{d}\langle B\rangle(r) \\ &\quad + \int_{-\infty}^{t+s} U(t+s,r) h_1(r, X(r))\,\mathrm{d}B(r) - \int_{-\infty}^{t} U(t,r) h_1(r, X(r))\,\mathrm{d}B(r)\Bigg\|^2. \end{aligned} \tag{6}$$

Because $D_1(t,x) \in SAA\Big(\mathbb{R} \times L_G^2(\mathcal{F}), L_G^2(\mathcal{F})\Big)$, it deduces

$$\lim_{s\to 0} \mathbb{E}\Big\|D_1(t+s,\ X(t+s)) - D_1(t,\ X(t))\,\Big\|^2 = 0. \tag{7}$$

By means of the properties of evolution family $U(t,r)$ and elementary inequality, we get

$$\begin{aligned}&\mathbb{E}\Bigg\|\int_{-\infty}^{t+s} U(t+s,r) f_1(r, X(r))\,\mathrm{d}r - \int_{-\infty}^{t} U(t,r) f_1(r, X(r))\,\mathrm{d}r\Bigg\|^2 \\ &= \mathbb{E}\Bigg\|\int_{-\infty}^{t} (U(t+s,t) - I)\, U(t,r) f_1(r, X(r))\,\mathrm{d}r + \int_{t}^{t+s} U(t+s,t) f_1(r, X(r))\,\mathrm{d}r\Bigg\|^2 \\ &\leq 2\mathbb{E}\Bigg\|\int_{-\infty}^{t} (U(t+s,t) - I)\, U(t,r) f_1(r, X(r))\,\mathrm{d}r\Bigg\|^2 + 2\mathbb{E}\Bigg\|\int_{t}^{t+s} U(t+s,t) f_1(r, X(r))\,\mathrm{d}r\Bigg\|^2. \end{aligned}$$

Due to the dominated convergence theorem, it shows

$$\lim_{s\to 0} \mathbb{E}\Bigg\|\int_{-\infty}^{t+s} U(t+s,r) f_1(r, X(r))\,\mathrm{d}r - \int_{-\infty}^{t} U(t,r) f_1(r, X(r))\,\mathrm{d}r\Bigg\|^2 = 0. \tag{8}$$

Combining the properties of evolution family $U(t,r)$ with Lemma 2.11, we have

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t+s} U(t+s,r)g_1(r,X(r))\,\mathrm{d}\langle B\rangle(r)-\int_{-\infty}^{t} U(t,r)g_1(r,X(r))\,\mathrm{d}\langle B\rangle(r)\right\|^2\\
&=\mathbb{E}\left\|\int_{-\infty}^{t}(U(t+s,t)-I)\,U(t,r)g_1(r,X(r))\,\mathrm{d}\langle B\rangle(r)+\int_{t}^{t+s} U(t+s,t)g_1(r,X(r))\,\mathrm{d}\langle B\rangle(r)\right\|^2\\
&\le 2\mathbb{E}\left\|\int_{-\infty}^{t}(U(t+s,t)-I)\,U(t,r)g_1(r,X(r))\,\mathrm{d}\langle B\rangle(r)\right\|^2\\
&+2\mathbb{E}\left\|\int_{t}^{t+s} U(t+s,t)g_1(r,X(r))\,\mathrm{d}\langle B\rangle(r)\right\|^2\\
&\le 2\overline{\sigma}^4\mathbb{E}\left(\int_{-\infty}^{t}\left\|(U(t+s,t)-I)\,U(t,r)g_1(r,X(r))\right\|\mathrm{d}r\right)^2\\
&+2\overline{\sigma}^4\mathbb{E}\left(\int_{t}^{t+s} U(t+s,t)\left\|g_1(r,X(r))\right\|\mathrm{d}r\right)^2.
\end{aligned}$$

And

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t+s} U(t+s,r)h_1(r,X(r))\,\mathrm{d}B(r)-\int_{-\infty}^{t} U(t,r)h_1(r,X(r))\,\mathrm{d}B(r)\right\|^2\\
&=\mathbb{E}\left\|\int_{-\infty}^{t}(U(t+s,t)-I)\,U(t,r)h_1(r,X(r))\,\mathrm{d}B(r)+\int_{t}^{t+s} U(t+s,t)h_1(r,X(r))\,\mathrm{d}B(r)\right\|^2\\
&\le 2\overline{\sigma}^2\int_{-\infty}^{t}\mathbb{E}\left\|(U(t+s,t)-I)\,U(t,r)h_1(r,X(r))\right\|^2\mathrm{d}r+2\overline{\sigma}^2\int_{t}^{t+s}U(t+s,t)\mathbb{E}\left\|h_1(r,X(r))\right\|^2\mathrm{d}r.
\end{aligned}$$

So, it follows

$$\lim_{s\to 0}\mathbb{E}\left\|(\Phi_1X)(t+s)-(\Phi_1X)(t)\right\|^2=0.$$

Step 2. Because $D(t,x)$, $f(t,x)$, $g(t,x)$ and $h(t,x)$ are the functions of $SAA\Big(\mathbb{R}\times L_G^2(\mathcal{F}),L_G^2(\mathcal{F})\Big)$, thus, there exists a subsequence $\{r_n\}$ of any real numbers $\{r'_n\}_{n\in\mathbb{N}}$, for some stochastic process $\widetilde{D_1}$, $\widetilde{f_1}$, $\widetilde{g_1}$ and $\widetilde{h_1}:\mathbb{R}\times L_G^2(\mathcal{F})\to L_G^2(\mathcal{F})$, such that

$$\lim_{n\to\infty}\mathbb{E}\left\|D_1(t+r_n,\ X(t+r_n))-\widetilde{D_1}(t,\ X(t))\right\|^2=0 \text{ and } \lim_{n\to\infty}\mathbb{E}\left\|\widetilde{D_1}(t-r_n,\ X(t-r_n))-D_1(t,\ X(t))\right\|^2=0,$$

$$\lim_{n\to\infty}\mathbb{E}\left\|f_1(t+r_n,\ X(t+r_n))-\widetilde{f_1}(t,\ X(t))\right\|^2=0 \text{ and } \lim_{n\to\infty}\mathbb{E}\left\|\widetilde{f_1}(t-r_n,\ X(t-r_n))-f_1(t,\ X(t))\right\|^2=0,$$

$$\lim_{n\to\infty}\mathbb{E}\left\|g_1(t+r_n,\ X(t+r_n))-\widetilde{g_1}(t,\ X(t))\right\|^2=0 \text{ and } \lim_{n\to\infty}\mathbb{E}\left\|\widetilde{g_1}(t-r_n,\ X(t-r_n))-g_1(t,\ X(t))\right\|^2=0,$$

$$\lim_{n\to\infty}\mathbb{E}\left\|h_1(t+r_n,\ X(t+r_n))-\widetilde{h_1}(t,\ X(t))\right\|^2=0 \text{ and } \lim_{n\to\infty}\mathbb{E}\left\|\widetilde{h_1}(t-r_n,\ X(t-r_n))-h_1(t,\ X(t))\right\|^2=0,$$

for each $t\in\mathbb{R}$ and $X(t)\in L_G^2(\mathcal{F})$.

In order to verify that $(\Phi_1X)(t)$ is a square mean almost automorphic process, we consider the operator

$$\begin{aligned}
\left(\widetilde{\Phi}_1X\right)(t) &= \widetilde{D_1}(t,\ X(t))+\int_{-\infty}^{t}U(t,r)\widetilde{f_1}(r,X(r))\,\mathrm{d}r+\int_{-\infty}^{t}U(t,r)\widetilde{g_1}(r,X(r))\,\mathrm{d}\langle B\rangle(r)\\
&\quad+\int_{-\infty}^{t}U(t,r)\widetilde{h_1}(r,X(r))\,\mathrm{d}B(r).
\end{aligned}\tag{9}$$

Then, we have

$$
\begin{aligned}
&\mathbb{E}\left\|\left(\Phi_1 X\right)(t+r_n)-\left(\widetilde{\Phi_1} X\right)(t)\right\|^2 \\
&=\mathbb{E}\left\|D_1\left(t+r_n,\ X(t+r_n)\right)+\int_{-\infty}^{t+r_n} U(t+r_n,r) f_1\left(r, X(r)\right) \mathrm{d} r+\int_{-\infty}^{t+r_n} U(t+r_n,r) g_1\left(r, X(r)\right) \mathrm{d}\langle B\rangle(r)\right. \\
&+\int_{-\infty}^{t+r_n} U(t+r_n,r) h_1\left(r, X(r)\right) \mathrm{d} B(r)-\widetilde{D_1}\left(t,\ X(t)\right)-\int_{-\infty}^{t} U(t,r) \widetilde{f_1}\left(r, X(r)\right) \mathrm{d} r \\
&\left.-\int_{-\infty}^{t} U(t,r) \widetilde{g_1}\left(r, X(r)\right) \mathrm{d}\langle B\rangle(r)-\int_{-\infty}^{t} U(t,r) \widetilde{h_1}\left(r, X(r)\right) \mathrm{d} B(r)\right\|^2 \\
&\leq 4 \mathbb{E}\left\|D_1\left(t+r_n,\ X(t+r_n)\right)-\widetilde{D_1}\left(t,\ X(t)\right)\right\|^2 \\
&+4 \mathbb{E}\left\|\int_{-\infty}^{t+r_n} U(t+r_n,r) f_1\left(r, X(r)\right) \mathrm{d} r-\int_{-\infty}^{t} U(t,r) \widetilde{f_1}\left(r, X(r)\right) \mathrm{d} r\right\|^2 \\
&+4 \mathbb{E}\left\|\int_{-\infty}^{t+r_n} U(t+r_n,r) g_1\left(r, X(r)\right) \mathrm{d}\langle B\rangle(r)-\int_{-\infty}^{t} U(t,r) \widetilde{g_1}\left(r, X(r)\right) \mathrm{d}\langle B\rangle(r)\right\|^2 \\
&+4 \mathbb{E}\left\|\int_{-\infty}^{t+r_n} U(t+r_n,r) h_1\left(r, X(r)\right) \mathrm{d} B(r)-\int_{-\infty}^{t} U(t,r) \widetilde{h_1}\left(r, X(r)\right) \mathrm{d} B(r)\right\|^2 . \qquad (10)
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t+r_n} U(t+r_n,r) f_1\left(r, X(r)\right) \mathrm{d} r-\int_{-\infty}^{t} U(t,r) \widetilde{f_1}\left(r, X(r)\right) \mathrm{d} r\right\|^2 \\
&=\mathbb{E}\left\|\int_{-\infty}^{t} U(t,r) f_1\left(r+r_n, X(r+r_n)\right) \mathrm{d} r-\int_{-\infty}^{t} U(t,r) \widetilde{f_1}\left(r, X(r)\right) \mathrm{d} r\right\|^2 \\
&\leq \int_{-\infty}^{t} U(t,r) \mathrm{d} r \int_{-\infty}^{t} U(t,r) \mathbb{E}\left\|f_1\left(r+r_n, X(r+r_n)\right)-\widetilde{f_1}\left(r, X(r)\right)\right\|^2 \mathrm{d} r, \qquad (11)
\end{aligned}
$$

where the last estimate converges to zero as $n \to \infty$.

Noting that, for any $t \in \mathbb{R}$, $\widetilde{\langle B\rangle}(t) := \langle B\rangle(t+r_n) - \langle B\rangle(r_n)$ has the same distribution with $\langle B\rangle(t)$ and taking the Cauchy-Schwarz inequality again, we have

$$
\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t+r_n} U(t+r_n,r) g_1\left(r, X(r)\right) \mathrm{d}\langle B\rangle(r)-\int_{-\infty}^{t} U(t,r) \widetilde{g_1}\left(r, X(r)\right) \mathrm{d}\langle B\rangle(r)\right\|^2 \\
&=\mathbb{E}\left\|\int_{-\infty}^{t} U(t,r)\left[g_1\left(r+r_n, X(r+r_n)\right)-\widetilde{g_1}\left(r, X(r)\right)\right] \mathrm{d}\langle B\rangle(r)\right\|^2 \\
&\leq \bar{\sigma}^4 \mathbb{E}\left\|\int_{-\infty}^{t} U(t,r)\left[g_1\left(r+r_n, X(r+r_n)\right)-\widetilde{g_1}\left(r, X(r)\right)\right] \mathrm{d} r\right\|^2 \\
&\leq \bar{\sigma}^4 \int_{-\infty}^{t} U(t,r) \mathrm{d} r \int_{-\infty}^{t} U(t,r) \mathbb{E}\left\|g_1\left(r+r_n, X(r+r_n)\right)-\widetilde{g_1}\left(r, X(r)\right)\right\|^2 \mathrm{d} r. \qquad (12)
\end{aligned}
$$

Therefor, we have

$$
\lim_{n\to\infty} \mathbb{E}\left\|\int_{-\infty}^{t+r_n} U(t+r_n,r) g_1\left(r, X(r)\right) \mathrm{d}\langle B\rangle(r)-\int_{-\infty}^{t} U(t,r) \widetilde{g_1}\left(r, X(r)\right) \mathrm{d}\langle B\rangle(r)\right\|^2=0.
$$

Let $\tilde{B}(t) = B(t + r_n) - B(r_n)$ for each $t \in \mathbb{R}$, then $\tilde{B}(t)$ is also a G-Brownian motion with the same distribution as $B(t)$, we obtain

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t+r_n} U(t+r_n, r)h_1(r, X(r))\,\mathrm{d}B(r) - \int_{-\infty}^{t} U(t,r)\widetilde{h_1}(r, X(r))\,\mathrm{d}B(r)\right\|^2 \\
&= \mathbb{E}\left\|\int_{-\infty}^{t} U(t,r)\left[h_1(r+r_n, X(r+r_n)) - \widetilde{h_1}(r, X(r))\right]\mathrm{d}\tilde{B}(r)\right\|^2 \\
&\le \bar{\sigma}^2 \int_{-\infty}^{t} \|U(t,r)\|^2\, \mathbb{E}\left\|h_1(r+r_n, X(r+r_n)) - \widetilde{h_1}(r, X(r))\right\|^2 \mathrm{d}r,
\end{aligned} \tag{13}$$

where the last estimate converges to zero as $n \to \infty$.

Therefore, we can conclude that

$$\lim_{n\to\infty} \mathbb{E}\left\|(\Phi_1 X)(t+r_n) - \left(\widetilde{\Phi}_1 X\right)(t)\right\|^2 = 0.$$

By an analogous arguments as above, we have

$$\lim_{n\to\infty} \mathbb{E}\left\|\left(\widetilde{\Phi}_1 X\right)(t-r_n) - (\Phi_1 X)(t)\right\|^2 = 0.$$

From the Steps 1 and 2, we have $(\Phi_1 X)(t) \in SAA\left(\mathbb{R},\ L_G^2(\mathcal{F})\right)$.

Step 3. As the similar way as Step 1, we can prove that $(\Phi_2 X)(t)$ is stochastically continuous process. According to the functions D_2, F_2, G_2 and $H_2 \in SBC_0\left(\mathbb{R}\times L_G^2(\mathcal{F}),\ L_G^2(\mathcal{F})\right)$, it follows that $(\Phi_2 X)(t)$ is stochastically bounded. In what follows, we aim to prove

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbb{E}\,\|(\Phi_2 X)(t)\|^2\,\mathrm{d}t = 0.$$

From the definition of $(\Phi_2 X)(t)$, we have

$$
\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left\|(\Phi_2 X)(t)\right\|^2 \mathrm{d}t\\
&\le 4\left\{\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left\|D_2(t,\ X(t))\right\|^2\mathrm{d}t+\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left\|\int_{-\infty}^{t}U(t,r)f_2(r,X(r))\,\mathrm{d}r\right\|^2\mathrm{d}t\right.\\
&\left.+\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left\|\int_{-\infty}^{t}U(t,r)g_2(r,X(r))\,\mathrm{d}\langle B\rangle(r)\right\|^2\mathrm{d}t+\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left\|\int_{-\infty}^{t}U(t,r)h_2(r,X(r))\,\mathrm{d}B(r)\right\|^2\mathrm{d}t\right\}\\
&\le 4\left\{\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left\|D_2(t,\ X(t))\right\|^2\mathrm{d}t\right.\\
&+\frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{t}U(t,r)\mathrm{d}r\int_{-\infty}^{t}U(t,r)\mathbb{E}\left\|f_2(r,X(r))\right\|^2\mathrm{d}r\right]\mathrm{d}t\\
&+\bar{\sigma}^4\frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{t}U(t,r)\mathrm{d}r\int_{-\infty}^{t}U(t,r)\mathbb{E}\left\|g_2(r,X(r))\right\|^2\mathrm{d}r\right]\mathrm{d}t\\
&\left.+\bar{\sigma}^2\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t}U^2(t,r)\mathbb{E}\left\|h_2(r,X(r))\right\|^2\mathrm{d}r\mathrm{d}t\right\}\\
&\le 4\left\{\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left\|D_2(t,\ X(t))\right\|^2\mathrm{d}t\right.\\
&+\frac{M^2}{\mu}\times\frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{t}e^{-\mu(t-r)}\mathbb{E}\left\|f_2(r,X(r))\right\|^2\mathrm{d}r\right]\mathrm{d}t\\
&+\frac{M^2\bar{\sigma}^4}{\mu}\times\frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{t}e^{-\mu(t-r)}\mathbb{E}\left\|g_2(r,X(r))\right\|^2\mathrm{d}r\right]\mathrm{d}t\\
&\left.+\frac{M^2\bar{\sigma}^2}{\mu}\times\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t}e^{-2\mu(t-r)}\mathbb{E}\left\|h_2(r,X(r))\right\|^2\mathrm{d}r\mathrm{d}t\right\}.
\end{aligned}
$$

As to the second part of the last inequality, it follows

$$
\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\int_{-\infty}^{t}e^{-\mu(t-r)}\mathbb{E}\left\|f_2(r,X(r))\right\|^2\mathrm{d}r\\
&=\frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\int_{-T}^{t}e^{-\mu(t-r)}\mathbb{E}\left\|f_2(r,X(r))\right\|^2\mathrm{d}r+\frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\int_{-\infty}^{-T}e^{-\mu(t-r)}\mathbb{E}\left\|f_2(r,X(r))\right\|^2\mathrm{d}r\\
&=\frac{1}{2T}\int_{-T}^{T}\mathrm{d}r\int_{r}^{T}e^{-\mu(t-r)}\mathbb{E}\left\|f_2(r,X(r))\right\|^2\mathrm{d}t+\frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\int_{-\infty}^{-T}e^{-\mu(t-r)}\mathbb{E}\left\|f_2(r,X(r))\right\|^2\mathrm{d}r\\
&\le\frac{1}{\mu}\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left\|f_2(r,X(r))\right\|^2\mathrm{d}r+\frac{1}{\mu^2}\frac{1}{2T}\left\|f_2(r,X(r))\right\|_\infty^2\to 0 \qquad (14)
\end{aligned}
$$

as $T\to\infty$.

Taking the similar method, we have

$$
\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{t}e^{-\mu(t-r)}\mathbb{E}\left\|g_2(r,X(r))\right\|^2\mathrm{d}r\right]\mathrm{d}t=0
$$

and

$$
\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t}e^{-2\mu(t-r)}\mathbb{E}\left\|h_2(r,X(r))\right\|^2\mathrm{d}r\mathrm{d}t=0.
$$

Thus, we proved that $(\Phi_2 X)(t) \in SBC_0\left(\mathbb{R},\ L_G^2(\mathcal{F})\right)$. According to the above three steps, we could demonstrate $(\Phi X)(t) \in SPAA\left(\mathbb{R},\ L_G^2(\mathcal{F})\right)$.

Uniqueness In the following parts, we will introduce that Φ has a unique fixed point. If $X(t)$ and $Y(t)$ are the solutions of (1), we have

$$\begin{aligned}
&\mathbb{E}\left\|(\Phi X)(t)-(\Phi Y)(t)\right\|^2\\
&=\mathbb{E}\left\|D(t,\ X(t))+\int_{-\infty}^{t}U(t,r)f(r,X(r))\,\mathrm{d}r+\int_{-\infty}^{t}U(t,r)g(r,X(r))\,\mathrm{d}\langle B\rangle(r)\right.\\
&+\int_{-\infty}^{t}U(t,r)h(r,X(r))\,\mathrm{d}B(r)-D(t,\ Y(t))-\int_{-\infty}^{t}U(t,r)f(r,Y(r))\,\mathrm{d}r\\
&\left.-\int_{-\infty}^{t}U(t,r)g(r,Y(r))\,\mathrm{d}\langle B\rangle(r)-\int_{-\infty}^{t}U(t,r)h(r,Y(r))\,\mathrm{d}B(r)\right\|^2\\
&\le 4\mathbb{E}\left\|D(t,\ X(t))-D(t,\ Y(t))\right\|^2+4\mathbb{E}\left\|\int_{-\infty}^{t}U(t,r)\left[f(r,X(r))-f(r,Y(r))\right]\mathrm{d}r\right\|^2\\
&+4\mathbb{E}\left\|\int_{-\infty}^{t}U(t,r)\left[g(r,X(r))-g(r,Y(r))\right]\mathrm{d}\langle B\rangle(r)\right\|^2\\
&+4\mathbb{E}\left\|\int_{-\infty}^{t}U(t,r)\left[h(r,X(r))-h(r,Y(r))\right]\mathrm{d}B(r)\right\|^2:=4\sum_{i=1}^{4}\Pi_i(t)
\end{aligned}$$

From the assumption (H2), we get

$$\Pi_1(t)=\mathbb{E}\left\|D(t,\ X(t))-D(t,\ Y(t))\right\|^2\le L_D\sup_{t\in\mathbb{R}}\mathbb{E}\left\|X(t)-Y(t)\right\|^2. \tag{15}$$

By Cauchy-Schwarz inequality, (H1) and (H2), we obtain

$$\begin{aligned}
\Pi_2(t) &= \mathbb{E}\left\|\int_{-\infty}^{t}U(t,r)\left[f(r,X(r))-f(r,Y(r))\right]\mathrm{d}r\right\|^2\\
&\le \int_{-\infty}^{t}U(t,r)\mathrm{d}r\,\mathbb{E}\int_{-\infty}^{t}U(t,r)\left\|f(r,X(r))-f(r,Y(r))\right\|^2\mathrm{d}r\\
&\le \frac{M^2L_f}{\mu}\int_{-\infty}^{t}e^{-\mu(t-r)}\mathbb{E}\left\|X(r)-Y(r)\right\|^2\mathrm{d}r\\
&\le \frac{M^2L_f}{\mu^2}\sup_{t\in\mathbb{R}}\mathbb{E}\left\|X(t)-Y(t)\right\|^2.
\end{aligned} \tag{16}$$

From Lemma 2.11, Cauchy-Schwarz inequality, (H1) and (H2), one can prove that

$$\begin{aligned}
\Pi_3(t) &= \mathbb{E}\left\|\int_{-\infty}^{t}U(t,r)\left[g(r,X(r))-g(r,Y(r))\right]\mathrm{d}\langle B\rangle(r)\right\|^2\\
&\le \bar{\sigma}^4\mathbb{E}\left\|\int_{-\infty}^{t}U(t,r)\left|g(r,X(r))-g(r,Y(r))\right|\mathrm{d}r\right\|^2\\
&\le \bar{\sigma}^4\int_{-\infty}^{t}U(t,r)\mathrm{d}r\int_{-\infty}^{t}U(t,r)\mathbb{E}\left\|g(r,X(r))-g(r,Y(r))\right\|^2\mathrm{d}r\\
&\le \frac{\bar{\sigma}^4M^2L_g}{\mu^2}\sup_{t\in\mathbb{R}}\mathbb{E}\left\|X(t)-Y(t)\right\|^2.
\end{aligned} \tag{17}$$

From Lemma 2.11, (H1) and (H2), we can verify

$$\begin{aligned}\Pi_4(t) &= \mathbb{E}\left\|\int_{-\infty}^{t} U(t,r)\left[h\left(r,X(r)\right)-h\left(r,Y(r)\right)\right]\mathrm{d}B(r)\right\|^2\\&= \mathbb{E}\int_{-\infty}^{t}\left\|U(t,r)\left[h\left(r,X(r)\right)-h\left(r,Y(r)\right)\right]\right\|^2\mathrm{d}\left\langle B\right\rangle(r)\\&\le \bar{\sigma}^2M^2L_h\int_{-\infty}^{t}e^{-2\mu(t-r)}\mathbb{E}\left\|X(r)-Y(r)\right\|^2\mathrm{d}r\\&\le \frac{\bar{\sigma}^2M^2L_h}{2\mu}\sup_{t\in\mathbb{R}}\mathbb{E}\left\|X(t)-Y(t)\right\|^2. \qquad (18)\end{aligned}$$

It follows from (15) to (18), we deduce

$$\mathbb{E}\left\|\left(\Phi X\right)(t)-\left(\Phi Y\right)(t)\right\|^2 \le \left[4L_D+\frac{4M^2L_f}{\mu^2}+\frac{\bar{\sigma}^4M^2L_g}{\mu^2}+\frac{2\bar{\sigma}^2M^2L_h}{\mu}\right]\sup_{t\in\mathbb{R}}\mathbb{E}\left\|X(t)-Y(t)\right\|^2. \qquad (19)$$

So,

$$\left\|\left(\Phi X\right)(t)-\left(\Phi Y\right)(t)\right\|_{SPAA}^2 \le \left[4L_D+\frac{4M^2L_f}{\mu^2}+\frac{\bar{\sigma}^4M^2L_g}{\mu^2}+\frac{2\bar{\sigma}^2M^2L_h}{\mu}\right]\left\|X(t)-Y(t)\right\|_{SPAA}^2. \qquad (20)$$

Consequently, Φ has a unique fixed point in $SPAA\left(\mathbb{R},\ L_G^2\left(\mathcal{F}\right)\right)$, which shows that (1) has unique square mean pseudo almost automorphic mild solution.

4. Stability of square mean pseudo almost automorphic solution

In this section, we firstly introduce the definitions of exponential stability. In order to obtain the main results, we let $D(t,0)=f(t,0)=g(t,0)=h(t,0)=0$.

Definition 4.1. The square mean pseudo almost automorphic mild solution $X(t)$ of (1) is

(1) exponentially stable in mean square if for any initial value $X(t_0)$, the solution $X(t)$ satisfies

$$\mathbb{E}\left\|X(t)\right\|^2 \le C\mathbb{E}\left\|X(t_0)\right\|^2 e^{-\lambda(t-t_0)},$$

where λ and C are positive constants independent of t_0.

(2) quasi sure exponentially stable if for any initial value $X(t_0)$, the solution $X(t)$ satisfies

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}\log\left\|X(t)\right\| \le -\lambda,\ q.s.,$$

where $\lambda > 0$.

Theorem 4.2. *Assuming that the conditions* (H1)-(H3) *are satisfied and*

$$5L_D+\frac{5M^2L_f}{\mu^2}+\frac{5\bar{\sigma}^4M^2L_g}{\mu^2}+\frac{5\bar{\sigma}^2M^2L_h}{2\mu} < 1.$$

Then the square mean pseudo almost automorphic mild solution $X(t)$ of (1) *is exponentially stable.*

Proof: From the definition of solution, we have

$$\begin{aligned}\mathbb{E}\left\|X(t)\right\|^2 &\leq 5\mathbb{E}\left\|D\left(t,\ X(t)\right)\right\|^2 + 5\mathbb{E}\left\|\int_{t_0}^{t} U(t,r)f\left(r,X(r)\right)\mathrm{d}r\right\|^2 \\ &+5\mathbb{E}\left\|\int_{t_0}^{t} U(t,r)g\left(r,X(r)\right)\mathrm{d}\left\langle B\right\rangle(r)\right\|^2 + 5\mathbb{E}\left\|\int_{t_0}^{t} U(t,r)h\left(r,X(r)\right)\mathrm{d}B(r)\right\|^2 \\ &+5\mathbb{E}\left\|U(t,\ t_0)\left[X(t_0)-D\left(t_0,\ X(t_0)\right)\right]\right\|^2.\end{aligned}$$

Similar to the proof of **Uniqueness**, we have

$$\begin{aligned}\mathbb{E}\left\|X(t)\right\|^2 &\leq \left[5L_D + \frac{5M^2L_f}{\mu^2} + \frac{5\bar{\sigma}^4M^2L_g}{\mu^2} + \frac{5\bar{\sigma}^2M^2L_h}{2\mu}\right]\sup_{t\in\mathbb{R}}\mathbb{E}\left\|X(t)\right\|^2 \\ &+10M^2(1+L_D)e^{-\mu(t-t_0)}\mathbb{E}\left\|X(t_0)\right\|^2.\end{aligned}$$

Therefore, we get

$$\mathbb{E}\left\|X(t)\right\|^2 \leq C\mathbb{E}\left\|X(t_0)\right\|^2 e^{-\mu(t-t_0)}.$$

where $C = 10M^2\left(1+L_D\right)/\left(1 - 5L_D - \frac{5M^2L_f}{\mu^2} - \frac{5\bar{\sigma}^4M^2L_g}{\mu^2} - \frac{5\bar{\sigma}^2M^2L_h}{2\mu}\right)$ So, we can find that the square mean pseudo almost automorphic mild solution $X(t)$ of (1) is exponentially stable.

Theorem 4.3. *Assuming that all the conditions of 4.2 are satisfied. Then the square mean pseudo almost automorphic mild solution $X(t)$ of* (1) *is said to be quasi sure exponentially stable.*

Proof: From the Theorem 4.2, we have

$$\mathbb{E}\left\|X(t)\right\|^2 \leq C\mathbb{E}\left\|X(t_0)\right\|^2 e^{-\mu(t-t_0)}. \tag{21}$$

By the elementary inequality, we obtain

$$\begin{aligned}\left\|X(t+s)\right\|^2 &= \Big\|D\left(t+s,X(t+s)\right) + \int_{t}^{t+s} U(t+s,r)f\left(r,X(r)\right)\mathrm{d}r \\ &+\int_{t}^{t+s} U(t+s,r)g\left(r,X(r)\right)\mathrm{d}\left\langle B\right\rangle(r) + \int_{t}^{t+s} U(t+s,r)h\left(r,X(r)\right)\mathrm{d}B(r) \\ &+U(t+s,t)\left[X(t)-D\left(t,\ X(t)\right)\right]\Big\|^2 \\ &\leq 5\left\|D\left(t+s,X(t+s)\right)\right\|^2 + 5\left\|\int_{t}^{t+s} U(t+s,r)f\left(r,X(r)\right)\mathrm{d}r\right\|^2 \\ &+5\left\|\int_{t}^{t+s} U(t+s,r)g\left(r,X(r)\right)\mathrm{d}\left\langle B\right\rangle(r)\right\|^2 + 5\left\|\int_{t}^{t+s} U(t+s,r)h\left(r,X(r)\right)\mathrm{d}B(r)\right\|^2 \\ &+5\left\|U(t+s,t)\left[X(t)-D\left(t,\ X(t)\right)\right]\right\|^2 \\ &:= 5\sum_{i=1}^{5}\Upsilon_i(t).\end{aligned} \tag{22}$$

From assumption (H1) and (H2), we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq\tau}\Upsilon_1(s)\right] &= \mathbb{E}\left[\sup_{0\leq s\leq\tau}\left\|D\left(t+s,\ X(t+s)\right)\right\|^2\right] \\ &\leq L_D\mathbb{E}\left[\sup_{0\leq s\leq\tau}\left\|X(t+s)\right\|^2\right].\end{aligned} \tag{23}$$

By Cauchy-Schwarz inequality, (H1) and (H2), we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le\tau}\Upsilon_2(s)\right] &= \mathbb{E}\left[\sup_{0\le s\le\tau}\left\|\int_t^{t+s}U(t+s,r)f(r,X(r))\,\mathrm{d}r\right\|^2\right]\\
&\le \mathbb{E}\sup_{0\le s\le\tau}\left\{\int_t^{t+s}\|U(t+s,r)\|^2\mathrm{d}r\int_t^{t+s}\left\|f(r,X(r))\right\|^2\mathrm{d}r\right\}\\
&\le \frac{M^2L_f}{2\mu}\mathbb{E}\left[\sup_{0\le s\le\tau}\int_t^{t+s}\|X(r)\|^2\mathrm{d}r\right]\\
&\le \frac{M^2L_f}{2\mu}\int_t^{t+\tau}\mathbb{E}\|X(r)\|^2\mathrm{d}r\\
&\le \frac{M^2L_f}{2\mu}C\mathbb{E}\left\|X(t_0\right\|^2\int_t^{t+\tau}e^{-\mu(r-t_0)}\mathrm{d}r\\
&\le \frac{M^2L_f}{2\mu^2}C\mathbb{E}\left\|X(t_0)\right\|^2e^{-\mu(t-t_0)}. \qquad (24)
\end{aligned}$$

From (H1), (H2) and Lemma 2.13, it shows

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le\tau}\Upsilon_3(s)\right] &\le M^2\bar{\sigma}^2\tau^2\int_t^{t+\tau}e^{-2\mu(t-r)}\mathbb{E}\left\|g(r,X(r))\right\|^2\mathrm{d}r\\
&\le M^2\bar{\sigma}^2\tau^2L_g\int_t^{t+\tau}e^{-2\mu(t-r)}\mathbb{E}\left\|X(r)\right\|^2\mathrm{d}r\\
&\le M^2\bar{\sigma}^2\tau^2L_gC\mathbb{E}\left\|X(t_0)\right\|^2\int_t^{t+\tau}e^{-2\mu(t-r)}e^{-\mu(r-t_0)}\mathrm{d}r\\
&\le \frac{M^2\bar{\sigma}^2\tau^2L_g}{\mu}e^{\tau}C\mathbb{E}\left\|X(t_0)\right\|^2e^{-\mu(t-t_0)}. \qquad (25)
\end{aligned}$$

From Lemma 2.11 and 2.12, (H1) and (H2),

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le\tau}\Upsilon_4(s)\right] &\le 4\bar{\sigma}^2M^2L_h\int_t^{t+\tau}e^{-2\mu(t-r)}\mathbb{E}\|X(r)\|^2\mathrm{d}r\\
&\le 4\bar{\sigma}^2M^2L_hC\mathbb{E}\left\|X(t_0)\right\|^2\int_t^{t+\tau}e^{-2\mu(t-r)}e^{-\mu(r-t_0)}\mathrm{d}r\\
&\le \frac{4\bar{\sigma}^2M^2L_h}{\mu}e^{\tau}C\mathbb{E}\left\|X(t_0)\right\|^2e^{-\mu(t-t_0)}. \qquad (26)
\end{aligned}$$

By elementary inequality, (H1) and (H2), we have

$$\begin{aligned}
5\mathbb{E}\left[\sup_{0\le s\le\tau}\Upsilon_5(s)\right] &= 5\mathbb{E}\left\|U(t+s,t)\left[X(t)-D\left(t,\ X(t)\right)\right]\right\|^2\\
&\le 10M^2(1+L_D)C\mathbb{E}\left\|X(t_0)\right\|^2e^{-\mu(t-t_0)}. \qquad (27)
\end{aligned}$$

It follows from (23) to (27), we deduce

$$\begin{aligned}
&(1-5L_D)\,\mathbb{E}\left[\sup_{t_0\le s\le t}\left\|X(s)\right\|^2\right]\\
&\le\left[\frac{5M^2L_f}{2\mu^2}+\frac{5M^2\bar{\sigma}^2\tau^2L_g}{\mu}e^{\tau}+\frac{20\bar{\sigma}^2M^2L_h}{\mu}e^{\tau}+10M^2(1+L_D)\right]C\mathbb{E}\left\|X(t_0)\right\|^2e^{-\mu(t-t_0)}.
\end{aligned} \qquad (28)$$

So, by virtue of $5L_D + \frac{5M^2L_f}{\mu^2} + \frac{5\bar{\sigma}^4M^2L_g}{\mu^2} + \frac{5\bar{\sigma}^2M^2L_h}{2\mu} < 1$, we get

$$\mathbb{E}\left[\sup_{0\le s\le\tau}\left\|X(t+s)\right\|^2\right] \le M_0e^{-\mu(t-t_0)}, \tag{29}$$

where $M_0 = \left[\frac{5M^2L_f}{2\mu^2} + \frac{5M^2\bar{\sigma}^2\tau^2L_g}{\mu}e^{\tau} + \frac{20\bar{\sigma}^2M^2L_h}{\mu}e^{\tau} + 10M^2(1+L_D)\right]C\mathbb{E}\left\|X(t_0)\right\|^2/(1-5L_D)$.

Consequently, for any $\epsilon \in (0,\ \mu)$,

$$\begin{aligned}\bar{C}\left(\omega : \sup_{0\le s\le\tau}\left\|X(n\tau+s)\right\|^2 \ge e^{-n(\mu-\epsilon)\tau}\right) \quad &\le \quad \frac{\mathbb{E}\left(\sup_{0\le s\le\tau}\left\|X(n\tau+s)\right\|^2\right)}{e^{-n(\mu-\epsilon)\tau}} \\ &\le M_0e^{-n\epsilon\tau}\mathbb{E}\left\|X(t_0)\right\|^2.\end{aligned}$$

According to Borel-Cantelli Lemma, we can conclude there exists a $k_0(\omega)$ such that for almost all $\omega \in \Omega$, $k \ge k_0(\omega)$,

$$\sup_{0\le s\le\tau}\left\|X(n\tau+s)\right\|^2 \le e^{-n(\mu-\epsilon)\tau}.$$

This implies

$$\limsup_{n\to\infty}\frac{\log\sup_{n\tau\le t\le(n+1)\tau}\left\|X(t)\right\|}{n\tau} \le -\frac{\mu-\epsilon}{2},\ q.\,s..$$

Therefore, we can obtain

$$\limsup_{t\to\infty}\frac{\log\left\|X(t)\right\|}{t} \le -\frac{\mu-\epsilon}{2},\ q.\,s..$$

Letting $\epsilon \to 0$, we obtain the desired results.

5. Conclusion

In this paper, a class of neutral stochastic evolution equations driven by G-Brownian motion has been studied. Firstly, under classical Lipchitz conditions, the existence and uniqueness of square mean pseudo almost automorphic mild solutions to the stochastic system. Next, the quasi sure exponential stability of square mean pseudo almost automorphic mild solutions to neutral stochastic evolution equations is investigated based stochastic analysis theory and Borel-Cantelli Lemma. Moreover, we obtained the exponential stability of square mean pseudo almost automorphic mild solutions.

Availability of supporting data
The data sets supporting the results are included within the article.

Competing interests
The author declare that he has no competing interests.

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