Meromorphic Solutions of Difference Equations Originated From Schwarzian Differential Equation

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Abstract. Let $f(z)$ be a meromorphic functions with finite order, $R(z)$ be a nonconstant rational function and $k$ be a positive integer. In this paper, we consider the difference equation originated from Schwarzian differential equation, which is of form

$$\left[\Delta^3 f(z)\Delta f(z) - \frac{3}{2} (\Delta^2 f(z))^2\right]^k = R(z) (\Delta f(z))^2.$$  

We investigate the uniqueness of meromorphic solution $f(z)$ of difference Schwarzian equation if $f(z)$ shares three values with any meromorphic function. The exact forms of meromorphic solutions $f(z)$ of difference Schwarzian equation are also presented.

1. Introduction and main results

In this paper, we use the basic notions of Nevanlinna’s theory, see [12, 28]. In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$. Let $S(r, f)$ denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all $r$ outside of a set with finite logarithmic measure.

Let $f(z)$ and $g(z)$ be two meromorphic functions, $a$ be a small function relative to both $f$ and $g$. We say that $f$ and $g$ share a CM if $f-a$ and $g-a$ have the same zeros with the same multiplicities, $f$ and $g$ are said to share a IM if $f-a$ and $g-a$ have the same zeros ignoring multiplicities. Nevanlinna’s four values theorem (see [26]) says that if two nonconstant meromorphic functions $f$ and $g$ share four values CM, then $f \equiv g$ or $f$ is a Möbius transformation of $g$. The condition ‘$f$ and $g$ share four values CM’ has been weakened to ‘$f$ and $g$ share two values CM and two values IM’ by Gundersen [9, 10], as well as by Mues [25].

For Schwarzian differential equation

$$\left[\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2\right]^k = R(z, f) = \frac{P(z, f)}{Q(z, f)},$$

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Ishizaki [18] showed that if the Schwarzian equation (1) possesses an admissible solution, then \( d + 2k \sum_{j=1}^{n} \delta(a_j f) \leq 4k \), where \( a_j \) are distinct complex constants, and \( d = \deg(R(z, f)) = \max\{\deg(P(z, f), \deg(Q(z, f))\} \).

In particular, when \( R(z, f) \) is independent of \( z \), it is shown that if (1) possesses an admissible solution \( f \), then by some Möbius transformation \( w = (af + b)/(cf + d)(ad - bc \neq 0) \), \( R(z, f) \) can be reduced to some special forms, see [18, Theorem 3]. Liao and Ye [23] considered differential equations, see, e.g., [3, 4, 11, 19–21]. Some papers studied uniqueness of meromorphic functions concerning the unicity of meromorphic functions sharing values with their shifts or differences, see, e.g., [8, 15, 27]. Others considered the value distribution and the growth of order of meromorphic solutions of difference equations, see, e.g., [3, 4, 11, 19–21].

Chen and Li [4], Lan and Chen [20] considered the difference counterpart of (3), which is a special type of the Schwarzian differential equation,

\[
\left[ f''' - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \right] = R(z),
\]

and gave the order of meromorphic solutions as follows.

**Theorem 1.1.** [23, Theorem 3] Let \( P(z) \) and \( Q(z) \) be polynomials with \( \deg P = m \) and \( \deg Q = n \), and let \( R(z) = P(z)/Q(z) \). If \( f \) is a transcendental meromorphic solution of (2), then \( m - n + 2k > 0 \) and the order \( \sigma(f) = (m-n+2k)/2k \).

For every positive integer \( n \), the forward differences \( \Delta^nf(z) \) are defined as

\[
\Delta f(z) = f(z + c) - f(z), \quad \Delta^{n+1} f(z) = \Delta^n f(z + c) - \Delta^n f(z).
\]

We know that \( \Delta f(z) \) is considered as difference counterpart of \( f' \). Recently, a number of papers focus on unicity of meromorphic functions sharing values with their shifts or difference operators, see, e.g., [1, 2, 5–8, 13–17, 22, 24, 27, 30]. Some papers studied uniqueness of meromorphic functions concerning meromorphic solutions of difference equations, see, e.g., [8, 15, 27]. Others considered the value distribution and the growth of order of meromorphic solutions of difference equations, see, e.g., [3, 4, 11, 19–21].

Chen and Li [4], Lan and Chen [20] considered the difference counterpart of form

\[
\left[ \frac{\Delta^2 f(z) - 3 \left( \frac{\Delta f(z)}{f(z)} \right)^2}{\Delta f(z) - 2 \left( \frac{\Delta f(z)}{f(z)} \right)} \right] = R(z, f),
\]

which is originated from the Schwarzian differential equation (1), they obtained that the value distribution of meromorphic solutions of (3). Furthermore, Lan and Chen [21] considered the difference equation

\[
\left[ \frac{\Delta^2 f(z)}{\Delta f(z)} - \frac{3}{2} \left( \frac{\Delta f(z)}{f(z)} \right)^2 \right] = R(z),
\]

which is a special type of equation (3), where \( k \) is a positive integer and \( R(z) \) is a nonconstant rational function. They obtain

**Theorem 1.2.** [21, Theorem 1.3] Let \( R(z) = \frac{P(z)}{Q(z)} \) be an irreducible rational function with \( \deg P(z) = p \) and \( \deg Q(z) = q \). Then

(i) every transcendental meromorphic solution of (4) satisfies \( \sigma(f) \geq 1 \); if \( p - q + 2k > 0 \), then (4) has no rational solutions;

(ii) if \( f(z) \) is a meromorphic solution of (4) with finite order, then \( \frac{\Delta^2 f(z)}{\Delta f(z)} \) and \( \frac{\Delta f(z)}{f(z)} \) in (4) are nonconstant rational functions;

(iii) every transcendental meromorphic solution \( f(z) \) with finite order has at most one Borel exceptional value unless

\[
f(z) = b + R_0(z)e^{\alpha z},
\]

where \( a, b \) are complex numbers with \( a \neq 0 \) and \( R_0(z) \) is a nonzero rational function.
If \( p - q + 2k > 0 \), then \( \sigma(f) < \infty \), then \( \Delta f(z) \) has at most one Borel exceptional value unless

\[
\Delta f(z) = R_1(z)e^a,
\]

where \( a \) is a complex number with \( a \neq i2k_1\pi \) for any \( k_1 \in \mathbb{Z} \), and \( R_1(z) \) is a nonzero rational function.

**Remark 1.3.** From Theorem 1.2, we see if \( f(z) \) is a transcendental meromorphic solution of (4) with finite order, then \( f(z) \) cannot have two Borel exceptional values.

We note that \( \Delta f(z) \) lies in the denominator in (4), and so \( \Delta f(z) \neq 0 \). Thus, \( f(z) \) cannot be a meromorphic function with period \( c \). If we remove this restriction, we investigate the properties of meromorphic solutions of equation

\[
\left[ \Delta^3 f(z) \Delta f(z) - \frac{3}{2} (\Delta^2 f(z))^2 \right]^k = R(z)(\Delta f(z))^{2k},
\]

and obtain

**Theorem 1.4.** Let \( f(z) \) be a transcendental meromorphic solution of equation (5) with finite order, where \( R(z) \) is a nonconstant rational function. Let \( g(z) \) be a meromorphic function and \( a, b \) be two distinct constants. If \( f(z) \) and \( g(z) \) share \( a, b, \infty \) CM, then one of the following statements holds:

(i) \( f(z) \equiv g(z) \);

(ii) \( f(z) = Ae^{mx} + B, g(z) = L(f) \), where \( A \neq 0, B \) are constants, \( mc = 2k_1\pi i \) for some nonzero integer \( k_1 \), \( L(f) \) is a M"obius transformation of \( f; \)

(iii) \( f(z) = a + (b-a)\frac{e^{my}}{e^{my}-1}, g = b + \frac{(b-a) e^{my}-\eta}{A e^{my}-1}, \) where \( A, B \) are nonzero constants, \( \frac{n}{m}(\neq 1) \) means a rational constant, \( mc = 2k_1\pi i \) for some nonzero integer \( k_1 \).

2. Lemmas

We now give some preparations.

**Lemma 2.1.** [3, 11] Let \( f(z) \) be a meromorphic function with order \( \sigma = \sigma(f), \sigma < \infty \), and let \( \eta \) be a fixed nonzero complex number, then for each \( \varepsilon > 0 \),

\[
T(r, f(z + \eta)) = T(r, f(z)) + O \left( r^{\sigma-1+\varepsilon} \right) + O(\log r).
\]

**Lemma 2.2.** [3] Let \( A_0(z), \ldots, A_n(z) \) be entire functions such that there exists an integer \( l, 0 \leq l \leq n, \) such that

\[
\sigma(A_l) = \max_{1 \leq j \leq n} \sigma(A_j).
\]

If \( f(z) \) is a meromorphic solution to

\[
A_n(z)y(z + n) + \cdots + A_1(z)y(z + 1) + A_0(z)y(z) = 0,
\]

then we have \( \sigma(f) \geq \sigma(A_l) + 1 \).

**Lemma 2.3.** [29] Suppose that \( n \geq 2 \), and let \( f_j(z)(j = 1, \ldots, n) \) be meromorphic functions and \( g_j(z)(j = 1, \ldots, n) \) be entire functions such that

(i) \( \sum_{j=1}^n f_j(z)e^{\sigma(j)} \equiv 0; \)

(ii) when \( 1 \leq j < k \leq n, g_j(z) - g_k(z) \) is not a constant;
(iii) when $1 \leq j \leq n, 1 \leq h < k \leq n$,

$$T(r, f_j) = o(T(r, e^{\omega r})) \quad (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite logarithmic measure.

Then $f_j(z) \equiv 0$. ($j = 1, \ldots, n$)

**Lemma 2.4.** Let $f(z)$ be a finite order meromorphic solution of equation (4), then $\Delta f(z)$ is a meromorphic solution of equation

$$w(z) = Q(z)w(z),$$

where $Q(z)$ is a nonconstant rational function.

**Proof.** Set

$$Q(z) = \frac{\Delta f(z + c)}{\Delta f(z)}.$$  \hspace{1cm} (6)

We then prove that $Q(z)$ is a nonconstant rational function.

Since $f(z)$ is of finite order, (6) shows $Q(z)$ is also of finite order and

$$\Delta f(z + c) = Q(z)\Delta f(z), \quad \Delta f(z + 2c) = Q(z + c)\Delta f(z + c) = Q(z + c)Q(z)\Delta f(z).$$

Hence,

$$\begin{cases}
\Delta^2 f(z) = \Delta f(z + c) - \Delta f(z) = (Q(z) - 1)\Delta f(z), \\
\Delta^3 f(z) = \Delta^2(\Delta f(z)) = \Delta f(z + 2c) - 2\Delta f(z + c) + \Delta f(z) = (Q(z + c)Q(z) - 2Q(z) + 1)\Delta f(z).
\end{cases}$$  \hspace{1cm} (7)

We see from (4) that

$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2} \left( \frac{\Delta^2 f(z)}{\Delta f(z)} \right)^2 = R_1(z),$$  \hspace{1cm} (8)

where $R_1(z)$ is some nonconstant rational function. Thus, (7) and (8) show that

$$Q(z + c)Q(z) - 2Q(z) + 1 - \frac{3}{2}(Q(z) - 1)^2 = R_1(z),$$  \hspace{1cm} (9)

that is,

$$Q(z + c) = \frac{\frac{3}{2}Q^2(z) - Q(z) + R_1(z) + \frac{1}{2}}{Q(z)}.$$  \hspace{1cm} (10)

Since $R_1(z)$ is a nonconstant rational function, we deduce from (9) that $Q(z)$ cannot be a constant. If $Q(z)$ is transcendental, noting that $\frac{3}{2}Q^2(z) - Q(z) + R_1(z) + \frac{1}{2}$ and $Q(z)$ are irreducible, then we apply Valiron-Mohon’ko Theorem to (10), and deduce

$$T(r, Q(z + c)) = 2T(r, Q(z)) + S(r, Q),$$

which contradicts to Lemma 2.1. Hence, $Q(z)$ is a nonconstant rational function. \hfill $\Box$

**Lemma 2.5.** Let $a, b$ be two distinct constants, $\beta, \gamma$ be nonconstant polynomials with $\deg \beta \neq \deg \gamma$, and

$$f(z) = a + (b - a)\frac{e^\beta - 1}{e^\gamma - 1}.$$  \hspace{1cm} (11)

Then $f(z)$ cannot be a meromorphic solution of equation (4).
Proof. Assume that $f$ is a meromorphic solution of equation (4). Lemma 2.4 shows
\[ \Delta f(z + c) = Q(z)\Delta f(z). \] (12)
Without loss of generality, we assume $Q(z)$ is a nonconstant polynomial. Otherwise, we just multiply the dominator of $Q(z)$ of both sides of (12). We now divide our proof into two cases.

**Case 2.1.** $\deg \beta > \deg \gamma$. Rewriting (11) as
\[ f(z) = a_{01}(z)e^{\beta(z)} + a_{00}(z), \] (13)
where
\[ a_{01}(z) = \frac{b - a}{e^z - 1}, \quad a_{00}(z) = \frac{a - b}{e^z - 1}. \]
Obviously,
\[ \sigma(a_{01}) = \sigma(a_{00}) = \deg \gamma < \deg \beta. \] (14)
Since $e^\beta$ is of regular growth order $\deg \beta$, we see $a_{01}, a_{00}$ are small functions of $e^\beta$. We conclude from (13) that
\[ \Delta f(z) = a_{01}(z + c)e^{\beta(z+c)} + a_{00}(z + c) - a_{01}(z)e^{\beta(z)} - a_{00}(z) \\
= (a_{01}(z + c)e^{\beta(z+c) - \beta(z)} - a_{01}(z))e^{\beta(z)} + a_{00}(z + c) - a_{00}(z) \\
= a_{11}(z)e^{\beta(z)} + a_{10}(z), \] (15)
where
\[ \begin{cases} a_{11}(z) = a_{01}(z + c)e^{\beta(z+c) - \beta(z)} - a_{01}(z), \\ a_{10}(z) = a_{01}(z + c) - a_{00}(z). \end{cases} \] (16)
We deduce from (14), (16), Lemma 2.1 and $\deg(\beta(z + c) - \beta(z)) = \deg \beta - 1$ that
\[ \sigma(a_{11}) \leq \max(\sigma(a_{01}), \deg \beta - 1) < \deg \beta, \quad \sigma(a_{10}) \leq \sigma(a_{00}) < \deg \beta. \] (17)
We assert that $a_{11}(z) \not\equiv 0$. Otherwise, (16) shows
\[ a_{01}(z + c)e^{\beta(z+c) - \beta(z)} - a_{01}(z) = 0. \] (18)
Applying Lemma 2.2 to equation (18), we have
\[ \sigma(a_{01}) \geq \sigma(e^{\beta(z+c) - \beta(z)}) + 1 = (\deg \beta - 1) + 1 = \deg \beta, \]
which contradicts with (14).
Substituting (15) into (12), we obtain
\[ (a_{11}(z + c)e^{\beta(z+c) - \beta(z)} - Q(z)a_{11}(z))e^{\beta(z)} + a_{10}(z + c) - Q(z)a_{10}(z) = 0. \]
By (17) and $\deg(\beta(z + c) - \beta(z)) = \deg \beta - 1$, applying Lemma 2.3 to the last equality, we have
\[ a_{11}(z + c)e^{\beta(z+c) - \beta(z)} - Q(z)a_{11}(z) = 0. \] (19)
Applying Lemma 2.2 to equation (19), we get
\[ \sigma(a_{11}) \geq \sigma(e^{\beta(z+c) - \beta(z)}) + 1 = (\deg \beta - 1) + 1 = \deg \beta, \]
which contradicts with (17).
Case 2.2. \( \text{deg} \beta < \text{deg} \gamma \). Rewriting (11) as

\[
f(z) = a + \frac{b_{00}(z)}{e^{\sigma(z)} - 1},
\]

where

\[
b_{00}(z) = (b - a)(e^{\theta(z)} - 1).
\]

Thus, we conclude from (20) that

\[
A f(z) = \frac{b_{00}(z + c)}{e^{\sigma(z+c)} - 1} \frac{b_{00}(z) - b_{10}(z)}{e^{\sigma(z)} - 1} = \frac{b_{11}(z)e^{\sigma(z)} + b_{10}(z)}{(e^{\sigma(z+c)} - 1)(e^{\sigma(z)} - 1)},
\]

where

\[
\begin{aligned}
 b_{11}(z) &= b_{00}(z + c) - b_{00}(z) \\
 b_{10}(z) &= b_{00}(z + c) + b_{00}(z)
\end{aligned}
\]

By (21), (23) and Lemma 2.1, we have

\[
\begin{aligned}
\sigma(b_{11}) &\leq \sigma(b_{00}) = \text{deg} \beta < \text{deg} \gamma \\
\sigma(b_{11}) &\leq \max\{\sigma(b_{00}), \sigma(e^{(\text{deg} \gamma - 1)}z)\} = \max\{\text{deg} \beta, \text{deg} \gamma - 1\} < \text{deg} \gamma.
\end{aligned}
\]

We again assert that \( b_{11}(z) \neq 0 \). Otherwise, (23) shows

\[
b_{00}(z + c) - e^{(\text{deg} \gamma - 1)}z b_{00}(z) = 0.
\]

Applying Lemma 2.2 to equation (25), we have

\[
\sigma(b_{00}) \geq \sigma(e^{(\text{deg} \gamma - 1)}z) + 1 = (\text{deg} \gamma - 1) + 1 = \text{deg} \gamma,
\]
a contradiction. Substituting (22) into (12), we have

\[
\frac{b_{11}(z + c)e^{\sigma(z+c)} + b_{10}(z + c)}{(e^{\sigma(z+c+2)} - 1)(e^{\sigma(z+c)} - 1)} = Q(z) \frac{b_{11}(z)e^{\sigma(z)} + b_{10}(z)}{(e^{\sigma(z+c)} - 1)(e^{\sigma(z)} - 1)}
\]

or

\[
\frac{b_{11}(z + c)e^{\sigma(z+c)} + b_{10}(z + c)}{(e^{\sigma(z+c+2)} - 1)} = Q(z) \frac{b_{11}(z)e^{\sigma(z)} + b_{10}(z)}{(e^{\sigma(z)} - 1)},
\]

or

\[
\frac{b_{11}(z + c)e^{\sigma(z+c)+\gamma z}}{Q(z)b_{11}(z)e^{\sigma(z+2z)+\gamma z}} - b_{11}(z + c)e^{\sigma(z+2z)} - e^{\sigma(z)} + Q(z)b_{10}(z) - b_{10}(z + c) = 0.
\]

That is,

\[
A_2(z)e^{\gamma z} + A_1(z)e^{\sigma(z)} + A_0(z)e^0 = 0,
\]

where

\[
\begin{aligned}
 A_0(z) &= Q(z)b_{10}(z) - b_{10}(z + c) \\
 A_1(z) &= -Q(z)b_{10}(z)e^{(\text{deg} \gamma - 1)z} - b_{11}(z + c)e^{(\text{deg} \gamma - 1)z} + Q(z)b_{11}(z) + b_{10}(z + c), \\
 A_2(z) &= b_{11}(z + c)e^{(\text{deg} \gamma - 1)z} - Q(z)b_{11}(z)e^{(\text{deg} \gamma - 1)z}.
\end{aligned}
\]
Lemma 2.6. By (24), (27) and Lemma 2.1, we have

\[
\begin{cases}
\sigma(A_0) \leq \sigma(b_{10}) < \deg \gamma \\
\sigma(A_1) \leq \max\{\sigma(b_{10}), \sigma(b_{11}), \sigma(e^{(e^{(z+2c)}-\gamma(z+1)})}, \sigma(e^{(e^{(z+2c)}-\gamma(z+1)})\} = \max\{\sigma(b_{10}), \sigma(b_{11}), \deg \gamma - 1\} < \deg \gamma, \\
\sigma(A_2) \leq \max\{\sigma(b_{11}), \sigma(e^{(e^{(z+2c)}-\gamma(z+1)})}, \sigma(e^{(e^{(z+2c)}-\gamma(z+1)})\} = \max\{\sigma(b_{11}), \deg \gamma - 1\} < \deg \gamma.
\end{cases}
\]

Thus, \(\sigma(A_j) < \deg \gamma\) (\(j = 0, 1, 2\)). Since \(e^x\) is of regular growth order \(\deg \gamma\), we obtain

\[
T(r, A_j) = o(T(r, e^\gamma)) = o(T(r, e^{2\gamma})), \quad j = 0, 1, 2.
\]

Applying Lemma 2.3 to (26), we have

\[
A_2(z) \equiv 0, \quad A_1(z) \equiv 0, \quad A_0(z) \equiv 0.
\]

By \(A_2(z) \equiv 0\) and (27), we obtain

\[
b_{11}(z + c)e^{(e^{(z+2c)}-\gamma(z+1)}) - Q(z)b_{11}(z)e^{(e^{(z+2c)}-\gamma(z+1)}) \equiv 0,
\]

or

\[
b_{11}(z + c) - Q(z)e^{(e^{(z+2c)}-\gamma(z+1))}b_{11}(z) \equiv 0,
\]

(28)

Applying Lemma 2.2 to equation (28), we have

\[
\sigma(b_{11}) \geq \sigma(e^{(e^{(z+2c)}-\gamma(z+1)}) + 1 = (\deg \gamma - 1) + 1 = \deg \gamma.
\]

which contradicts with (24).

Thus, \(f(z)\) of the form (12) cannot be a meromorphic solution of equation (4). \(\square\)

Lemma 2.6. [19] Let \(A_0(z), \ldots, A_n(z)\) be entire functions of finite order such that among those coefficients having the maximal order \(\sigma = \max\{\sigma(A_k), 0 \leq k \leq n\}\), exactly one has its type strictly greater than the others. If \(f(z) \neq 0\) is a meromorphic solution of equation

\[
A_n(z)f(z + \omega_n) + \cdots + A_1(z)f(z + \omega_1) + A_0(z)f(z) = 0,
\]

(29)

then \(\sigma(f) \geq \sigma + 1\).

Lemma 2.7. [11, 19] Let \(w\) be a transcendental meromorphic solution with finite order of difference equation

\[
P(z, w) = 0,
\]

where \(P(z, w)\) is a difference polynomial in \(w(z)\). If \(P(z, w) \neq 0\) for a meromorphic function \(a\), where \(a\) is a small function with respect to \(w\), then

\[
m \left( r, \frac{1}{w - a} \right) = S(r, w).
\]

3. Proof of Theorem 1.4

Proof. (i) We first support that \(\Delta f(z) \neq 0\). Then equation (5) can be changed into equation (4).

Since \(f(z)\) and \(g(z)\) share \(a, b, \infty\) CM, we have

\[
N \left( r, \frac{1}{f - a} \right) = N \left( r, \frac{1}{g - a} \right), \quad N \left( r, \frac{1}{f - b} \right) = N \left( r, \frac{1}{g - b} \right), \quad N(r, f) = N(r, g).
\]
By the second fundamental Nevanlinna Theorem, we have

\[ T(r, g) \leq N(r, f) + N\left(r, \frac{1}{g-a}\right) + N\left(r, \frac{1}{g-b}\right) + S(r, g) \]

\[ = N(r, f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r, g) \]

\[ \leq 3T(r, f) + S(r, g). \]

Thus, \( g(z) \) is of finite order.

Since \( f(z) \) and \( g(z) \) share \( a, b, \infty \) CM, we see again that

\[ \frac{f(z) - a}{g(z) - a} = e^{\alpha(z)}, \quad (30) \]

and

\[ \frac{f(z) - b}{g(z) - b} = e^{\beta(z)}, \quad (31) \]

where \( \alpha(z) \) and \( \beta(z) \) are polynomials.

Assume, to the contrary, that \( f(z) \neq g(z) \). Then from (30) and (31), we obtain

\[ e^\alpha \neq 1, \quad e^\beta \neq 1, \quad e^\alpha \neq e^\beta, \quad \alpha(z) \neq \beta(z). \]

Again by (30) and (31), we get

\[ f(z) = a + (b - a) \frac{e^\beta - 1}{e^\beta - 1}, \quad (32) \]

or

\[ f(z) = a + (b - a) \frac{A - 1}{Ae^{\gamma} - 1}, \quad (33) \]

where \( \gamma = \beta - \alpha \) is a nonzero polynomial.

If \( \beta \) and \( \gamma \) are both constants, then \( f \) is a constant from (33), a contradiction.

If \( \beta \) is a constant and denoting \( A = e^\beta \), then \( A \neq 1 \). (32) shows

\[ f(z) = a + (b - a) \frac{A - 1}{Ae^{\gamma} - 1}. \]

Hence, \( f(z) \) has two distinct finite Borel exceptional values \( a \) and \( a + (b - a)(1 - A) \), which contradicts with Remark 1.3.

If \( \alpha \) is a constant and denoting \( B = e^{-\alpha} \), then \( B \neq 1 \). (32) shows

\[ f(z) = a + (b - a) \frac{A - 1}{Be^{\gamma} - 1}. \]

Thus, \( f(z) \) has two distinct finite Borel exceptional values \( b \) and \( a + \frac{b - a}{B} \), which contradicts with Remark 1.3 again.

If \( \gamma \) is a constant and denoting \( A = \frac{A - 1}{Be^{\gamma} - 1}, B = a - A \), then \( A, B \) are constants. By (33), we have

\[ f(z) = a + Ae^\gamma - A = Ae^\gamma + B. \]

It is easy to see that \( f(z) \) has two Borel values \( B \) and \( \infty \). Theorem 1.2 (iii) shows \( \deg \beta = 1 \). Without loss of generality, we assume \( \beta(z) = mz \), then \( f(z) = Ae^{mc} + B \), where \( m \) is a nonzero constant. Thus,

\[ \Delta f(z) = A(e^{mc} - 1)e^{mc}, \quad \Delta f(z + c) = Ae^{mc}(e^{mc} - 1)e^{mc}. \quad (34) \]
We note that $\Delta f(z) \neq 0$ from (4). Thus, $\gamma mc - 1 \neq 0$ and $\Delta f(z + c) = e^{mc} \Delta f(z)$, which contradicts with Lemma 2.4.

We deduce from (33) and Lemma 2.5 that $\deg \beta = \deg \gamma$, and

$$
\Delta f(z) = \left( b \right) \left( \frac{e^{d(z+c)} - 1}{e^{\gamma(z+c)} - 1} - \frac{e^{d(z)} - 1}{e^{\gamma(z)} - 1} \right).
$$

Without loss of generality, we assume $Q(z)$ is a nonconstant polynomial in Lemma 2.4. By (35) and Lemma 2.4, we conclude that

$$
\frac{e^{d(z+2c)} - 1}{e^{\gamma(z+2c)} - 1} - \frac{e^{d(z+c)} - 1}{e^{\gamma(z+c)} - 1} = Q(z) \left( \frac{e^{d(z+c)} - 1}{e^{\gamma(z+c)} - 1} - \frac{e^{d(z)} - 1}{e^{\gamma(z)} - 1} \right),
$$

or

$$
\frac{e^{d(z+2c)} - 1}{e^{\gamma(z+2c)} - 1} + Q(z) \frac{e^{d(z)} - 1}{e^{\gamma(z)} - 1} = (Q(z) + 1) \frac{e^{d(z+2c)} - 1}{e^{\gamma(z+2c)} - 1},
$$

that is,

$$
\frac{e^{d(z+2c)+\gamma(z+c)+\gamma(z)}}{e^{\gamma(z+2c)} - 1} + Q(z) \frac{e^{d(z)+\gamma(z+c)+\gamma(z)}}{e^{\gamma(z)} - 1} - (Q(z) + 1) e^{d(z)+\gamma(z+c)+\gamma(z)}
- e^{d(z+2c)+\gamma(z+c)} - Q(z)e^{d(z)+\gamma(z+c)} - Q(z)e^{d(z)+\gamma(z+c)}
+ (Q(z) + 1) e^{d(z)+\gamma(z+c)} + (Q(z) + 1) e^{d(z)+\gamma(z+c)} - e^{d(z)} - Q(z) e^{d(z)} - Q(z) e^{d(z)} = 0.
$$

Rewriting the above equality as

$$
A_4(z) e^{d(z)+2\gamma(z)} + A_3(z) e^{d(z)+\gamma(z)} + A_2(z) e^{d(z)} + A_1(z) e^{d(z)} + A_0(z) e^{d(z)} = 0,
$$

where

$$
A_4(z) = e^{d(z+2c)+\beta(z)+\gamma(z)-\gamma(z)} + Q(z) e^{d(z+2c)+\gamma(z+c)-\gamma(z)} - (Q(z) + 1) e^{d(z+c)+\beta(z)+\gamma(z+2c)+\gamma(z)},
$$

$$
A_3(z) = e^{d(z+2c)+\beta(z)+\gamma(z+c)-\gamma(z)} - Q(z) e^{d(z)+\gamma(z+c)-\gamma(z)} - e^{d(z+2c)+\beta(z)} - Q(z) e^{d(z+2c)-\gamma(z)}
+ (Q(z) + 1) e^{d(z)+\beta(z)+\gamma(z+2c)-\gamma(z)} + (Q(z) + 1) e^{d(z)+\beta(z)},
$$

$$
A_2(z) = e^{d(z)+\beta(z)} - Q(z) e^{d(z)+\gamma(z+c)-2\gamma(z)} + (Q(z) + 1) e^{d(z)+\gamma(z+c)-2\gamma(z)},
$$

$$
A_1(z) = e^{d(z)+\beta(z)} - (Q(z) + 1) e^{d(z)+\beta(z)} + Q(z),
$$

$$
A_0(z) = e^{d(z)+\gamma(z)} + (Q(z) + 1) e^{d(z)+\gamma(z)} - Q(z).
$$

Obviously,

$$
\sigma(A_4) \leq \max\{\deg \beta - 1, \deg \gamma - 1\}, \quad \sigma(A_3) \leq \max\{\deg \beta - 1, \deg \gamma - 1\},
\sigma(A_2) \leq \deg \gamma - 1, \quad \sigma(A_1) \leq \deg \beta - 1, \quad \sigma(A_0) \leq \deg \gamma - 1.
$$

That is,

$$
\sigma(A_j) < \deg \beta = \deg \gamma, \quad (j = 0, 1, 2, 3, 4).
$$

Thus, equation (36) can be rewritten as

$$
A_4(z) e^{d(z)+\gamma(z)} + A_3(z) e^{d(z)} + A_2(z) e^{d(z)} + A_1(z) e^{d(z)+\gamma(z)} + A_0(z) = 0.
$$
In the following, we divide our proof into four cases.

**Case 3.1.** $\deg(\beta + \gamma) < \deg \gamma$. Combining this with $\deg \beta = \deg \gamma$, we get

$$\deg(\beta - \gamma) = \deg \gamma, \quad \deg(\beta - 2\gamma) = \deg \gamma.$$ 

Thus, $e^{\beta}, e^{\gamma}, e^{\beta-\gamma}, e^{\beta-2\gamma}$ are of regular growth order $\deg \gamma$.

Equation (41) shows that

$$A_3(z)e^{\beta(z)} + A_2(z)e^{\gamma(z)} + A_1(z)e^{\beta(z)-\gamma(z)} + B_0(z) = 0,$$

where

$$B_0(z) = A_4(z)e^{\beta(z)+\gamma(z)} + A_0(z).$$

By this and (40), we obtain $\sigma(B_0) \leq \max(\sigma(A_4), \sigma(A_0), \deg(\beta + \gamma)) < \deg \gamma = \deg \beta$. Then

$$\begin{align*}
T(r, A_j) &= o(T(r, e^{\beta})) = o(T(r, e^{\gamma})) = o(T(r, e^{\beta-2\gamma})) \quad (j = 1, 2, 3) \\
T(r, B_0) &= o(T(r, e^{\beta})) = o(T(r, e^{\gamma})) = o(T(r, e^{\beta-2\gamma}))
\end{align*}$$

Together with (42) and Lemma 2.3, we have

$$B_0(z) \equiv 0, \quad A_j(z) \equiv 0, \quad j = 1, 2, 3.$$

By $A_2(z) \equiv 0$ and (37), we have

$$-e^{\gamma(z+c)-\gamma(z)} - Q(z)e^{\gamma(z+c)+\gamma(z+c)-2\gamma(z)} + (Q(z) + 1)e^{\gamma(z+c-2\gamma(z)} \equiv 0.$$

or

$$-Q(z)e^{\gamma(z+c-2\gamma(z)} + (Q(z) + 1)e^{\gamma(z+c-2\gamma(z)} - 1 \equiv 0. \quad (43)$$

In Case 3.1, we again split two subcases.

**Subcase 3.1.1.** $\deg \gamma \geq 2$. Let $H(z) = e^{\gamma(z)}$, then

$$e^{\gamma(z+c)-\gamma(z)} = e^{\gamma(z+c)+\gamma(z+c)-\gamma(z)} = H(z+c)H(z).$$

Thus, equation (43) can be written as

$$-Q(z)H(z+c)H(z) + (Q(z) + 1)H(z+c) - 1 = 0.$$

For any given meromorphic function $w(z)$, set

$$P(z, w) = -Q(z)w(z+c)w(z) + (Q(z) + 1)w(z+c) - 1.$$

Then $P(z, H(z)) \equiv 0$. Moreover, $P(z, 0) = -1 \equiv 0$. By this and Lemma 2.7, we have $m\left(r, \frac{1}{H}\right) = S(r, H)$. But

$$m\left(r, \frac{1}{H}\right) = m(r, e^{\gamma(z)-\gamma(z+c)}) = T\left(r, \frac{1}{H}\right) = T(r, H) + O(1).$$

Thus, $T(r, H) = S(r, H)$, a contradiction.

**Subcase 3.1.2.** $\deg \gamma = 1$. Let $\gamma(z) = mz + n_1$, where $m \neq 0, n_1$ are complex constants. Then $\gamma(z+2c) - \gamma(z+c) = mc, \gamma(z+2c) - \gamma(z) = 2mc$. Substituting these into (43), we have

$$(e^{mc} - 1)(e^{mc}Q(z) - 1) = 0.$$
By combining this with (46) and Lemma 2.3, it follows

\[ \deg\beta = \deg\gamma, \quad \deg(\beta + \gamma) < \deg\beta, \]

we may assume \( \beta(z) = -mz + n_2 \), where \( n_2 \) is a complex constant. So, \( \theta(z + r) = \theta(z) \). By \( \deg(z + r) = \deg(z) \) and (32), we see \( f(z + c) = f(z) \). Thus, \( \Delta f(z) = 0 \). This contradicts with \( \Delta f(z) \neq 0 \).

**Case 3.2.** \( \deg(\beta - \gamma) < \deg\gamma \). Equation (41) shows that

\[
\left( A_4(z)e^{\delta - \gamma} + A_3(z)e^{\delta - \gamma} + A_2(z) \right) e^{\gamma} + \left( A_1(z)e^{\delta - \gamma} + A_0(z) \right) e^{\delta} = 0, \tag{44}
\]

By (40), (44), \( \deg(\beta - \gamma) < \deg\gamma \) and Lemma 2.3, we obtain

\[
A_4(z)e^{\delta - \gamma} + A_3(z)e^{\delta - \gamma} + A_2(z) = 0, \quad A_1(z)e^{\delta - \gamma} + A_0(z) = 0.
\]

Substituting (38), (39) and \( \beta(z) = \alpha(z) + \gamma(z) \) into the last equality \( A_1(z)e^{\delta - \gamma} + A_0(z) \equiv 0 \), we have

\[
e^\gamma(t) = (e^{\gamma(t + 2c) - 1} - (Q + 1)e^{\gamma(t + 2c) - \gamma(t)}(e^{\gamma(t + 2c)} - 1) + Q(e^{\gamma(t)} - 1) = 0.
\]

That is to say, \( y(z) = e^{\alpha(z)} - 1 \) is a meromorphic solution of equation

\[
e^{\gamma(t + 2c) - \gamma(t)}y(z + 2c) = (Q + 1)e^{\gamma(t + 2c) - \gamma(t)}y(z + c) + Qy(z) = 0. \tag{45}
\]

Since \( \alpha \) cannot be a constant, by \( \deg(\beta - \gamma) = \deg\alpha < \deg\gamma \), then \( \deg\gamma \geq 2 \). Set

\[
y(z) = a_kz^k + a_{k-1}z^{k-1} + \cdots + a_0,
\]

where \( k \geq 2 \) is an integer, \( a_k \neq 0, a_{k-1}, \ldots, a_0 \) are constant. Then

\[
y(z + 2c) - \gamma(z) = 2kca_kz^{k-1} + \cdots, \quad y(z + c) - \gamma(z) = kca_kz^{k-1} + \cdots.
\]

By these, we see in the equation (45), the coefficient \( e^{\gamma(t + 2c) - \gamma(t)} \) is of order \( k - 1 \) with type \( [2kca_k] \), the coefficient

\(-Q+1\)

\(e^{\gamma(t + 2c) - \gamma(t)} \) is of order \( k - 1 \) with type \( [kca_k] \). By these and applying Lemma 2.6 to equation (45), we have \( \sigma(y) \geq (k - 1) + 1 = k = \deg\gamma \). But \( \sigma(y) = \sigma(e^{\gamma}) - 1 = \deg\alpha = \deg(\beta - \gamma) < \deg\gamma \), a contradiction.

**Case 3.3.** \( \deg(\beta - 2\gamma) < \deg\gamma \). Equation (41) can be rewritten as

\[
A_4(z)e^{\delta - \gamma} + A_3(z)e^{\delta - \gamma} + A_0(z)e^{\gamma - \gamma} + (A_2(z) + A_1(z)e^{\delta - 2\gamma(z)} = 0. \tag{46}
\]

By \( \deg\beta = \deg\gamma \) and \( \deg(\beta - 2\gamma) < \deg\gamma \), we have \( \deg(\beta - \gamma) = \deg(\beta + \gamma) = \deg\gamma \). By this and (40), we have

\[
\begin{align*}
T(r, A_j) = o[T(r, e^\delta)] = o[T(r, e^{\delta - \gamma})] = o[T(r, e^{\delta + \gamma})] & \quad (j = 0, 3, 4) \\
T(r, A_2 + A_1e^{\delta - 2\gamma}) = o[T(r, e^\delta)] = o[T(r, e^{\delta - \gamma})] = o[T(r, e^{\delta + \gamma})].
\end{align*}
\]

Combining this with (46) and Lemma 2.3, it follows

\[
A_4(z) \equiv 0, \quad A_3(z) \equiv 0, \quad A_0(z) \equiv 0, \quad A_2(z) + A_1(z)e^{\delta - 2\gamma(z)} = 0.
\]

By \( A_0(z) \equiv 0 \) and (39), we have

\[
-e^{\gamma(t + 2c) - \gamma(t)} + (Q + 1)e^{\gamma(t + 2c) - \gamma(t)} - Q(z) \equiv 0. \tag{47}
\]

If \( \gamma \geq 2 \), then \( \deg(\gamma(z + 2c) - \gamma(z)) = \deg(y(z + c) - \gamma(z)) = \deg\gamma - 1 \geq 1 \). Set \( H(z) = e^{\gamma(t + 2c) - \gamma(t)} \), then \( e^{\gamma(t + 2c) - \gamma(t)} = H(z + c)H(z) \). Equation (47) can be written as

\[-H(z + c)H(z) + (Q + 1)H(z) - Q(z) = 0.\]

For any given meromorphic function \( w(z) \), set

\[
P(z, w) = -w(z + c)w(z) + (Q + 1)w(z) - Q(z).
\]
Hence, \( P(z, H(z)) = 0 \). It is easy to see \( P(z, 0) = -Q(z) \neq 0 \), by this and Lemma 2.7, we have \( m(r, H) = S(r, H) \). Thus, \( N(r, \frac{1}{H}) = T(r, H) + S(r, H) \). But \( N(r, \frac{1}{H}) = N(r, \frac{1}{e^{mc}}) = 0 \), a contradiction.

If \( \deg \gamma = 1 \), let \( \gamma(z) = mz + n_1 \), where \( m \neq 0, n_1 \) are constants. Hence, \( \gamma(z + 2c) - \gamma(z) = 2mc \), substituting these into (47), we get

\[
(e^mc - 1)(Q(z) - e^{mc}) = 0.
\]

Thus, \( e^{mc} = 1 \). So, \( e^{(z+c)} = e^z \).

By \( \deg (\beta - 2 \gamma) < \deg \beta = \deg \gamma \), we may assume \( \beta(z) = 2mz + n_2 \), where \( n_2 \) is a constant. Then \( e^{(z+c)} = e^{2mc+2mz+n_2} = e^{2mc+n_2} = e^{\beta(z)} \). By \( e^{(z+c)} = e^{\beta(z)} = e^{\gamma(z+c)} = e^{\gamma(z)} \) and (32), we see \( f(z + c) = f(z) \). Then \( \Delta f(z) \equiv 0 \), a contradiction again.

**Case 3.4.** \( \deg(\beta + \gamma) = \deg(\beta - \gamma) = \deg(\beta - 2\gamma) = \deg \gamma \). By this and (40), for \( j = 0, 1, 2, 3, 4 \), we have

\[
T(r, A_j) = o[T(r, e^\beta)] = o[T(r, e^\beta')] = o[T(r, e^{\beta+c})] = o[T(r, e^{\beta+2\gamma})] = o[T(r, e^{\beta+3\gamma})].
\]

Combining this with Lemma 2.3, we have

\[
A_j(z) = 0, \quad j = 0, 1, 2, 3, 4.
\]

By \( A_2(z) \equiv 0 \) and (37), we also obtain (43).

If \( \deg \gamma \geq 2 \), using the same method as the above **Case 3.1.1**, we get a contradiction.

If \( \deg \gamma = 1 \), then \( \deg \beta = \deg \gamma = 1 \). Let \( \gamma(z) = mz + n_1, \beta(z) = nz + n_2 \), where \( m \neq 0, n \neq 0, n_1, n_2 \) are complex constants. Then \( \gamma(z + 2c) - \gamma(z + c) = mc, \gamma(z + 2c) - \gamma(z) = 2mc \). Substituting these into (43), we have

\[
(e^{mc} - 1)(e^{mc} - Q(z) - 1) = 0.
\]

Since \( Q(z) \) is a nonconstant polynomial, we have \( e^{mc} = 1 \). Then \( e^{\gamma(z+c)} = e^{\gamma(z)} \).

By \( A_1(z) \equiv 0 \), (38) and \( \beta(z + 2c) - \beta(z) = 2nc, \beta(z + c) - \beta(z) = nc \), we have

\[
(e^{mc} - 1)(e^{mc} - Q(z)) = 0.
\]

Since \( Q(z) \) is a nonconstant polynomial, we have \( e^{mc} = 1 \). Then \( e^{\beta(z+c)} = e^{\beta(z)} \). By \( e^{\beta(z+c)} = e^{\beta(z)} = e^{\gamma(z+c)} = e^{\gamma(z)} \) and (32), we see \( f(z + c) = f(z) \). Then \( \Delta f(z) \equiv 0 \), a contradiction.

**(ii)** We second support that \( \Delta f(z) \equiv 0 \). By checking the proof of Theorem 1.4 (i), we also obtain (30)–(34). Thus, we deduce from (34) and \( \Delta f(z) \equiv 0 \) that \( e^{mc} = 1 \), and \( mc = 2k_1\pi i \) for some nonzero integer \( k_1 \).

Therefore, we obtain from (31), \( \beta(z) = mz \) and \( f(z) = Ae^{mc} + B \) that

\[
g(z) = \frac{(b + A)f - b(A + B)}{f - B} = L(f),
\]

where \( L(f) \) is a Mobius transformation of \( f \). Thus, (ii) holds.

**(iii)** We third support that \( \Delta f(z) \equiv 0 \). By checking the proof of **subcase 3.1.2, Case 3.3 and Case 3.4** in the Theorem 1.4 (i), we see \( \gamma(z) = mz + n_1, \beta(z) = nz + n_2 \), where \( mc = 2k_1\pi i, nc = 2k_2\pi i \) for some nonzero integer \( k_1, k_2 \). Substituting \( \gamma(z) = mz + n_1, \beta(z) = nz + n_2 \) into (33), we have

\[
f(z) = a + (b - a) \frac{e^{mc} - 1}{e^{mc} - n_1 - 1} = a + (b - a) \frac{Ae^{mc} - 1}{Be^{mc} - 1}.
\]

where \( A = e^{zi}, B = e^{ni} \) are nonzero constants, and \( \frac{n_1}{n_2} = \frac{k_1}{k_2} \) is a rational number. Substituting (48), \( \beta(z) = nz + n_2 \) into (31), we have

\[
g = b + (b - a) \frac{A - Be^{mc-n_2i}}{A - Be^{mc-1}}.
\]

By \( a(z) = \beta(z) - \gamma(z) \) cannot be a constant, we see \( \frac{n_1}{n_2} \neq 1 \). Thus, (iii) holds. \( \square \)
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References