For a finite group $G$, let $Z(G)$ denote the center of $G$ and $\text{cs}^*(G)$ be the set of non-trivial conjugacy class sizes of $G$. In this paper, we show that if $G$ is a finite group such that for some odd prime power $q \geq 4$, $\text{cs}^*(G) = \text{cs}^*(\text{PGL}_2(q))$, then either $G \cong \text{PGL}_2(q) \times Z(G)$ or $G$ contains a normal subgroup $N$ and a non-trivial element $t \in G$ such that $N \cong \text{PSL}_2(q) \times Z(G)$, $t^2 \in N$ and $G = N \langle t \rangle$. This shows that the almost simple groups cannot be determined by their set of conjugacy class sizes (up to an abelian direct factor).

1. Introduction

Throughout this paper, $G$ is a finite group, $Z(G)$ is the center of $G$ and for $a \in G$, $\text{cl}_G(a)$ is the conjugacy class in $G$ containing $a$ and $C_G(a)$ denotes the centralizer of the element $a$ in $G$. We denote by $\text{cs}^*(G)$, the set of non-trivial conjugacy class sizes of $G$. Studying the interplay between the structure of a group and the set of its conjugacy class sizes is one of the interesting concepts in group theory. For instance, J. Thompson in 1988 conjectured that:

Thompson’s conjecture. Let $S$ be a simple group. If $G$ is a finite centerless group with $\text{cs}^*(G) = \text{cs}^*(S)$, then $G \cong S$.

In a series of papers, it has been proved that Thompson’s conjecture is true for many families of finite simple groups (see [1]-[6], [9], [11], [13], [16]).

$G$ is named an almost simple group when there exists a simple group $S$ such that $S \trianglelefteq G \precsim \text{Aut}(S)$. In [14] and [17], it has been shown that Thompson’s conjecture is true for some almost simple groups. Inspired by Thompson’s conjecture, A. Camina and R. Camina come up with the following problem [10]:

Problem. If $S$ is a simple group and $G$ is a finite group with $\text{cs}^*(G) = \text{cs}^*(S)$, then is it true that $G \cong S \times Z(G)$?

In 2015, it has been investigated that the above problem is true when $S \cong \text{PSL}_2(q)$ [8]. Then, in [7], it has been proven that the answer of the above problem is true for many families of finite simple groups. Naturally, one can ask what happens for $G$ in the above problem when $S$ is an almost simple group. So, in this paper, we prove that:

Main theorem. Let $q > 4$ be an odd prime power. If $G$ is a finite group with $\text{cs}^*(G) = \text{cs}^*(\text{PGL}_2(q))$, then either $G \cong \text{PGL}_2(q) \times Z(G)$ or $G$ contains a normal subgroup $N$ and a non-trivial element $t \in G$ such that $N \cong \text{PSL}_2(q) \times Z(G)$, $t^2 \in N$ and $G = N \langle t \rangle$. 

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In this paper, all groups are finite. For simplicity of notation, throughout this paper let \( p > 4 \) be a power of an odd prime \( p \), \( GF(q) \) be a field with \( q \) elements and \( G \) be a group with \( cs'(G) = cs'(PGL_2(q)) \). Throughout this paper, we use the following notation: For a natural number \( n \), let \( \pi(n) \) be the set of prime divisors of \( n \), \( C_n \) denote a cyclic group of order \( n \) and for a group \( H \), let \( \pi(H) = \pi(\langle H \rangle) \). Also, \( H.G \) denotes an extension of \( H \) by \( G \). For a prime \( r \) and natural numbers \( a \) and \( b \), \( |a|_r \) is the \( r \)-part of \( a \), i.e., \( |a|_r = r^i \) when \( r \nmid a \) and \( r^{i+1} \nmid a \) and, \( \gcd(a, b) \) and \( \text{lcm}(a, b) \) are the greatest common divisor of \( a \) and \( b \) and the lowest common multiple of \( a \) and \( b \), respectively. For the set \( \pi \) of some primes, \( x \) is named a \( \pi \)-element (\( \pi' \)-element) of a group \( H \) if \( \pi(o(x)) \subseteq \pi(\pi(o(x)) \subseteq \pi(H) - \pi) \).

2. Definitions and preliminary results

Lemma 2.1. [12, Proposition 4] Let \( H \) be a group. If there exists \( p \in \pi(H) \) such that \( p \) does not divide any conjugacy class sizes of \( H \), then the \( p \)-Sylow subgroup of \( H \) is central in \( H \).

Definition 2.2. For a group \( H \), the prime graph \( GK(H) \) of \( H \) is a simple graph whose vertices are the prime divisors of the order of \( H \) and two distinct prime numbers \( p \) and \( q \) are joined by an edge if \( G \) contains an element of order \( pq \). Denote by \( t(H) \) the number of connected components of the graph \( GK(H) \) and denote by \( \pi_i = \pi_i(H) \), \( i = 1, \ldots, t(H) \), the \( i \)-th connected component of \( GKH \). For a group \( H \) of an even order, let \( 2 \in \pi_1 \). If \( GK(H) \) is disconnected, then \( |H| \) can be expressed as a product of co-prime positive integers \( m_i(H) \), \( i = 1, 2, \ldots, t(H) \), where \( \pi(m_i(H)) = \pi_i(H) \), and if there is no ambiguity write \( m_i \) for showing \( m_i(H) \). These \( m_i \)s are called the order components of \( H \) and the set of order components of \( H \) will be denoted by \( OC(H) \). The list of all simple groups with disconnected prime graph and the sets of their order components have been obtained in [15] and [18].

Lemma 2.3. [14] If \( H \) is a group with \( OC(H) = OC(PGL_2(q)) \), then \( H \cong PGL_2(q) \).

Lemmas 2.4, 2.5 and 2.6 are easy to prove for a group \( H \):

Lemma 2.4. For \( x \in H - Z(H) \), let \( C/Z(H) = C_{H/Z(H)}(xZ(H)) \). Then \( C_{H}(x) \subseteq C \).

Lemma 2.5. For every \( x \in H \) and natural number \( n \),

(i) \( C_H(x) \leq C_H(x^n) \) and \( |C_H(x^n)| \mid |C_H(x)| \);

(ii) if \( |C_H(x)| \) is maximal in \( cs'(H) \) by divisibility and \( \pi = \pi(o(x)) \), then for every \( \pi' \)-element \( y \in C_H(x) \), \( C_H(xy) = C_H(x) \). In particular, if \( |C_H(x)| \) is maximal and minimal in \( cs'(H) \) by divisibility and \( \pi = \pi(o(x)) \), then for every \( \pi' \)-element \( y \in C_H(x) - Z(H) \), \( C_H(y) = C_H(x) \).

Lemma 2.6. Let \( K \) be a normal subgroup of \( H \) and \( \Pi = H/K \). Let \( \overline{x} \) be the image of the element \( x \) of \( H \) in \( \Pi \). Then,

(i) \( |C_K(\overline{x})| \mid |C_H(x)| \);

(ii) \( |C_\Pi(\overline{x})| \mid |C_H(x)| \);

(iii) for every abelian group \( A \), \( cs'(H \times A) = cs'(H) \).

Lemma 2.7. For a group \( H \), \( \text{lcm}(\alpha : \alpha \in cs'(H)) \mid [H : Z(H)] \).

Proof. Since for every \( x \in H \), \( Z(H) \leq C_H(x) \), we get that \( |C_H(x)| \mid [H : Z(H)] \). Thus, \( \text{lcm}(\alpha : \alpha \in cs'(H)) \mid [H : Z(H)] \), as wanted. \( \square \)

Lemma 2.8. Let \( \pi \) be a set of primes, \( x \) be a non-central \( \pi \)-element of the group \( H \) and \( C/Z(H) = C_{H/Z(H)}(xZ(H)) \). Then, for a \( \pi' \)-element \( y \in H \), \( y \in C \) if and only if \( y \in C_H(x) \).

Proof. Obviously, \( C_H(x) \leq C \). Now let \( y \in C \) be a \( \pi' \)-element. Then, \( yZ(H) \in C/Z(H) \), so there exists \( z \in Z(H) \) such that \( y^{-1}x = xz \). This shows that \( o(x) = o(xz) = \text{lcm}(o(x), o(z)) \), hence \( o(z) \mid o(x) \). On the other hand, \( xy^{-1} = yz \). Thus, \( o(y) = o(yz) = \text{lcm}(o(y), o(z)) \), so \( o(z) \mid o(y) \). This forces \( o(z) \mid \gcd(o(x), o(y)) = 1 \). Therefore, \( z = 1 \). Consequently, \( y^{-1}xy = x \). This shows that \( y \in C_H(x) \), as desired. \( \square \)
Lemma 2.9. For a group $H$, let $t, s \in \pi(H)$ and $S \in \text{Syl}_1(H)$. If for every $t$-element $y \in H - Z(H)$, $|c_H(y)|_t > 1$ and if $x$ is a $t$-element of $H$ such that $|c_H(x)|$ is maximal and minimal in $cs^*(H)$ by divisibility, then either $|H/Z(H)|_t = |c_H(x)|_t$ or $C_H(S) \leq Z(H)$.

Proof. Let $C_H(S) \not\subseteq Z(H)$. Thus, by assumption and Lemma 2.5(i), there exists a $t'$-element $z \in C_H(S) - Z(H)$. Now we claim that $\frac{|H/Z(H)|_t}{|c_H(x)|_t} = |c_H(z)|_t$. If not, then $C_H(x)$ contains a non-central $s$-element $w$. Hence, by Lemma 2.5(ii), $C_H(x) = C_H(w)$. Obviously, $z \in C_H(S) \subseteq C_H(w) = C_H(x)$. Consequently, Lemma 2.5(ii) forces $C_H(x) = C_H(z)$. Therefore, $|c_H(x)|_t = |c_H(z)|_t = 1$, which is a contradiction. So, $|H/Z(H)|_t = |c_H(x)|_t$, as claimed. □

Lemma 2.10. For a group $H$ and $t \in \pi(H)$, let $|\{c_H(x) : x \in H - Z(H), o(x) = \text{a power of } t\}| = |\alpha|$ and $|cs^*(H)| > 1$. If $\alpha$ is maximal and minimal in $cs^*(H)$ by divisibility, then $|H/Z(H)|_t = \text{Max}[|\beta| : \beta \in cs^*(H)]$.

Proof. Working towards a contradiction, let $|H/Z(H)|_t \neq \text{Max}[|\beta| : \beta \in cs^*(H)]$. Thus for every $y \in cs^*(H) - |\alpha|$, $|\gamma|_t < |H/Z(H)|_t$. Let $\gamma = |c_H(y)|_t$ for some $y \in H - Z(H)$. Then, by our assumption and Lemma 2.5(i), we can assume that $y$ is a $t'$-element. Also, $|c_H(y)|_t < |H/Z(H)|_t$. Hence, $C_H(y)$ contains a non-central $t$-element $z$. Since $|c_H(z)|_t = \alpha$, Lemma 2.5(ii) shows that $|c_H(y)|_t = |c_H(z)|_t = \alpha$, which is a contradiction. This completes the proof. □

Lemma 2.11. For a group $H$, $\pi(H/Z(H)) = \bigcup_{\alpha \in cs^*(H)} \pi(\alpha)$.

Proof. By Lemma 2.7, $\bigcup_{\alpha \in cs^*(H)} \pi(\alpha) \subseteq \pi(H/Z(H))$. Now if there exists $t \in \pi(H/Z(H)) - \bigcup_{\alpha \in cs^*(H)} \pi(\alpha)$, then for every $\alpha \in cs^*(H), t \nmid \alpha$. Therefore, Lemma 2.1 forces the $t$-Sylow subgroup $T$ of $H$ to be an abelian direct factor of $H$. Thus, $T \subseteq Z(H)$ and hence, $t \nmid |H/Z(H)|$, which is a contradiction. This shows that $\pi(H/Z(H)) = \bigcup_{\alpha \in cs^*(H)} \pi(\alpha)$. □

Lemma 2.12. For a group $H$, if there exists $\alpha \in cs^*(H)$ and $p, q \in \pi(H/Z(H))$ such that $|\alpha|_p < |H/Z(H)|_p$ and $|\alpha|_q < |H/Z(H)|_q$, then there exists a path between $p$ and $q$ in $\text{GK}(H/Z(H))$.

Proof. Let $x \in H - Z(H)$ with $\alpha = |c_H(x)|$. By Lemma 2.5(i), we can assume that $x$ is of the prime power order. Since $|\alpha|_p < |H/Z(H)|_p$ and $|\alpha|_q < |H/Z(H)|_q$, we get that $p, q \parallel |c_H(x)|/Z(H)$. Thus, $C_H(x)$ contains a non-central $p$-element $x_1$ and a non-central $q$-element $x_2$. If $p \nmid o(x)$, then since $x_2 \in C_H(x)$, we get that $xx_2(Z(H) \subseteq H/Z(H))$ is of order $pq$, so the proof is complete. The same reasoning completes the proof when $q \nmid o(x)$. Now let $o(x)$ be a power of a prime $r$, where $r \nmid |p, q|$. The same reasoning as above shows that $H/Z(H)$ contains elements of order $pr$ and $rq$, so $p - r = q$ is a path in $\text{GK}(H/Z(H))$, as wanted. □

3. Main results

Theorem 3.1. $\text{OC}(G/Z(G)) = \text{OC}(\text{PGL}_2(q))$.

Proof. We are going to prove this theorem in the following steps:

Step 1. $|\text{PGL}_2(q)| \parallel |G : Z(G)|$.

Proof. From Lemma 2.7, $\text{lcm}(\alpha : \alpha \in cs^*(G)) = |G : Z(G)|$. On the other hand, $cs^*(G) = cs^*(\text{PGL}_2(q)) = |q^2 - 1, q(q + 1), q(q - 1)|/2$.

Therefore, $|\text{PGL}_2(q)| \parallel |G : Z(G)|$.

Step 2. For every $p$-element $x \in G - Z(G)$, $|c_G(x)| = q^2 - 1$ and $|c_G(x)| = q^2 - 1$, where $G = G/Z(G)$ and $x$ is the image of $x$ in $G$.

Proof. We first show that for every $p$-element $x \in G - Z(G)$, $|c_G(x)| = q^2 - 1$. Working towards a contradiction, assume that $G$ contains a non-central $p$-element $x$ such that $|c_G(x)| \neq q^2 - 1$. Thus, by (1)

$$|c_G(x)|_p = |\text{PGL}_2(q)|_p.$$
Also, \( q^2 - 1 \in \text{cs}^*(G) \), so there exists a non-central element \( y \in G \) such that \( |c_{\hat{G}}(y)| = q^2 - 1 \). Hence, we can assume that there exists a \( p \)-Sylow subgroup \( P \) of \( G \) such that \( x \in P \) and \( P \leq C_G(y) \). Since \( q^2 - 1 \) is maximal in \( \text{cs}^*(G) \) by divisibility, Lemma 2.5 leads us to assume that \( y \) is of the prime power order. If \( y \) is a \( p' \)-element, then since \( x \in C_G(y) \), we get from maximality and minimality of \( q^2 - 1 \) in \( \text{cs}^*(G) \), and Lemma 2.5(ii) that \( |c_{\hat{G}}(x)| = q^2 - 1 \), which is a contradiction. This forces \( y \) to be a \( p \)-element and for every \( p' \)-element \( z \in G \), \( |c_{\hat{G}}(z)| \neq q^2 - 1 \). Thus,

\[
y \in Z(P) - Z(G). \tag{3}
\]

Also, \( x \in C_G(x) - Z(G) \). Thus, \( p \mid |C_G(x)/Z(G)| \) and hence, \( (2) \) forces \( |G/Z(G)|_p > |PGL_2(q)|_p \). Now let \( z \) be a \( p' \)-element of \( G - Z(G) \). Then, the above statements show that \( p \mid |C_G(z)/Z(G)| \), so \( C_G(z) \) contains a non-central \( p \)-element \( w \). We can assume that \( w \in P \) and \( P \cap C_G(z) \in \text{Syl}_p(C_G(z)) \). Moreover, Lemma 2.5(ii) shows that \( |c_{\hat{G}}(zw)|, |c_{\hat{G}}(w)| \neq q^2 - 1 \), so \( (1) \) forces \( |C_G(w)|_p = |C_G(z)|_p \). Since \( C_G(z) \leq C_G(w), C_G(z) \), we get from (3) that \( y \in P \cap C_G(w) = P \cap C_G(z) \leq C_G(z) \). Thus, Lemma 2.5(ii) shows that \( |c_{\hat{G}}(z)| = |c_{\hat{G}}(y)| = q^2 - 1 \), which is a contradiction. This shows that for every \( p' \)-element \( x \in G - Z(G) \), \( |c_{\hat{G}}(x)| = q^2 - 1 \).

Let \( x \in G - Z(G) \) be a \( p \)-element and \( C(Z(G)) = C_G(x) \). Thus, by the above statements, \( |c_{\hat{G}}(x)| = q^2 - 1 \) and hence if \( u \in C_G(x) - C_C(x) \), then Lemmas 2.4 and 2.8 show that \( \phi(uC_G(x)) \) is a power of \( p \). So, Lemma 2.4 guarantees that \( p \mid |C_C(x)| \). However, \( C(x)\leq G \) and hence, \( |C_G(x)| \mid |G : C_G(x)| = |c_{\hat{G}}(x)| \). This forces \( p \mid |c_{\hat{G}}(x)| \), which is a contradiction. Therefore, \( C = C_G(x) \) and hence, \( |c_{\hat{G}}(x)| = |c_{\hat{G}}(y)| = q^2 - 1 \), as desired.

**Step 3.** \( G/Z(G) = \text{PGL}_2(q) \).

*Proof.* From Step 1, \( |PGL_2(q)| \mid |G : Z(G)| \). Let \( s \in \pi(G/Z(G)) \). Since by Lemma 2.11, \( \pi(G/Z(G)) = \pi(PGL_2(q)) \), we have \( s \in \pi(PGL_2(q)) \). Let \( S_1 \subseteq \text{Syl}_p(G) \) and \( S \subseteq \text{Syl}_p(PGL_2(q)) \). Since \( Z(S) \neq 1 \) and \( Z(PGL_2(q)) = \{1\} \), we get that there exists \( a \in cs^*(PGL_2(q)) = cs^*(G) \) such that \( |a| = q^2 - 1 \). This forces \( C_G(S_1) \neq Z(G) \). Thus, if \( s \neq \bar{u} \), then Step 2 and Lemma 2.9 show that \( |G/Z(G)|_p = |PGL_2(q)|_p \) for some \( \beta \in cs^*(G) \). So \( |G/Z(G)|_p \leq |PGL_2(q)|_p \). Also, Lemma 2.10 guarantees that \( |G/Z(G)|_p \mid |PGL_2(q)|_p \) and hence, \( |G/Z(G)| \mid |PGL_2(q)| \). Therefore, \( G/Z(G) \mid |PGL_2(q)| \).

**Step 4.** \( OC(G/Z(G)) = OC(PGL_2(q)) \).

*Proof.* If there exists \( t \in \pi(G/Z(G)) - \{p\} \) such that \( t \) and \( p \) are adjacent in \( G/K(Z(G)) \), then there exist a non-central \( p \)-element \( x \) and a non-central \( t \)-element \( y \) such that \( x = y \). So, \( y \in C_G(x) - Z(G) \) and hence \( t \mid |C_G(x)/Z(G)| \). On the other hand, Steps 2 and 3 show that \( |c_{\hat{G}}(x)| = q^2 - 1 \) and \( |G/Z(G)|_p = |PGL_2(q)|_p \). Thus, \( t \in \pi(q^2 - 1) \) and \( |G/Z(G)|_p = |c_{\hat{G}}(x)||c_G(x)/Z(G)|_p > |q^2 - 1|_p = |PGL_2(q)|_p \), which is a contradiction. This forces \( |p| \) to be an odd connected component of \( G/K(Z(G)) \). Also, for every \( t, s \in \pi(PGL_2(q)) \) which are adjacent in \( G/K(Z(G)) \), Step 3 and Lemma 2.12 show that there exists a path between \( t \) and \( s \) in \( G/K(Z(G)) \). Now since \( \pi_1(PGL_2(q)) = \pi(q^2 - 1) \) is a connected component in \( G/K(Z(G)) \), \( |G/Z(G)| = |PGL_2(q)| \) and \( |p| \) is an odd connected component of \( G/K(Z(G)) \), we get that \( \pi(q^2 - 1) \) is a component of \( G/Z(G) \). Hence, \( OC(G/Z(G)) = OC(PGL_2(q)) \).

**Corollary 3.2.** \( G/Z(G) \cong PGL_2(q) \).

*Proof.* Since by Theorem 3.1, \( OC(G/Z(G)) = OC(PGL_2(q)) \), Lemma 2.3 shows that \( G/Z(G) \cong PGL_2(q) \).

**Lemma 3.3.** For every subgroup \( Z_1 \) of \( Z(G) \), \( cs^*(G/Z_1) = cs^*(PGL_2(q)) \).

*Proof.* Let \( Z_1 \) be a subgroup of \( Z(G) \). Put \( \hat{G} = G/Z_1 \) and \( \hat{G} = (G/Z_1)/(Z(G)/Z_1) \). For every \( x \in G \), let \( \hat{x} \) and \( \hat{x} \) be the images of \( x \) in \( \hat{G} \) and \( \hat{G} \), respectively. By Corollary 3.2, \( \hat{G} \cong G/Z(G) \cong PGL_2(q) \). By (1), \( x_1, x_2, x_3 \in G \) such that \( |c_{\hat{G}}(x_1)| = q^2 - 1, |c_{\hat{G}}(x_2)| = q(q - 1) \) and \( |c_{\hat{G}}(x_3)| = q(q + 1) \) also. For every \( 1 \leq i \leq 3 \), Lemma 2.6 implies that \( |c_{\hat{G}}(x_1)|, |c_{\hat{G}}(x_2)| \) and \( |c_{\hat{G}}(x_3)| \) are maximal in \( cs^*(\hat{G}) \). Thus, for every \( 1 \leq i \leq 3 \), \( |c_{\hat{G}}(x_1)|, |c_{\hat{G}}(x_2)| \) and \( |c_{\hat{G}}(x_3)| \) are maximal in \( cs^*(\hat{G}) \). Therefore, \( q^2 - 1, q(q \pm 1) \in cs^*(\hat{G}) \). On the other hand, for \( e \in \{0, 1, 2\} \), there exists \( y_0, y_1, y_2 \in G \) such that \( |c_{\hat{G}}(y_0)| = q^{e+1}/2 \). Since \( |c_{\hat{G}}(y_0)|, |c_{\hat{G}}(y_1)| \) and \( |c_{\hat{G}}(y_2)| \) are minimal in \( cs^*(\hat{G}) \), we get that \( |c_{\hat{G}}(y_0)| = |c_{\hat{G}}(y_1)| \). Therefore, \( q(q \pm 1) \in cs^*(G) \) and hence, \( cs^*(G) \subseteq cs^*(G) \). Now if \( y \in G \) such that \( |c_{\hat{G}}(y)| \in cs^*(G) \), then since \( |c_{\hat{G}}(y)| \neq |c_{\hat{G}}(y_0)| \), \( |c_{\hat{G}}(y)| \) and \( |c_{\hat{G}}(y_1)| \) are minimal in \( cs^*(\hat{G}) = cs^*(PGL_2(q)) = cs^*(G) \), we get, by considering the maximal elements of \( cs^*(G) \), that \( |c_{\hat{G}}(y)| \in |q(q \pm 1)|/2 \).
Thus, there exists $1 \geq \gamma \geq 1$, and hence $\text{lcm}(\gamma) \in \{q(q \pm 1), q(q \pm 1)/2\} \subseteq \text{cs}(G)$, a contradiction. This implies that $\text{cs}(G) = \text{cs}(G)$. □

**Lemma 3.4.**  If $M$ is a normal subgroup of $G$ with $M/Z(M) \cong \text{PGL}_2(q)$, then $\text{cs}(M) = \text{cs}(\text{PGL}_2(q))$.

**Proof.** Put $\tilde{M} = M/Z(M)$ and for $x \in M$, let $\tilde{x}$ be the image of $x$ in $\tilde{M}$. Then, since $|\text{cl}_M(x)| = |\text{cl}_M(x)|$ and $|\text{cl}_M(x)| = |\text{cl}_M(x)|$, arguing by analogy as the proof of Lemma 3.3 completes the proof. □

**Lemma 3.5.**  For a group $H$, if $x \in H$ and $Z(H) \leq \langle x \rangle$, then $C_H(x) \leq N_H(\langle x \rangle)/Z(H)$, where $H = H/Z(H)$ and $x$ is the image of $x$ in $H$.

**Proof.** Let $y = yZ(H) \in C_H(x)$. Then, there exists $z \in Z(H)$ such that $y^{-1}xy = xz \in \langle x \rangle$. Thus, $y \in N_H(\langle x \rangle)$. Therefore, $yZ(H) \in N_H(\langle x \rangle)/Z(H)$, as wanted. □

**Lemma 3.6.**  Let $Z = Z(\text{GL}_2(q))$ and let $x$ be the image of $x \in \text{GL}_2(q)$ in $\text{PGL}_2(q)$. If $q \equiv \varepsilon \pmod{4}$ and $|\text{cl}_{\text{PGL}_2(q)}(x)| = q(q + \varepsilon)$, then either $|\text{cl}_{\text{PGL}_2(q)}(x)| = q(q + \varepsilon)/2$ or $x \in \text{SL}_2(q)/Z$ and $|\text{cl}_{\text{PGL}_2(q)}(x)| = q(q + \varepsilon)/2$.

**Proof.** Let $|\text{cl}_{\text{PGL}_2(q)}(x)| = q(q + \varepsilon)$ and $|\text{cl}_{\text{PGL}_2(q)}(x)| \neq q(q + \varepsilon)$. Then, $|\text{cl}_{\text{PGL}_2(q)}(x)| = q(q + \varepsilon)/2$ and hence, $|\text{cl}_{\text{PGL}_2(q)}(x)| = 2(q - \varepsilon)$. Thus, $x$ is a semi-simple element in $\text{PGL}_2(q)$ and hence $o(x) \mid |q - \varepsilon|$. So, one of the following cases holds:

I. $\varepsilon = +$. Then, we can assume that for some $\mu \in G F(q) - \{0\}$, $x = \text{diag}(\mu, 1)$. Since $|\text{cl}_{\text{PGL}_2(q)}(x)| = 2(q - \varepsilon)$, we can check at once that $wZ \in C_{\text{PGL}_2(q)}(x)$, where

\[
w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Thus, there exists $1 \neq z \in Z$ such that

\[
x^{-1}wz = wz
\]

and hence $\text{lcm}(o(z), o(w)) = o(wz) = o(w) = 2$. This forces $o(z) = 2$. Therefore, $z = \text{diag}(-1, -1)$. So, (4) guarantees that $\mu = \mu^{-1} = -1$. On the other hand, for a generator $d$ of $GF(q) - \{0\}$, $d^{(q-1)/2} = -1$. However, $(q - 1)/2$ is even. Hence, there exists $d' \in GF(q) - \{0\}$ such that $d'^2 = -1$. Therefore, $x = \text{diag}(d'^2, 1) = \text{diag}(d', d^-1)d(1, d') \in \text{SL}_2(q)Z$. This shows that $x \in \text{SL}_2(q)Z$.

II. $\varepsilon = -$. Let $\alpha \in GF(q^2) - \{0\}$ such that $o(\alpha) = o(x)$. Let $\sigma$ be a Frobenius automorphism of $GL_2(GF(q))$ such that $(GL_2(GF(q)))_{\sigma} = GL_2(q)$, where $GF(q)$ is an algebraic closure of $GF(q)$. Then, there exists $g \in GL_2(GF(q))$ such that $g^{-1}g^t = w$, where

\[
w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Set $t = \text{diag}(\alpha, \alpha^t)$. We can check at once that $w^t, t^t \in GL_2(q)$ and $N_{\text{GL}_2(q)}(t^t) = C_{\text{GL}_2(q)}(t)$ such that $Z \leq C_{\text{GL}_2(q)}$ and $t^t \in C_{\text{GL}_2(q)}$. Without loss of generality, let $t^t = x$. Thus by Lemma 3.5, $w^tZ \in C_{\text{PGL}_2(q)}(t)$. However, $o(w^t) = 2$ and $[x, w^t] = z \in Z$. So, $o(zw^t) = o(w^t) = 2$ and hence $o(z) = 2$. Therefore, $z = \text{diag}(-1, -1)$. Since $w^{-t}w^t = x^t w t = tz$, consequently, $\alpha^t = -\alpha$. This forces $\alpha^{2(q-1)} = 1$. Thus, $o(x) = o(t^t) = 2$. Since $[PGL_2(q) : SL_2(q)Z] = 2$, we get from 4 that $x \in SL_2(q)Z$, as wanted. □

**Lemma 3.7.**  If $G = (PSL_2(q) \times Z(G))(t)$, where $t \in G - (PSL_2(q) \times Z(G))$ and $t^2 \in PSL_2(q) \times Z(G)$, then $\text{cs}(G/Z(G)) = \text{cs}(G/Z(G)) = \text{cs}(G)$.

**Proof.** Since $PSL_2(q) \leq PSL_2(q) \times Z(G)$, for every $\sigma \in \text{Aut}(PSL_2(q) \times Z(G))$, $\sigma(PSL_2(q)) \cap PSL_2(q) \leq PSL_2(q)$. However, $PSL_2(q)$ is simple. Thus, $\sigma(PSL_2(q)) \subseteq PSL_2(q) = \{1\}$ or $PSL_2(q)$. In the first case, $PSL_2(q) \times \sigma(PSL_2(q)) \leq PSL_2(q) \times Z(G)$, which is impossible. Consequently, $\sigma(PSL_2(q)) = PSL_2(q)$. This shows that $PSL_2(q)$ is a characteristic subgroup of $PSL_2(q) \times Z(G)$. On the other hand, $|G : PSL_2(q) \times Z(G)| = 2$. Therefore, $PSL_2(q) \times Z(G) \leq G$ and hence $PSL_2(q) \leq G$. Thus, for every $x \in G$ and $y \in PSL_2(q)$, $x^{-1}yx \in PSL_2(q)$. This forces $C_G(Z(G))(yZ(G)) = C_G(y)/Z(G)$. Consequently, $|\text{cl}(G/Z(G))(yZ(G))| = |\text{cl}(G)|$. 

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Now let \( y \in G - (PSL_2(q) \times Z(G)) \). So, \( y = gt \) for some \( g \in PSL_2(q) \times Z(G) \). Without loss of generality, let \( g \in PSL_2(q) \). Then, since \( PSL_2(q) \subseteq G \), we can see at once that there do not exist \( g' \in PSL_2(q) \times Z(G) \) and \( z' \in Z(G) - \{1\} \) such that \( yg'z'^{-1} = g'z' \). Also, if there exists \( g' \in PSL_2(q) \) and \( z', z'' \in Z(G) \) such that \((g'z')^{-1}y(g'z') = yz'' \), then \( t^{-1}g'z'^{-1}yt = yz'' \), so \( t^{-1}g'z'^{-1}gt = gz'' \). However, \( g'^{-1}g \in PSL_2(q) \), \( g \not\subseteq G \). Therefore, \( t^{-1}g^{-1}gt = g' \in PSL_2(q) \) and hence, \( g'g = gz'' \). This forces \( z'' \in Z(G) \cap PSL_2(q) = \{1\} \), so \( z'' = 1 \). This shows that \( C_{G/Z(G)}(yZ(G)) = C_G(y)/Z(G) \) and consequently, \( |c_{G/Z(G)}(yZ(G))| = |c_G(y)| \). This guarantees that \( cs'(G/Z(G)) = cs'(G) \), as wanted. □

**Proof of the main theorem.** Let \( G \) be the smallest counterexample. Then, it is obvious that \( Z(G) \neq 1 \). We claim that \( Z(G) \) is prime. If not, \( Z(G) \) contains a non-trivial subgroup \( Z_1 \) of the prime order. Thus, by Lemma 3.3, \( cs'(G/Z_1) = cs'(PSL_2(q)) \). On the other hand, \( (G/Z_1)/(Z(G)/Z_1) \cong G/Z(G) \cong PSL_2(q) \), by Corollary 3.2. Consequently, \( Z(G/Z_1) = Z(G)/Z_1 \). Also, \( |G/Z_1| < |G| \). Hence, our assumption shows that one of the following cases occurs:

**Case 1.** \( G/Z_1 \cong PSL_2(q) \times Z(G)/Z_1 \). Then, \( G \) contains a non-trivial normal subgroup \( M/Z_1 \cong PSL_2(q) \). Thus, \( Z(M) = Z_1 \) and Lemma 3.4 shows that \( cs'(M) = cs'(PSL_2(q)) \). Hence, our assumption shows that \( M \) is as follows:

(i) \( M \cong PSL_2(q) \times Z_1 \). Thus, \( M \) contains a normal subgroup \( N \) such that \( N \not\subseteq PSL_2(q) \) and \( M = N \times Z_1 \). So, \( G = MZ_1 = NZ_1 \). However, \( N \not\subseteq Z(G) \) and \( (M \cap Z(G)) = N \cap Z_1 = \{1\} \). Therefore, \( G = N \times Z(G) \cong PSL_2(q) \times Z(G) \), a contradiction.

(ii) \( M \cong (PSL_2(q) \times Z_1)Z_1 \). Then, \( M \) contains a characteristic subgroup \( N \) such that \( N \not\subseteq PSL_2(q) \) and \( M = N \times Z_1 \). Since \( N \not\subseteq G \), \( G \not\subseteq Z(G) \). Thus, \( NZ_1 \not\subseteq G \) and \( N \not\subseteq Z(G) \). Hence, \( N \cap Z(G) = N \cap Z_1 = \{1\} \). Consequently, \( N \not\subseteq Z(G) \). Since \( |G : N \times Z(G)| = 2 \), we get that \( G \) contains a 2-element \( t \) such that \( t^2 \in N \times Z(G) \) and \( G = (N \times Z(G)) \cdot t \cong (PSL_2(q) \times Z_1)Z_1 \), a contradiction.

**Case 2.** \( G/Z_1 \cong (PSL_2(q) \times Z(G)/Z_1) \). Then, \( G \) contains a normal subgroup \( M \) and a subgroup \( N \) such that \( Z_1 \not\subseteq N \), \( N/Z_1 \not\subseteq PSL_2(q) \) and \( M/Z_1 = N/Z_1 \times Z(G)/Z_1 \). Since \( N/Z_1 \not\subseteq PSL_2(q) \), we have \( Z(N) = Z_1 \). Also, \( Z_1 \) is prime. Thus, \( N' \cap Z_1 = Z_1 \) or \( \{1\} \). If \( N' \cap Z_1 = \{1\} \), then \( N' \times Z_1 \not\subseteq N \). However, \( N' \not\subseteq N/Z_1 \not\subseteq PSL_2(q) \). Hence, \( N \not\subseteq PSL_2(q) \). Therefore, \( G = M \cdot (t) \cong (PSL_2(q) \times Z(G))Z_1 \), a contradiction. This forces \( N' \cap Z_1 = Z_1 \). Thus, \( N \not\subseteq N' \). If \( |Z_1| = 1 \), then we have \( N \not\subseteq PSL_2(q) \). Hence, the above argument leads us to get a contradiction. Now let \( |Z_1| = 2 \) and \( N = \text{Schur cover of } PSL_2(q) \). Therefore, \( G \not\subseteq L_2(q) \), \( Z_1 = Z(N) \) and \( M \not\subseteq L_2(q)Z(G) \). On the other hand, \( |G : M| = |G/Z_1 : M/Z_1| = 2 \). This shows that \( G \) contains a 2-element \( t \) such that \( t^2 \in M \) and \( G \cong (SL_2(q)Z_1) \cdot t \). It is known that

\[
 cs'(SL_2(q)) = \{q(q \pm 1), q^2 - 1\}.
\]

Let \( q \equiv 1 \pmod{4} \). Then, since \( q(q + 1)/2 \in cs'(G) \), we get that \( G \) contains an element \( x \) with \( |c_{G}(x)| = q(q + 1)/2 \). Now we have two following possibilities:

- \( x \in N \). Then, since \( N \not\subseteq SL_2(q) \) and \( |c_{N}(x)| = |c_{G}(x)| \), we get from (5) that \( |c_{N}(x)| = 1 \), so \( x \in Z(N) = Z_1 \subseteq Z(G) \), a contradiction.

- \( x \in G - NZ(G) \). Then, \( xZ(G) \in G/Z(G) \cong PSL_2(q) \). Thus, Lemma 3.6 shows that \( |c_{G/Z(G)}(xZ(G))| = q(q + 1) \). So, by Lemma 2.6, \( |c_{G/Z(G)}(xZ(G))| \), which is impossible.

The above contradictions show that \( Z(G) \) is prime. Thus, we apply the same reasoning as one used in Case 2 as follows: Since \( G/Z(G) \cong PSL_2(q) \) and \( PSL_2(q) \) contains a normal subgroup of index 2 which is isomorphic to \( PSL_2(q) \), we can assume that \( G \) contains a normal subgroup \( N \) containing \( Z(G) \) such that \( N/Z(G) \not\subseteq PSL_2(q) \). Since \( Z(G) \) is prime, we have \( N' \cap Z(G) = \{1\} \) or \( N' \cap Z(G) = Z(G) \). If \( N' \cap Z(G) = \{1\} \), then \( N' \times Z(G) \not\subseteq N \). However, \( N' \not\subseteq N/Z(G) \) and \( PSL_2(q) \) is simple, so \( N' \not\subseteq PSL_2(q) \). Consequently, \( N \not\subseteq PSL_2(q) \times Z(G) \). Moreover, \( |G : N| = 2 \) and hence, \( G \) contains a 2-element \( t \) such that \( t^2 \in M \) and \( G = N \cdot t \cong (PSL_2(q) \times Z(G))Z_1 \), a contradiction. This forces \( N' \cap Z(G) = Z(G) \). Thus, \( Z(G) \not\subseteq N' \).
\[ |Z(G)| = 2 \text{ and } N \text{ is a Schur cover of } PSL_2(q). \text{ Therefore, } N \cong SL_2(q) \text{ and } Z(G) = Z(N). \] It follows that 
\[ [G : N] = [G/Z(G) : N/Z(G)] = 2. \] This shows that \( G \) contains a 2-element \( t \in G \) such that \( t^2 \in N \) and 
\[ G = SL_2(q).t. \] It is known that 
\[ cs'(SL_2(q)) = \{ q(q \pm 1), q^2 - 1 \}. \] (6)

Let \( q \equiv \varepsilon (\text{mod} \ 4) \). Then, since \( q(q + \varepsilon)/2 \in cs'(G) \), we get that \( G \) contains an element \( x \) with \( |cl_G(x)| = q(q + \varepsilon)/2 \). Now we have two following possibilities:

- \( x \in N. \) Then, since \( N \cong SL_2(q) \) and \( |cl_N(x)| | |cl_G(x)| \), we get from (6) that \( |cl_N(x)| = 1. \) So \( x \in Z(N) = Z(G) \), a contradiction.

- \( x \in G - NZ(G). \) Then, \( xZ(G) \in G/Z(G) \cong PGL_2(q) \). Thus, Lemma 3.6 shows that \( |cl_{G/Z(G)}(xZ(G))| = q(q + 1). \) So, by Lemma 2.6, \( q(q + 1) | |cl_G(x)| \), which is impossible.

The above contradictions complete the proof as well.

**Remark 3.8.** Let \( A \) be an abelian group containing a proper subgroup, say \( A' \), and \( a \in A - A' \) such that \( 1 \neq a^2 \in A' \) and \( A = A'.(a) \). Also, let \( \sigma \) be a diagonal automorphism of \( PSL_2(q) \). Set \( t = (a, a) \) and \( H = (PSL_2(q) \times A').t. \) Then, since \( 1 \neq t^2 = (a^2, a^2) \in PSL_2(q) \times A' \) and \( A' = Z(H) \), Lemma 3.7 shows that \( cs'(H) = cs'(H/Z(H)) = cs'(PGL_2(q)). \) Note that \( H \neq B \times PGL_2(q) \), for every abelian group \( B \). Also, if \( H \neq PGL_2(q) \times Z(H) \), then it is obvious that \( cs'(H) = cs'(PGL_2(q)). \) Thus, if \( q > 5 \) is odd, then \( PGL_2(q) \) cannot be determined uniquely by its conjugacy class sizes under an abelian direct factor.

**Remark 3.9.** If \( G \cong (PSL_2(q) \times Z(G)).C_2 \), then we can check easily that \( G \cong ((PSL_2(q) \times Z(G)).C_2) \times Z(G)' \), where \( Z(G) \in Syl_2(Z(G)) \) and \( Z(G)' = (\pi(Z(G)) - \{ 2 \}) \)-Hall subgroup of \( Z(G) \).

**References**


