Filomat 34:4 (2020), 1147–1156
https://doi.org/10.2298/FIL2004147G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Parabolic Local Campanato Estimates for Commutators of Parabolic Fractional Maximal and Integral Operators With Rough Kernel

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Abstract. In this paper, the author introduces parabolic generalized local Morrey spaces and parabolic local Campanato spaces, respectively and also establishes parabolic local Campanato estimates for commutators of parabolic fractional maximal and integral operators with rough kernel on parabolic generalized local Morrey spaces.

1. Introduction

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ denote the unit sphere on $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma(x')$, where x' denotes the unit vector in the direction of x and $\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq 1$ be fixed real numbers.

Note that for each fixed $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the function

$$F(x,\rho) = \sum_{i=1}^{n} \frac{x_i^2}{\rho^{2\alpha_i}}$$

is a strictly decreasing function of $\rho > 0$. Hence, there exists a unique $\rho = \rho(x)$ such that $F(x,\rho) = 1$. It is clear that for each fixed $x \in \mathbb{R}^n$, the function $F(x,\rho)$ is a decreasing function in $\rho > 0$. Fabes and Riviére [4] showed that $(\mathbb{R}^n,\rho)$ is a metric space which is often called the mixed homogeneity space related to $\{\alpha_i\}_{i=1}^n$. For $t > 0$, we let A_t be the diagonal $n \times n$ matrix

$$A_t = diag\left[t^{\alpha_1}, \ldots, t^{\alpha_n}\right] = \begin{pmatrix} t^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & t^{\alpha_n} \end{pmatrix}.$$

2010 *Mathematics Subject Classification*. Primary 42B20; Secondary 42B25, 42B35

Keywords. Parabolic fractional integral operator, parabolic fractional maximal operator, rough kernel, parabolic generalized local Morrey space, parabolic local Campanato space, commutator

Received: 10 August 2016; Accepted: 23 March 2020

Communicated by Dragan S. Djordjević

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Let $\rho \in (0,\infty)$ and $0 \leq \varphi_{n-1} \leq 2\pi$, $0 \leq \varphi_i \leq \pi$, $i = 1,\dots,n-2$. For any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, set

$$\begin{aligned} x_1 &= \rho^{\alpha_1} \cos\varphi_1 \dots \cos\varphi_{n-2} \cos\varphi_{n-1}, \\ x_2 &= \rho^{\alpha_2} \cos\varphi_1 \dots \cos\varphi_{n-2} \sin\varphi_{n-1}, \\ &\vdots \\ x_{n-1} &= \rho^{\alpha_{n-1}} \cos\varphi_1 \sin\varphi_2, \\ x_n &= \rho^{\alpha_n} \sin\varphi_1. \end{aligned}$$

Thus $dx = \rho^{\alpha-1} J(x')\, d\rho d\sigma(x')$, where $\alpha = \sum\limits_{i=1}^{n} \alpha_i$, $x' \in S^{n-1}$, $J(x') = \sum\limits_{i=1}^{n} \alpha_i \left(x_i'\right)^2$, $d\sigma$ is the element of area of S^{n-1} and $\rho^{\alpha-1} J(x')$ is the Jacobian of the above transform. Obviously, $J(x') \in C^\infty\left(S^{n-1}\right)$ function and that there exists $M > 0$ such that $1 \leq J(x') \leq M$ and $x' \in S^{n-1}$.

Let P be a real $n \times n$ matrix, whose all the eigenvalues have positive real part. Let $A_t = t^P$ $(t > 0)$, and set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that (see [3])

$(1-1)$ $\rho(A_t x) = t\rho(x)$, $t > 0$, for every $x \in \mathbb{R}^n$,

$(1-2)$ $\rho(0) = 0$, $\rho(x-y) = \rho(y-x) \geq 0$, and $\rho(x-y) \leq k(\rho(x-z) + \rho(y-z))$,

$(1-3)$ $dx = \rho^{\gamma-1} d\sigma(w)\, d\rho$, where $\rho = \rho(x)$, $w = A_{\rho^{-1}} x$ and $d\sigma(w)$ is a measure on the unit ellipsoid $\{w : \rho(w) = 1\}$.

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss (see [3]) and a homogeneous group in the sense of Folland-Stein (see [5]).

Denote by $E(x,r)$ the ellipsoid with center at x and radius r, more precisely, $E(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$. For $k > 0$, we denote $kE(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < kr\}$. Moreover, by the property of ρ and the polar coordinates transform above, we have

$$|E(x,r)| = \int\limits_{\rho(x-y)<r} dy = v_\rho r^{\alpha_1+\dots+\alpha_n} = v_\rho r^\gamma,$$

where $|E(x,r)|$ stands for the Lebesgue measure of $E(x,r)$ and v_ρ is the volume of the unit ellipsoid on $\mathbb{R}^n$. By $E^C(x,r) = \mathbb{R}^n \setminus E(x,r)$, we denote the complement of $E(x,r)$. If we take $\alpha_1 = \dots = \alpha_n = 1$ and $P = I$, then obviously $\rho(x) = |x| = \left(\sum\limits_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$, $\gamma = n$, $(\mathbb{R}^n, \rho) = (\mathbb{R}^n, |\cdot|)$, $E_I(x,r) = B(x,r)$, $A_t = tI$ and $J(x') \equiv 1$. Moreover, in the standard parabolic case $P_0 = diag[1,\dots,1,2]$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Suppose that $\Omega(x)$ is a real-valued and measurable function defined on $\mathbb{R}^n$. Suppose that S^{n-1} is the unit sphere on $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$. Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous of degree zero with respect to A_t ($\Omega(x)$ is A_t-homogeneous of degree zero), that is, $\Omega(A_t x) = \Omega(x)$, for any $t > 0, x \in \mathbb{R}^n$. We define $s' = \frac{s}{s-1}$ for any $s > 1$.

The parabolic fractional maximal and integral operators $M^P_{\Omega,\alpha} f$ and $I^P_{\Omega,\alpha} f$ by with rough kernels, $0 < \alpha < \gamma$, of a function $f \in L^{loc}(\mathbb{R}^n)$ are defined by

$$M^P_{\Omega,\alpha} f(x) = \sup_{t>0} |E(x,t)|^{-1+\frac{\alpha}{\gamma}} \int\limits_{E(x,t)} \left|\Omega(x-y)\right| |f(y)| dy,$$

$$I^P_{\Omega,\alpha} f(x) = \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{\rho(x-y)^{\gamma-\alpha}} f(y) dy. \tag{1}$$

It is obvious that when $\Omega \equiv 1$, $M^P_{1,\alpha} \equiv M^P_\alpha$ and $I^P_{1,\alpha} \equiv I^P_\alpha$ are the parabolic fractional maximal operator and the parabolic fractional integral operator, respectively. If $P = I$, then $M^I_{\Omega,\alpha} \equiv M_{\Omega,\alpha}$ and $I^I_{\Omega,\alpha} \equiv I_{\Omega,\alpha}$ are the fractional maximal operator with rough kernel and fractional integral operator with rough kernel, respectively. It is well known that the parabolic fractional maximal and integral operators play an important role in harmonic analysis (see [2, 5, 7]).

We notice that when $\alpha = 0$, the above operators become the parabolic Calderón–Zygmund singular integral operator with rough kernel $T^P_\Omega = T^P_{\Omega,0}$ and the corresponding parabolic maximal operator with rough kernel $M^P_{\Omega,0} \equiv M^P_\Omega$:

$$T^P_\Omega f(x) = p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{\rho(x-y)^\gamma} f(y)dy,$$

$$M^P_\Omega f(x) = \sup_{t>0} |E(x,t)|^{-1} \int\limits_{E(x,t)} \left|\Omega(x-y)\right| |f(y)|dy.$$

It is obvious that when $\Omega \equiv 1$, $T^P_\Omega \equiv T^P$ and $M^P_\Omega \equiv M^P$ are the parabolic singular operator and the parabolic maximal operator, respectively. If $P = I$, then $M^I_\Omega \equiv M_\Omega$ is the Hardy-Littlewood maximal operator with rough kernel, and $T^I_\Omega \equiv T_\Omega$ is the homogeneous singular integral operator. It is well known that the parabolic maximal and singular operators play an important role in harmonic analysis (see [2, 5, 6, 11]).

On the other hand let b be a locally integrable function on $\mathbb{R}^n$, then for $0 < \alpha < \gamma$, we define commutators generated by parabolic fractional maximal and integral operators with rough kernel and b as follows, respectively.

$$M^P_{\Omega,b,\alpha}(f)(x) = \sup_{t>0} |E(x,t)|^{-1+\frac{\alpha}{\gamma}} \int\limits_{E(x,t)} \left|b(x) - b(y)\right| \left|\Omega(x-y)\right| |f(y)|dy, \tag{2}$$

$$[b, I^P_{\Omega,\alpha}]f(x) \equiv b(x)I^P_{\Omega,\alpha}f(x) - I^P_{\Omega,\alpha}(bf)(x) = \int\limits_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x-y)}{\rho(x-y)^{\gamma-\alpha}} f(y)dy. \tag{3}$$

Similarly, for $\alpha = 0$, we define commutators generated by parabolic maximal and singular integral operators by with rough kernels and b as follows, respectively.

$$M^P_{\Omega,b}(f)(x) = \sup_{t>0} |E(x,t)|^{-1} \int\limits_{E(x,t)} \left|b(x) - b(y)\right| \left|\Omega(x-y)\right| |f(y)|dy, \tag{4}$$

$$[b, T^P_\Omega]f(x) \equiv b(x)T^P_\Omega f(x) - T^P_\Omega(bf)(x) = p.v. \int\limits_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x-y)}{\rho(x-y)^\gamma} f(y)dy. \tag{5}$$

Because of the need for the study of partial differential equations (PDEs), Morrey [8] introduced Morrey spaces $M_{p,\lambda}$ which naturally are generalizations of Lebesgue spaces.

A measurable function $f \in L_p(\mathbb{R}^n)$, $p \in (1,\infty)$, belongs to the parabolic Morrey spaces $M_{p,\lambda,P}(\mathbb{R}^n)$ with $\lambda \in [0,\gamma)$ if the following norm is finite:

$$\|f\|_{M_{p,\lambda,P}} = \left(\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^\lambda} \int\limits_{E(x,r)} \left|f(y)\right|^p dy \right)^{1/p},$$

where $E(x,r)$ stands for any ellipsoid with center at x and radius r. When $\lambda = 0$, $M_{p,\lambda,P}(\mathbb{R}^n)$ coincides with the parabolic Lebesgue space $L_{p,P}(\mathbb{R}^n)$.

If $P = I$, then $M_{p,\lambda,I}(\mathbb{R}^n) \equiv M_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,I}(\mathbb{R}^n) \equiv L_p(\mathbb{R}^n)$ are the classical Morrey and the Lebesgue spaces, respectively.

One of important issue in the study of operators is their boundedness. Spanne (published by Peetre [9]) and Adams [1] proved the following boundedness properties of the fractional integral operator I_α for $0 < \alpha < n$ on classical Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$.

For the parabolic fractional integral I_α^P, their results can be summarized as follows.

Theorem 1.1. *(Spanne, but published by Peetre [9]) Let $0 < \alpha < \gamma$, $1 < p < \frac{\gamma-\lambda}{\alpha}$, $0 < \lambda < \gamma - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$. Then, the operator I_α^P is bounded from $M_{p,\lambda,P}$ to $M_{q,\frac{\lambda q}{p},P}$.*

Theorem 1.2. *(Adams [1]) Let $0 < \alpha < \gamma$, $1 < p < \frac{\gamma-\lambda}{\alpha}$, $0 < \lambda < \gamma - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma-\lambda}$. Then the operator I_α^P is bounded from $M_{p,\lambda,P}$ to $M_{q,\lambda,P}$.*

By Hölder's inequality, one can observe that: The indices q_1, q_2 and μ satisfy the following relations:

$$\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{\gamma}, \frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{\gamma - \lambda}, \frac{\mu}{q_1} = \frac{\lambda}{p}.$$

Since $q_1 < q_2$, by Hölder's inequality we get

$$\left\|I_\alpha^P f\right\|_{M_{q_1,\mu,P}} \leq \left\|I_\alpha^P f\right\|_{M_{q_2,\lambda,P}}.$$

Thus, Theorem 1.2 is a sharper result than Theorem 1.1, in other words, Theorem 1.2 improves Theorem 1.1 when $1 < p < \frac{\gamma-\lambda}{\alpha}$:

$$\left\|I_\alpha^P f\right\|_{M_{q_1,\mu,P}} \leq \left\|I_\alpha^P f\right\|_{M_{q_2,\lambda,P}} \leq C \left\|f\right\|_{M_{p,\lambda,P}}.$$

Recall that, for $0 < \alpha < \gamma$,

$$M_\alpha^P f(x) \leq v_n^{\frac{\alpha}{\gamma}-1} I_\alpha^P\left(\left|f\right|\right)(x)$$

holds (see [7]). Hence Theorems 1.1 and 1.2 also imply boundedness of the parabolic fractional maximal operator M_α^P, where v_n is the volume of the unit ellipsoid on $\mathbb{R}^n$.

The following theorem is valid.

Theorem 1.3. *Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be A_t-homogeneous of degree zero. Let also $0 < \alpha < \gamma$, $1 < p, q < \infty$ and $0 < \lambda, \mu < \gamma$. If the operators $I_{\Omega,\alpha}^P$ and $M_{\Omega,\alpha}^P$ are bounded from $M_{p,\lambda,P}$ to $M_{q,\mu,P}$, then*

$$\frac{\gamma - \mu}{q} = \frac{\gamma - \lambda}{p} - \alpha, \tag{6}$$

and

$$\max\left(1, \frac{\gamma - \lambda}{\gamma - \mu + \alpha}\right) < p < \frac{\gamma - \lambda}{\alpha}. \tag{7}$$

Proof. Let $\tau > 0$ and $\psi_\tau f(x) = f(\tau x)$. Then, it is obvious that

$$I_{\Omega,\alpha}^P f(x) = \tau^\alpha I_{\Omega,\alpha}^P(\psi_\tau f)\left(\frac{x}{\tau}\right)$$

and

$$\left\|\psi_\tau f\right\|_{M_{p,\lambda,P}}=\tau^{-\frac{\gamma-\lambda}{p}}\left\|f\right\|_{M_{p,\lambda,P}}.$$

Since the operator $I^P_{\Omega,\alpha}$ is bounded from $M_{p,\lambda,P}$ to $M_{q,\mu,P}$, then

$$\begin{aligned}\left\|I^P_{\Omega,\alpha}f\right\|_{M_{q,\mu,P}}&=r^{-\frac{\mu}{q}}\left\|I^P_{\Omega,\alpha}f\right\|_{L_q(E(x,r))}\\&\leq\tau^{\alpha+\frac{\gamma-\mu}{q}}\left\|I^P_{\Omega,\alpha}\left(\psi_\tau f\right)\right\|_{M_{q,\mu,P}}\\&\lesssim\tau^{\alpha+\frac{\gamma-\mu}{q}}\left\|\psi_\tau f\right\|_{M_{p,\lambda,P}}\\&\lesssim\tau^{\alpha+\frac{\gamma-\mu}{q}-\frac{\gamma-\lambda}{p}}\left\|f\right\|_{M_{p,\lambda,P}}.\end{aligned}$$

Thus, (6) is hold and it follows that $p<\frac{\gamma-\lambda}{\alpha}$. Since $q>1$ and by (6), then

$$\begin{aligned}0&\leq(\gamma-\mu)-\frac{\gamma-\mu}{q}=(\gamma-\mu)-\left(\frac{\gamma-\lambda}{p}-\alpha\right)\\&=\gamma-\mu+\alpha-\left(\frac{\gamma-\lambda}{p}\right)\end{aligned}$$

and thus

$$p\geq\frac{\gamma-\lambda}{\gamma-\mu+\alpha}.$$

As a result, (7) is hold.

Set

$$\widetilde{I}^P_{|\Omega|,\alpha}\left(\left|f\right|\right)(x)=\int\limits_{\mathbb{R}^n}\frac{\left|\Omega(x-y)\right|}{\rho\left(x-y\right)^{\gamma-\alpha}}\left|f(y)\right|dy\qquad 0<\alpha<\gamma,$$

where $\Omega\in L_s(S^{n-1})$ $(s>1)$ is A_t-homogeneous of degree zero on $\mathbb{R}^n$. It is easy to see that, for $\widetilde{I}^P_{|\Omega|,\alpha}$, Theorem 7 is also hold. On the other hand, for any $t>0$, we have

$$\begin{aligned}\widetilde{I}^P_{|\Omega|,\alpha}\left(\left|f\right|\right)(x)&\geq\int\limits_{E(x,t)}\frac{\left|\Omega(x-y)\right|}{\rho\left(x-y\right)^{\gamma-\alpha}}\left|f(y)\right|dy\\&\geq\frac{1}{t^{\gamma-\alpha}}\int\limits_{E(x,t)}\left|\Omega(x-y)\right|\left|f(y)\right|dy.\end{aligned}$$

Taking the supremum for $t>0$ on the inequality above, we get

$$C^{-1}_{\gamma,\alpha}\widetilde{I}^P_{|\Omega|,\alpha}\left(\left|f\right|\right)(x)\geq M^P_{\Omega,\alpha}f(x)\qquad C_{\gamma,\alpha}=\left|E\left(0,1\right)\right|^{\frac{\gamma-\alpha}{\gamma}}.$$

□

Remark 1.4. *(6) is the sufficient condition in inequalities above (see Theorems 1.1 and 1.2). Indeed, if we take* $\mu=\frac{\lambda q}{p}$ *and* $\mu=\lambda$*, the remainder statement is the same and we omit it.*

We now recall the definition of parabolic generalized local (central) Morrey space $LM^{\{x_0\}}_{p,\varphi,P}$ in the following.

Definition 1.5. *[6, 7] **(parabolic generalized local (central) Morrey space)** Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM^{\{x_0\}}_{p,\varphi,P} \equiv LM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)$ the parabolic generalized local Morrey space, the space of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM^{\{x_0\}}_{p,\varphi,P}} = \sup_{r>0} \varphi(x_0,r)^{-1} |E(x_0,r)|^{-\frac{1}{p}} \|f\|_{L_p(E(x_0,r))} < \infty.$$

According to this definition, we recover the local parabolic Morrey space $LM^{\{x_0\}}_{p,\lambda,P}$ and weak local parabolic Morrey space $WLM^{\{x_0\}}_{p,\lambda,P}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-\gamma}{p}}$:

$$LM^{\{x_0\}}_{p,\lambda,P} = LM^{\{x_0\}}_{p,\varphi,P}\,|_{\varphi(x_0,r)=r^{\frac{\lambda-\gamma}{p}}}, \qquad WLM^{\{x_0\}}_{p,\lambda,P} = WLM^{\{x_0\}}_{p,\varphi,P}\,|_{\varphi(x_0,r)=r^{\frac{\lambda-\gamma}{p}}}.$$

Now, let us recall the defination of the space of $LC^{\{x_0\}}_{p,\lambda,P}$ (parabolic local Campanato space).

Definition 1.6. *[6, 7] Let $1 \le p < \infty$ and $0 \le \lambda < \frac{1}{\gamma}$. A parabolic local Campanato function $b \in L^{loc}_p(\mathbb{R}^n)$ is said to belong to the $LC^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n)$, if*

$$\|b\|_{LC^{\{x_0\}}_{p,\lambda,P}} = \sup_{r>0} \left(\frac{1}{|E(x_0,r)|^{1+\lambda p}} \int\limits_{E(x_0,r)} \left| b(y) - b_{E(x_0,r)} \right|^p dy \right)^{\frac{1}{p}} < \infty,$$

where

$$b_{E(x_0,r)} = \frac{1}{|E(x_0,r)|} \int\limits_{E(x_0,r)} b(y)\,dy.$$

Define

$$LC^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n) = \left\{ b \in L^{loc}_p(\mathbb{R}^n) : \|b\|_{LC^{\{x_0\}}_{p,\lambda,P}} < \infty \right\}.$$

In [6, 7] the boundedness of a class of parabolic sublinear operators with rough kernel and their commutators on the parabolic generalized local Morrey spaces under generic size conditions which are satisfied by most of the operators in harmonic analysis has been investigated, respectively.

Inspired by [6, 7], our main purpose in this paper is to consider the boundedness of above operators ($[b, T^P_\Omega]$, $M^P_{\Omega,b}$, $[b, I^P_{\Omega,\alpha}]$, $M^P_{\Omega,b,\alpha}$) on the parabolic generalized local Morrey spaces, respectively. But, the techniques and non-trivial estimates which have been used in the proofs of our main results are quite different from [6, 7]. For example, using inequality about the weighted Hardy operator H_w in [6, 7], in this paper we will only use the following relationship between essential supremum and essential infimum

$$\left(\operatorname*{essinf}_{x\in E} f(x) \right)^{-1} = \operatorname*{esssup}_{x\in E} \frac{1}{f(x)}, \tag{8}$$

where f is any real-valued nonnegative function and measurable on E (see [10], page 143).

Our main results can be formulated as follows.

Theorem 1.7. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, is A_t-homogeneous of degree zero. Let $0 < \alpha < \gamma$, $1 < p < \frac{\gamma}{\alpha}$, $b \in LC^{\{x_0\}}_{p_2,\lambda,P}(\mathbb{R}^n)$, $0 \le \lambda < \frac{1}{\gamma}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{\gamma}$, $M^P_{\Omega,b,\alpha}$, $[b, I^P_{\Omega,\alpha}]$ are defined as (2), (3) and $[b, I^P_{\Omega,\alpha}]$ satisfies Theorem 3.4 in [7].*

Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{q_1}+1-\gamma\lambda}} dt \leq C\,\varphi_2(x_0, r), \tag{9}$$

and for $q_1 < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{q_1}-\frac{\gamma}{s}+1-\gamma\lambda}} dt \leq C\,\varphi_2(x_0, r) r^{\frac{\gamma}{s}},$$

where C does not depend on r.

Then the operators $M^P_{\Omega,b,\alpha}$ and $[b, I^P_{\Omega,\alpha}]$ are bounded from $LM^{\{x_0\}}_{p_1,\varphi_1,P}$ to $LM^{\{x_0\}}_{q,\varphi_2,P}$. Moreover,

$$\left\|M^P_{\Omega,b,\alpha} f\right\|_{LM^{\{x_0\}}_{q,\varphi_2,P}} \lesssim \left\|[b, I^P_{\Omega,\alpha}]f\right\|_{LM^{\{x_0\}}_{q,\varphi_2,P}} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}}.$$

Using the idea of proving Theorem 1.3, we can obtain the following pointwise relation:

Lemma 1.8. *Let $0 < \alpha < \gamma$ and $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be A_t-homogeneous of degree zero. Then we have*

$$M^P_{\Omega,b,\alpha} f(x) \leq [b, \widetilde{I}^P_{|\Omega|,\alpha}]\left(\left|f\right|\right)(x) \qquad \text{for } x \in \mathbb{R}^n.$$

Proof. For any $t > 0$, we have

$$\begin{aligned}
[b, \widetilde{I}^P_{|\Omega|,\alpha}]\left(\left|f\right|\right)(x) &\geq \int_{\rho(x-y)<t} \left|b(x) - b(y)\right| \frac{\left|\Omega(x-y)\right|}{\rho(x-y)^{\gamma-\alpha}} \left|f(y)\right| dy \\
&\geq \frac{1}{t^{\gamma-\alpha}} \int_{E(x,t)} \left|b(x) - b(y)\right| \left|\Omega(x-y)\right| \left|f(y)\right| dy.
\end{aligned}$$

Taking the supremum for $t > 0$ on the inequality above, we get

$$[b, \widetilde{I}^P_{|\Omega|,\alpha}]\left(\left|f\right|\right)(x) \geq M^P_{\Omega,b,\alpha} f(x) \qquad \text{for } x \in \mathbb{R}^n.$$

□

Theorem 1.9. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is A_t-homogeneous of degree zero. Let $b \in LC^{\{x_0\}}_{p_2,\lambda,P}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{\gamma}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $M^P_{\Omega,b}$, $[b, T^P_\Omega]$ are defined as (4), (5) and $[b, T^P_\Omega]$ satisfies Theorem 3.2. in [6]. Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{p_1}+1-\gamma\lambda}} dt \leq C\,\varphi_2(x_0, r), \tag{10}$$

and for $p_1 < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{p_1}-\frac{\gamma}{s}+1-\gamma\lambda}} dt \leq C\,\varphi_2(x_0, r) r^{\frac{\gamma}{s}},$$

where C does not depend on r.

Then, the operators $[b, T^P_\Omega]$ and $M^P_{\Omega,b}$ are bounded from $LM^{\{x_0\}}_{p_1,\varphi_1,P}$ to $LM^{\{x_0\}}_{p,\varphi_2,P}$. Moreover,

$$\left\|M^P_{\Omega,b} f\right\|_{LM^{\{x_0\}}_{p,\varphi_2,P}} \lesssim \left\|[b, T^P_\Omega]f\right\|_{LM^{\{x_0\}}_{p,\varphi_2,P}} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}}.$$

Lemma 1.10. *Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be A_t-homogeneous of degree zero. Then we have*

$$M^P_{\Omega,b} f(x) \leq [b, \tilde{T}^P_{|\Omega|}]\left(\left|f\right|\right)(x) \qquad \textit{for } x \in \mathbb{R}^n.$$

Proof. For any $t > 0$, we have

$$\begin{aligned}
[b, \tilde{T}^P_{|\Omega|}]\left(\left|f\right|\right)(x) &\geq \int\limits_{\rho(x-y)<t} \left|b(x) - b(y)\right| \frac{\left|\Omega(x-y)\right|}{\rho(x-y)^{\gamma}} \left|f(y)\right| dy \\
&\geq \frac{1}{t^{\gamma}} \int\limits_{E(x,t)} \prod_{i=1}^{m} \left|b(x) - b(y)\right| \left|\Omega(x-y)\right| \left|f(y)\right| dy.
\end{aligned}$$

Taking the supremum for $t > 0$ on the inequality above, we get

$$[b, \tilde{T}^P_{|\Omega|}]\left(\left|f\right|\right)(x) \geq M^P_{\Omega,b} f(x) \qquad \text{for } x \in \mathbb{R}^n.$$

□

Throughout the paper all constants are denoted by C which may vary from one position to another. For two values F and G, $F \approx G$ means that there are positive constants C_1 and C_2 such that $C_1 G \leq F \leq C_2 G$. Also, $F \lesssim G$ means that there exists a positive constant C such that $F \leq CG$. For $s > 1$, we denote by $s' = \frac{s}{s-1}$ the conjugate exponent of s.

2. Proofs of the main results

2.1. Proof of Theorem 1.7

Proof. We consider $[b, I^P_{\Omega,\alpha}]$ firstly. Since $f \in LM^{\{x_0\}}_{p_1,\varphi_1,P}$, by (8) and it is also non-decreasing, with respect to t, of the norm $\left\|f\right\|_{L_{p_1}(E(x_0,t))}$, we get

$$\begin{aligned}
\frac{\left\|f\right\|_{L_{p_1}(E(x_0,t))}}{\operatorname*{essinf}\limits_{0<t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p_1}}} &\leq \operatorname*{esssup}\limits_{0<t<\tau<\infty} \frac{\left\|f\right\|_{L_{p_1}(E(x_0,t))}}{\varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p_1}}} \\
&\leq \operatorname*{esssup}\limits_{0<\tau<\infty} \frac{\left\|f\right\|_{L_{p_1}(E(x_0,\tau))}}{\varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p_1}}} \leq \|f\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}}.
\end{aligned} \tag{11}$$

For $s' \leq p$, since (φ_1, φ_2) satisfies (9) and by (11), we have

$$\begin{aligned}
&\int\limits_r^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\left\|f\right\|_{L_{p_1}(E(x_0,t))}}{t^{\frac{\gamma}{q_1}-\gamma\lambda+1}} dt \\
&\leq \int\limits_r^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\left\|f\right\|_{L_{p_1}(E(x_0,t))}}{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p_1}}} \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{q_1}-\gamma\lambda+1}} dt \\
&\lesssim \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}} \int\limits_r^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{q_1}-\gamma\lambda+1}} dt \\
&\lesssim \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}} \varphi_2(x_0,r).
\end{aligned} \tag{12}$$

Then by Definition 1.5, Theorem 3.4 in [7] and (12), we get

$$\begin{aligned}\left\|[b, I^{P}_{\Omega,\alpha}]f\right\|_{LM^{\{x_0\}}_{q,\varphi_2,P}} &= \sup_{r>0} \varphi_2(x_0, r)^{-1} |E(x_0, r)|^{-\frac{1}{q}} \left\|[b, I^{P}_{\Omega,\alpha}]f\right\|_{L_q(E(x_0,r))} \\ &\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \sup_{r>0} \varphi_2(x_0, r)^{-1} \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\left\|f\right\|_{L_{p_1}(E(x_0,t))}}{t^{\frac{\gamma}{q_1}-\gamma\lambda+1}} dt \\ &\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}}. \end{aligned} \tag{13}$$

From the process proving (13), it is easy to see that the conclusions of (13) also hold for $[b, \widetilde{I}^{P}_{|\Omega|,\alpha}]$. Combining this with Lemma 1.8, we can immediately obtain

$$\left\|M^{P}_{\Omega,b,\alpha} f\right\|_{LM^{\{x_0\}}_{q,\varphi_2,P}} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}}.$$

For the case of $q_1 < s$, we can also use the same method, so we omit the details. This completes the proof of Theorem 1.7. □

2.2. *Proof of Theorem 1.9*

Proof. Similar to the proof of Theorem 1.7, we consider $[b, T^{P}_{\Omega}]$ firstly.

For $s' \leq p$, since (φ_1, φ_2) satisfies (10) and by (11), we have

$$\begin{aligned} &\int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\left\|f\right\|_{L_{p_1}(E(x_0,t))}}{t^{\frac{\gamma}{p_1}+1-\gamma\lambda}} dt \\ &\leq \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\left\|f\right\|_{L_{p_1}(E(x_0,t))}}{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{\gamma}{p_1}}} \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{p_1}+1-\gamma\lambda}} dt \\ &\lesssim \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}} \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{p_1}+1-\gamma\lambda}} dt \\ &\lesssim \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}} \varphi_2(x_0, r). \end{aligned} \tag{14}$$

Then by Definition 1.5, Theorem 3.2. in [6] and (14), we get

$$\begin{aligned}\left\|[b, T^{P}_{\Omega}]f\right\|_{LM^{\{x_0\}}_{p,\varphi_2,P}} &= \sup_{r>0} \varphi_2(x_0, r)^{-1} |E(x_0, r)|^{-\frac{1}{p}} \left\|[b, T^{P}_{\Omega}]f\right\|_{L_p(E(x_0,r))} \\ &\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \sup_{r>0} \varphi_2(x_0, r)^{-1} \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\left\|f\right\|_{L_{p_1}(E(x_0,t))}}{t^{\frac{\gamma}{p_1}-\gamma\lambda+1}} dt \\ &\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}}. \end{aligned} \tag{15}$$

From the process proving (15), it is easy to see that the conclusions of (15) also hold for $[b, \tilde{T}^{P}_{|\Omega|}]$. Combining this with Lemma 1.10, we can immediately obtain

$$\left\|M^{P}_{\Omega,b} f\right\|_{LM^{\{x_0\}}_{p,\varphi_2,P}} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \left\|f\right\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}}.$$

For the case of $p_1 < s$, we can also use the same method, so we omit the details. This completes the proof of Theorem 1.9. □

References

[1] D.R. Adams, A note on Riesz potentials, Duke Math. J. 42 (1975) 765–778.

[2] A.P. Calderón, A. Torchinsky, Parabolic maximal functions associated with a distribution, Adv. Math. 16 (1975) 1–64.

[3] R.R. Coifman, G. Weiss, Analyse harmonique non-commutative sur certains espaces homogènes, (French) Étude de certaines intégrales singuliàres, Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971.

[4] E. Fabes, N. Riviére, Singular integrals with mixed homogeneity, Stud. Math. 27 (1966) 19–38.

[5] G.B. Folland, E.M. Stein, Hardy Spaces on homogeneous groups, Math. Notes, 28, Princeton Univ. Press, Princeton, 1982.

[6] F. Gürbüz, Parabolic sublinear operators with rough kernel generated by parabolic Calderón-Zygmund operators and parabolic local Campanato space estimates for their commutators on the parabolic generalized local Morrey spaces, Open Math. 14 (2016) 300–323.

[7] F. Gürbüz, Parabolic generalized local Morrey space estimates of rough parabolic sublinear operators and commutators, Adv. Math. (China) 46 (2017) 765–792.

[8] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. 43 (1938) 126–166.

[9] J. Peetre, On the theory of $\mathcal{L}_{p,\lambda}$, J. Funct. Anal. 4 (1969) 71–87.

[10] R.L. Wheeden, A. Zygmund, Measure and Integral: An Introduction to Real Analysis, vol. 43 of Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 1977.

[11] Q.Y. Xue, Y. Ding and K. Yabuta, Parabolic Littlewood-Paley g-function with rough kernel, Acta Math. Sin. (Engl. Ser.) 24 (2008) 2049–2060.