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Sum of K-Frames in Hilbert C^*-Modules

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Abstract. In this paper, we investigate some conditions under which the action of an operator on a K-frame, remain again a K-frame for Hilbert module E. We also give a generalization of Douglas theorem to prove that the sum of two K-frames under certain condition is again a K-frame. Finally, we characterize the K-frame generators in terms of operators.

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [6]. Frames can be viewed as redundant bases which are generalization of orthonormal bases. Many generalizations of frames were introduced, e.g., frames of subspaces [4], Pseudo-frames [1], G-frames [17], and fusion frames [3]. Recently, L. Gavruta introduced the concept of K-frame for a given bounded operator K on Hilbert space in [10]. Hilbert C^*-modules arose as generalizations of the notion of Hilbert space. The basic idea was to consider modules over C^*-algebras instead of linear spaces and to allow the inner product to take values in the C^*-algebra of coefficients being C^*-(anti-)linear in its arguments [13]. In [8] authors generalized frame concept for operators in Hilbert C^*-modules. The paper is organized as follows. In Section 2, some notations and preliminary results of Hilbert Modules, their frames and K-frames are given. In Section 3, we study the action of operators on K-frames and under certain conditions, we shall show that it is again a K-frame. The next section is devoted to sum of K-frames. In fact, to show that the sum of two K-frames under certain conditions is again a K-frame we need to say a generalization of the Douglas Theorem [18], which may interest by its own. Finally, in the last section, we consider a unitary system of operators and characterize the K-frame generators in terms of operators. We also look forward to sum of two K-frame generators to be a K-frame generator.

2. Preliminaries

In this section we give some preliminaries about frames, K-frames in Hilbert spaces and Hilbert modules and related operators which we need in the following sections. A finite or countable sequence $\{f_k\}_{k\in\mathbb{J}}$ is

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called a frame for a separable Hilbert space H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{J}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \forall f \in H.$$

Frank and Larson [8] introduced the notion of frames in Hilbert C^*-modules as a generalization of frames in Hilbert spaces. A (left) Hilbert C^*-module over the C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module E equipped with an $\mathcal{A}$-valued inner product satisfy the following conditions:

1. $\langle x, x\rangle \geq 0$ for every $x \in E$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,
2. $\langle x, y\rangle = \langle y, x\rangle^*$ for every $x, y \in E$,
3. $\langle \cdot, \cdot\rangle$ is $\mathcal{A}$-linear in the first argument,
4. E is complete with respect to the norm $\|x\|^2 = \|\langle x, x\rangle\|_{\mathcal{A}}$.

Given Hilbert C^*-modules E and F, we denote by $L_{\mathcal{A}}(E, F)$ or $L(E, F)$ the set of all adjointable operators from E to F i.e. the set of all maps $T : E \to F$ such that there exists $T^* : F \to E$ with the property

$$\langle Tx, y\rangle = \langle x, T^*y\rangle,$$

for all $x \in E$, $y \in F$. It is well-known that each adjointable operator is necessarily bounded $\mathcal{A}$-linear in the sense $T(ax) = aT(x)$, for all $a \in \mathcal{A}, x \in E$. We denote $L(E)$ for $L(E, E)$. In fact $L(E)$ is a C^*-algebra.

Let $\mathcal{A}$ be a C^*-algebra and consider

$$\ell^2(\mathcal{A}) := \{\{a_j\}_n \subseteq \mathcal{A} : \sum_j a_n a_j^* \quad \text{converges in norm in } \mathcal{A}\}.$$

It is easy to see that $\ell^2(\mathcal{A})$ with pointwise operations and the inner product

$$\langle \{a_j\}, \{b_j\}\rangle = \sum_j a_j {b_j}^*,$$

becomes a Hilbert C^*-module which is called the *standard Hilbert C^*-module* over $\mathcal{A}$. Throughout this paper, we suppose E is a Hilbert $\mathcal{A}$-module and $\mathbb{J}$ a countable index set. Also, we denote the range of $T \in L(E)$ by $R(T)$, and the kernel of T by $N(T)$. A Hilbert $\mathcal{A}$-module E is called finitely generated (countably generated) if there exists a finite subset $\{x_1, ..., x_j\}$ (countable set $\{x_j\}_{j\in\mathbb{J}}$) of E such that E equals the closed $\mathcal{A}$-linear hull of this set. The basic theory of Hilbert C^*-modules can be found in [13].

The following lemma found the relation between the range of an operator and the kernel of its adjoint operator.

Lemma 2.1. *([19], Lemma 15.3.5; [13], Theorem 3.2) Let $T \in L(E, F)$. Then*

1. $N(T) = N(|T|)$, $N(T^*) = R(T)^\perp$, $N(T^*)^\perp = R(T)^{\perp\perp} \supseteq \overline{R(T)}$;
2. *$R(T)$ is closed if and only if $R(T^*)$ is closed, and in this case $R(T)$ and $R(T^*)$are orthogonally complemented with $R(T) = N(T^*)^\perp$ and $R(T^*) = N(T)^\perp$.*

The following theorem is extended Douglas theorem [7] for Hilbert modules.

Theorem 2.2. *[18] Let $T' \in L(G, F)$ and $T \in L(E, F)$ with $\overline{R(T^*)}$ orthogonally complemented. The following statements are equivalent:*

1. *$T'T'^* \leq \lambda TT^*$ for some $\lambda > 0$;*
2. *There exists $\mu > 0$ such that $\|T'^*z\| \leq \mu\|T^*z\|$ for all $z \in F$;*
3. *There exists $D \in L(G, E)$such that $T' = TD$, i.e. the equation $TX = T'$ has a solution;*
4. *$R(T') \subseteq R(T)$.*

Here, we recall the concept of frame in Hilbert C^*-modules which is defined in [8]. Let E be a countably generated Hilbert module over a unital C^*-algebra $\mathcal{A}$. A sequence $\{x_j\}_{j\in\mathbb{J}} \subset E$ is said to be a *frame* if there exist two constant $C, D > 0$ such that

$$C\langle x, x\rangle \leq \sum_j \langle x, x_j\rangle\langle x_j, x\rangle \leq D\langle x, x\rangle, \text{ for all } x \in E. \tag{1}$$

The optimal constants (i.e. maximal for C and minimal for D) are called *frame bounds*. If the sum in (1) converges in norm, the frame is called *standard frame*. In this paper all frames consider standard frames. The sequence $\{x_j\}_{j\in\mathbb{J}}$ is called a *Bessel sequence* with bound D if the upper inequality in (1) holds for every $x \in E$.
Let $\{x_j\}_{j\in\mathbb{J}}$ be a Bessel sequence for Hilbert module E over $\mathcal{A}$. The operator $T : E \to \ell^2(\mathcal{A})$ defined by $Tx = \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}$ is called the *analysis operator*. The adjoint operator $T^* : \ell^2(\mathcal{A}) \to E$ which is given by

$$T^*\{c_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_j x_j,$$

is called the *pre-frame operator* or the *synthesis operator*. By composing T and T^*, we obtain the *frame operator* $S : E \to E$ given by

$$Sx = T^*Tx = \sum_{j\in\mathbb{J}} \langle x, x_j\rangle x_j, \ (x \in E).$$

By [8], if $\{x_j\}_{j\in\mathbb{J}}$ is a frame, the frame operator is positive and invertible. Also it is the unique operator in $L(E)$ such that the reconstruction formula

$$x = \sum_{j\in\mathbb{J}} \langle x, S^{-1}x_j\rangle x_j = \sum_{j\in\mathbb{J}} \langle x, x_j\rangle S^{-1}x_j, \ x \in E,$$

holds. It is easy to see that the sequence $\{S^{-1}x_j\}_{j\in\mathbb{J}}$, is a frame for E, and it is called the *canonical dual* frame of $\{x_j\}_{j\in\mathbb{J}}$.

Theorem 2.3. *[[14], Proposition 2.2] Let $\{x_j\}_{j\in\mathbb{J}}$ be a sequence in E such that $\sum_{j\in\mathbb{J}} c_j x_j$ converges for all $c = \{c_j\}_{j\in\mathbb{J}} \in \ell^2(\mathcal{A})$. Then $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence in E.*

Theorem 2.4. *[12] Let E be a finitely or countably generated Hilbert module over a unital C^*-algebra $\mathcal{A}$, and $\{x_j\}_{j\in\mathbb{J}}$ be a sequence in E. Then $\{x_j\}_{j\in\mathbb{J}}$ is a frame for E with bounds C and D if and only if*

$$C\|x\|^2 \leq \|\sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle\| \leq D\|x\|^2, (x \in E).$$

Najati in [14] extended the concept of atomic system and a K-frame to Hilbert modules.

Definition 2.5. *A sequence $\{x_j\}_{j\in\mathbb{J}}$ of E is called an* atomic system *for $K \in L(E)$ if the following statement hold:*

1. *The series $\sum_{j\in\mathbb{J}} c_j x_j$ converges for all $c = \{c_j\}_{j\in\mathbb{J}} \in \ell^2(\mathcal{A})$;*
2. *There exists $C > 0$ such that for every $x \in E$ there exists $\{a_{j,x}\}_{j\in\mathbb{J}} \in \ell^2(\mathcal{A})$ such that $\sum_{j\in\mathbb{J}} a_{j,x}a^*{}_{j,x} \leq C\langle x, x\rangle$ and $Kx = \sum_{j\in\mathbb{J}} a_{j,x}x_j$.*

By Theorem 2.3, the condition (1) in the above definition, actually says that $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence.

Theorem 2.6. *[14] If $K \in L(E)$, then there exists an atomic system for K.*

Theorem 2.7. *[14] Let $\{x_j\}_{j\in\mathbb{J}}$ be a Bessel sequence for E and $K \in L(E)$. Suppose that $T \in L(E, \ell^2(\mathcal{A}))$ is given by $T(x) = \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}$ and $\overline{R(T)}$ is orthogonally complemented. Then the following statements are equivalent:*

1. $\{x_j\}_{j\in\mathbb{J}}$ *is an atomic system for* K*;*
2. *There exist constants* $C, B > 0$ *such that*

$$B\|K^*x\|^2 \leq \|\sum_j \langle x, x_j\rangle\langle x_j, x\rangle\| \leq C\|x\|^2;$$

3. *There exists* $D \in L(E, \ell^2(\mathcal{A}))$ *such that* $K = T^*D$.

Definition 2.8. *Let* E *be a Hilbert* $\mathcal{A}$*-module,* $\{x_j\}_{j\in\mathbb{J}} \subset E$ *and* $K \in L(E)$*. The sequence* $\{x_j\}_{j\in\mathbb{J}}$ *is said to be a* K-frame *if there exist constants* $C, D > 0$ *such that*

$$C\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \leq D\langle x, x\rangle, \quad x \in E. \tag{2}$$

The following theorem gives a characterization of K-frames using linear adjiontable operators.

Theorem 2.9. *[14] Let* $K \in L(E)$ *and* $\{x_j\}_{j\in\mathbb{J}}$ *be a Bessel sequence for* E*. Suppose that* $T \in L(E, \ell^2(\mathcal{A}))$ *is given by* $T(x) = \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}$ *and* $\overline{R(T)}$ *is orthogonally complemented. Then* $\{x_j\}_{j\in\mathbb{J}}$ *is a K-frame for* E *if and only if there exists a linear bounded operator* $L : \ell^2(\mathcal{A}) \to E$ *such that* $Le_j = x_j$ *and* $R(K) \subseteq R(L)$*, where* $\{e_j\}_{j\in\mathbb{J}}$ *is the canonical orthonormal basis for* $\ell^2(\mathcal{A})$*.*

3. Operators On K-frames

In this section we study the action of an operator on a K-frame. The following lemma shows that the action of an adjointable operator on a Bessel sequence is again a Bessel sequence.

Lemma 3.1. *Let* E *be a Hilbert* $\mathcal{A}$*-module and* $\{x_j\}_{j\in\mathbb{J}}$ *be a Bessel sequence. Then* $\{Mx_j\}_{j\in\mathbb{J}}$ *is a Bessel sequence for every* $M \in L(E)$*.*

Proof. Since $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence there exists constant D such that

$$\sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \leq D\langle x, x\rangle,$$

for every $x \in E$. So

$$\begin{aligned}\sum_{j\in\mathbb{J}} \langle x, Mx_j\rangle\langle Mx_j, x\rangle &= \sum_{j\in\mathbb{J}} \langle M^*x, x_j\rangle\langle x_j, M^*x\rangle \\ &\leq D\langle M^*x, M^*x\rangle \\ &= D\langle MM^*x, x\rangle \\ &\leq D\|M\|^2\langle x, x\rangle,\end{aligned}$$

for every $x \in E$. □

Theorem 3.2. *Let* E *be a Hilbert* $\mathcal{A}$*-module,* $K \in L(E)$ *and* $\{x_j\}_{j\in\mathbb{J}}$ *be a K-frame for* E*. Let* $M \in L(E)$ *with* $R(M) \subset R(K)$ *and* $\overline{R(K^*)}$ *is orthogonally complemented. Then* $\{x_j\}_{j\in\mathbb{J}}$ *is an M-frame for* E*.*

Proof. Since $\{x_j\}_{j\in\mathbb{J}}$ *is a K-frame then there exist positive numbers* μ *and* λ *such that*

$$\lambda\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}} \langle x, x_n\rangle\langle x_j, x\rangle \leq \mu\langle x, x\rangle. \tag{3}$$

Using Theorem 2.2, the fact that $R(M) \subset R(K)$ shows that $MM^* \leq \lambda' KK^*$ for some $\lambda' > 0$. So

$$\langle MM^*x, x\rangle \leq \lambda' \langle KK^*x, x\rangle,$$

and hence,

$$\frac{\lambda}{\lambda'}\langle MM^*x, x\rangle \leq \lambda\langle K^*x, K^*x\rangle.$$

From (3), *we have*

$$\frac{\lambda}{\lambda'}\langle MM^*x, x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \leq \mu\langle x, x\rangle.$$

Therefore, $\{x_j\}_{j\in\mathbb{J}}$ is an M-frame with the bounds $\frac{\lambda}{\lambda'}$ and μ for E. □

In the following theorem, we obtain the result of the last theorem by different conditions.

Theorem 3.3. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for Hilbert $\mathcal{A}$-module E. Suppose that $T \in L(E, l^2(A))$ with $T(x) = \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}$ for every $x \in E$, $\overline{R(T)}$ is orthogonally complemented and $M \in L(E)$ such that $R(M) \subset R(K)$. Then $\{x_j\}_{j\in\mathbb{J}}$ is an M-frame for E.*

Proof. *By Theorem 2.9, there exists $L : \ell^2(\mathcal{A}) \to E$ such that $Le_j = f_j$, $j \in \mathbb{J}$ and $R(K) \subset R(L)$. So $R(M) \subset R(L)$. Now again by Theorem 2.9, we conclude that $\{x_j\}_{j\in\mathbb{J}}$ is an M-frame for E.* □

Theorem 3.4. *Let E be a Hilbert $\mathcal{A}$-module and $K \in L(E)$ with the dense range. Let $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for E and $T \in L(E)$ has closed range. If $\{Tx_j\}_{j\in\mathbb{J}}$ is a K-frame for E, then T is surjective.*

Proof. *Suppose that $K^*x = 0$ for $x \in E$. Then for each $y \in E$, $\langle Ky, x\rangle = \langle y, K^*x\rangle = 0$ and So $\langle z, x\rangle = 0$ for each $z \in E$. Since $R(K)$ is dense in E, hence $x = 0$ and so K^* is injective. Now, we show that T^* is injective too. Note that if $\{Tx_j\}_{j\in\mathbb{J}}$ is a K-frame for E with bounds λ and μ, then*

$$\lambda\|K^*x\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle x, Tx_j\rangle\langle Tx_j, x\rangle\| \leq \mu\|x\|^2,$$

and therefore,

$$\lambda\|K^*x\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle T^*x, x_n\rangle\langle x_n, T^*x\rangle\| \leq \mu\|x\|^2.$$

If $x \in N(T^*)$ then $T^*x = 0$. Hence $\langle T^*x, x_j\rangle = 0$ for each $j \in \mathbb{J}$, and so $K^*x = 0$, by the last inequality. Since K^* is injective, it follows that $x = 0$, and so T^* is injective. Therefore

$$E = N(T^*) + \overline{R(T)} = \overline{R(T)} = R(T),$$

and this completes the proof. □

Theorem 3.5. *Let $K \in L(E)$ and $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for E. If $T \in L(E)$ has closed range, $R(K^*) \subset R(T)$, $\overline{R(TK)}$ is orthogonal complemented and $KT = TK$, then $\{Tx_j\}_{j\in\mathbb{J}}$ is a K-frame for $R(T)$.*

Proof. *It was proved in [20] that if T has closed range, then T has the Moore-Penrose inverse operator $T^\dagger$ such that $TT^\dagger T = T$ and $T^\dagger TT^\dagger = T^\dagger$. So $TT^\dagger|_{R(T)} = I_{R(T)}$ and $(TT^\dagger)^* = I^* = I = TT^\dagger$. For every $x \in R(T)$ we have*

$$\begin{aligned}\langle K^*x, K^*x\rangle &= \langle (TT^\dagger)^*K^*x, (TT^\dagger)^*K^*x\rangle\\ &= \langle T^{\dagger^*}T^*K^*x, T^{\dagger^*}T^*K^*x\rangle\\ &\leq \|(T^\dagger)^*\|^2\langle T^*K^*x, T^*K^*x\rangle,\end{aligned}$$

and so

$$\|(T^{\dagger})^*\|^{-2}\langle K^*x, K^*x\rangle \leq \langle T^*K^*x, T^*K^*x\rangle.$$

Since $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame, with lower frame bound λ and $R(T^*K^*) \subset R(K^*T^*)$, then by Theorem 2.2, there exists some $\lambda' > 0$ such that

$$\begin{aligned}\sum_{j\in\mathbb{J}}\langle x, Tx_n\rangle\langle Tx_n, x\rangle &= \sum_{j\in\mathbb{J}}\langle T^*x, x_n\rangle\langle x_n, T^*x\rangle\\ &\geqslant \lambda\langle K^*T^*x, K^*T^*x\rangle\\ &\geqslant \lambda'\lambda\langle T^*K^*x, T^*K^*x\rangle\\ &\geqslant \lambda'\lambda\|(T^{\dagger})^*\|^2\langle K^*x, K^*x\rangle.\end{aligned}$$

This implies that $\{Tx_j\}_{j\in\mathbb{J}}$ satisfies in lower frame condition. On the other hand, by Lemma 3.1, $\{Tx_j\}_{j\in\mathbb{J}}$ is a Bessel sequence and therefore $\{Tx_j\}_{j\in\mathbb{J}}$ is a K-frame for Hilbert module $R(T)$. □

Theorem 3.6. *Let E be a Hilbert $\mathcal{A}$-module, $K \in L(E)$ and $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for E. Moreover, let $T \in L(E)$ be a co-isometry such that $R(T^*K^*) \subset R(K^*T^*)$ and $\overline{R(TK)}$ is orthogonal complemented. Then $\{Tx_j\}_{j\in\mathbb{J}}$ is a K-frame for E.*

Proof. *Using Lemma 3.1, $\{Tx_j\}_{j\in\mathbb{J}}$ is a Bessel sequence. Also, by Theorem 2.2, there exists $\lambda' > 0$ such that $\|T^*K^*x\|^2 \leq \lambda'\|K^*T^*x\|^2$, for each $x \in E$. Suppose λ is a lower bound for the K-frame $\{x_j\}_{j\in\mathbb{J}}$. Since T is a co-isometry, then*

$$\begin{aligned}\frac{\lambda}{\lambda'}\|K^*x\|^2 &= \frac{\lambda}{\lambda'}\|T^*K^*x\|^2\\ &\leq \lambda\|K^*T^*x\|^2\\ &\leq \sum_{j\in\mathbb{J}}\langle T^*x, x_n\rangle\langle x_n, T^*x\rangle\\ &= \sum_{j\in\mathbb{J}}\langle x, Tx_n\rangle\langle Tx_n, x\rangle,\end{aligned}$$

which implies that $\{Tx_j\}_{j\in\mathbb{J}}$ is a K-frame for E. □

Remark 3.7. *Consider $K \in L(E)$ with dense range, $T \in L(E)$ with closed range such that $TK = KT$ and $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for E. Then $\{Tx_j\}_{j\in\mathbb{J}}$ is a K-frame for E if and only if T is surjective.*

Theorem 3.8. *Let $K \in L(E)$ whose range is dense and $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for E. Moreover, let $T \in L(E)$ has closed the range. If $\{Tx_j\}_{j\in\mathbb{J}}$ and $\{T^*x_j\}_{j\in\mathbb{J}}$ are K-frames for E, then T is invertible.*

Proof. *By Theorem 3.4, T is surjective. Since $\{T^*x_j\}_{j\in\mathbb{J}}$ is a K-frame for E then there exist positive numbers μ and λ such that for every $x \in E$*

$$\lambda\|K^*x\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle x, T^*x_j\rangle\langle T^*x_j, x\rangle\| \leq \mu\|x\|^2.$$

So for $x \in N(T)$ we have

$$\lambda\|K^*x\|^2 \leq \sum_{j\in\mathbb{J}}\langle x, T^*x_j\rangle\langle T^*x_j, x\rangle = 0.$$

Then $\|K^*x\|^2 = 0$ and so $x \in N(K^*)$. On the other hand, $K \in L(E)$ has dense range. Hence K^* is injective and so T is also injective. □

4. Sums of K-frames

In this section we show that the sum of two K-frames in a Hilbert C^*-module under certain conditions is again a K-frame, it was proved in Hilbert space case by Ramu and Johnson [15]. In the proof of Theorem 3.2 of [13], it was indicated that if T has closed range then $R(T^*T)$ is closed and $R(T) = R(T^*T)$. The following theorem says that this result still holds for adjointable operators between Hilbert C^*-modules (even though $\overline{R(T^*)}$ may not be complemented).

Theorem 4.1. *[13] For T in $L(E, F)$, the sub-spaces $R(T^*)$ and $R(T^*T)$ have the same closure.*

In [16], Sharifi proved that the converse of above theorem is also true.

Theorem 4.2 (Lemma 1.1, [16]). *Suppose $T \in L(E)$. Then the operator T has closed range if and only if $R(TT^*)$ has closed rang. In this case, $R(T) = R(TT^*)$.*

Corollary 4.3. *Suppose $T \in L(E)^+$. Then $R(T)$ is closed if and only if $R(T^{1/2})$ is closed. In this case, $R(T) = R(T^{1/2})$.*

Proof. *The proof is immediately consequence of replacement T by $T^{1/2}$ in the above theorem.* □

Theorem 4.4. *Let E be a Hilbert module and $A, B \in L(E)$ such that $R(A) + R(B)$ is closed. Then*

$$R(A) + R(B) = R((AA^* + BB^*)^{\frac{1}{2}}).$$

Proof. *Define $T \in L(E \oplus E)$ by $T := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$. Then $T^* = \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}$, and*

$$TT^* = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} = \begin{bmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{bmatrix}.$$

So we have

$$(TT^*)^{1/2} = \begin{bmatrix} (AA^* + BB^*)^{1/2} & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand

$$T \begin{bmatrix} E \\ E \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E \\ E \end{bmatrix},$$

thus

$$R(T) = R(A) + R(B) \oplus \{0\}.$$

Since $R(T) = (R(A) + R(B))$ is closed then by Theorem 4.2, $R(T) = R(TT^*)$. But by Corollary 4.3, $R(TT^*) = R((TT^*)^{1/2})$. So we have

$$R(A) + R(B) = R((AA^* + BB^*)^{1/2}).$$

□

The following theorem is a generalization of Douglas theorem [Theorem 1.1, [18]], for Hilbert modules.

Theorem 4.5. *Let $A, B_1, B_2 \in L(E)$ and $R(B_1) + R(B_2)$ is closed. The following statements are equivalent.*

1. *$R(A) \subset R(B_1) + R(B_2)$;*
2. *$AA^* \leq \lambda(B_1B_1{}^* + B_2B_2{}^*)$ for some $\lambda > 0$;*
3. *There exist $X, Y \in L(E)$ such that $A = B_1X + B_2Y$.*

Proof. (1) $\Longrightarrow$ (2): *Suppose* $R(A) \subset R(B_1) + R(B_2)$. *Then by Theorem 4.4, we have*

$$\begin{aligned} R(A) &\subset R(B_1) + R(B_2) \\ &= R((B_1 B_1^* + B_2 B_2^*)^{1/2}), \end{aligned}$$

thus Theorem 2.2, implies $AA^* \leq \lambda(B_1 B_1^* + B_2 B_2^*)$ *for some* $\lambda > 0$.
(2) $\Longrightarrow$ (1): *By Theorems 2.2, and 4.5, it is clear.*
(3) $\Longrightarrow$ (1): *It is obvious.*
(1) $\Longrightarrow$ (3): *Define* $S, T \in L(E \oplus E)$ *by*

$$S = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}.$$

Then $R(S) \subset R(T)$ *by Theorem 2.2. Suppose*

$$X = \begin{bmatrix} X_1 & X_3 \\ X_2 & X_4 \end{bmatrix},$$

is the solution of $S = TX$, *so we have* $A = B_1 X_1 + B_2 X_2$. *This completes the proof.* □

Following lemma shows that the sum of two Bessel sequences is a Bessel sequence too.

Lemma 4.6. *Suppose that* $\{x_j\}_{j\in\mathbb{J}}$ *and* $\{y_j\}_{j\in\mathbb{J}}$ *are two Bessel sequences in Hilbert module E. Then, by the Minkowski's inequality,* $\{x_j + y_j\}_{j\in\mathbb{J}}$ *is also a Bessel sequence for E.*

Now we are going to show that under certain conditions the sum of two K-frame, is a K-frame.

Theorem 4.7. *Let* $\{x_j\}_{j\in\mathbb{J}}$ *and* $\{y_j\}_{j\in\mathbb{J}}$ *be two K-frames for E and also let the corresponding operators in Theorem 2.9, be* L_1 *and* L_2 *respectively. If* $L_1 L_2^*$ *and* $L_2 L_1^*$ *are positive operators and* $R(L_1) + R(L_2)$ *is closed, then* $\{x_j + y_j\}_{j\in\mathbb{J}}$ *is a K-frame for E.*

Proof. *By the hypothesis we have*

$$L_1 e_j = x_j, L_2 e_j = y_j, R(K) \subset R(L_1), R(K) \subset R(L_2),$$

where $\{e_j\}_{j\in\mathbb{J}}$ *is the canonical orthonormal basis of* $\ell^2(\mathcal{A})$. *So* $R(K) \subset R(L_1) + R(L_2)$, *by Theorem 4.5, and* $KK^* \leq \lambda(L_1 L_1^* + L_2 L_2^*)$ *for some* $\lambda > 0$. *On the other hand for each* $x \in E$,

$$\begin{aligned} \sum_{j\in\mathbb{J}} \langle x, x_j + y_j\rangle\langle x_j + y_j, x\rangle &= \sum_{j\in\mathbb{J}} \langle (L_1^* + L_2^*)x, e_j\rangle\langle e_j, L_1^* + L_2^*)x\rangle \\ &= \sum_{j\in\mathbb{J}} \langle (L_1 + L_2)^* x, e_j\rangle\langle e_j, (L_1 + L_2)^* x\rangle \\ &= \|(L_1 + L_2)^* x\|_{l^2(\mathcal{A})} \\ &= \langle (L_1 + L_2)^* x, (L_1 + L_2)^* x\rangle \\ &= \langle L_1^* x, L_1^* x\rangle + \langle L_1^* x, L_2^* x\rangle \\ &+ \langle L_2^* x, L_1^* x\rangle + \langle L_2^* x, L_2^* x\rangle \\ &\geqslant \langle (L_1 L_1^* + L_2 L_2^*)x, x\rangle \\ &\geqslant \frac{1}{\lambda}(\langle KK^* x, x\rangle \\ &\geqslant \frac{1}{\lambda}(\langle K^* x, K^* x\rangle. \end{aligned}$$

Thus $\{x_j + y_j\}_{j\in\mathbb{J}}$ *is a K–frame.* □

5. *K*-frame vectors for unitary systems

A unitary system is a set of unitary operators which contains the identity operator. A vector ψ in E is called a *complete K-frame* vector for a unitary system $\mathcal{U}$ on E if $\mathcal{U}\psi = \{U\psi \mid U \in \mathcal{U}\}$ is a K-frame for E. If $\mathcal{U}\psi$ is an orthonormal basis for E, then ψ is called a *complete wandering* vector for $\mathcal{U}$. The set of all complete K-frame vectors and complete wandering vectors for $\mathcal{U}$ is denoted by $\mathcal{F}_K(\mathcal{U})$ and $\omega(\mathcal{U})$, respectively. In this section we characterize $\mathcal{F}_K(\mathcal{U})$ in terms of operators and elements of $\omega(\mathcal{U})$.

Definition 5.1. *For a unitary system $\mathcal{U}$ on a Hilbert module E and $\psi \in E$, the local commutant $C_\psi(\mathcal{U})$ of $\mathcal{U}$ at ψ is defined by*

$$C_\psi(\mathcal{U}) = \{T \in L(E) \mid TU\psi = UT\psi, \quad U \in \mathcal{U}\}.$$

Also, let $\ell^2_{\mathcal{U}}(\mathcal{A})$ be the Hilbert $\mathcal{A}$-module defined by

$$\ell^2_{\mathcal{U}}(\mathcal{A}) = \{\{a_U\} \subset \mathcal{A} \quad : \sum a_U a_U^* \quad \text{converges in} \quad \|.\|\}.$$

The following theorem characterizes complete *K*-frame vectors in terms of operators on complete wandering vectors.

Theorem 5.2. *Suppose $\mathcal{U}$ is a unitary system of E, $K \in L(E)$, $\psi \in \omega(\mathcal{U})$ and $\eta \in E$. Moreover, suppose that $\psi_\eta \in L(E, \ell^2_{\mathcal{U}}(\mathcal{A}))$ is given by $T_\eta(x) = \{\langle x, U_\eta\rangle\}_{U\in\mathcal{U}}$ and $\overline{R(T_\eta{}^*)}$ is orthogonal complemented. Then $\eta \in \mathcal{F}_K(\mathcal{U})$ if and only if there exists an operator $A \in C_\psi(\mathcal{U})$ with $R(K) \subset R(A)$ such that $\eta = A\psi$.*

Proof. *($\Longrightarrow$) Suppose $\{e_U\}_{U\in\mathcal{U}}$ denote the standard orthonormal basis of $\ell^2_{\mathcal{U}}(\mathcal{A})$, where e_U takes value $1_{\mathcal{A}}$ at U and $o_{\mathcal{A}}$ at every where else. Now suppose $\eta \in \mathcal{F}_K(\mathcal{U})$. Define operator T_ψ from E to $\ell^2_{\mathcal{U}}(\mathcal{A})$ by $T_\psi x = \sum_{U\in\mathcal{U}}\langle x, U_\psi\rangle e_U$. It is easy to check that T_ψ is well defined, adjointable and invertible. Let $A = T_\eta{}^* T_\psi$. Then for any $x \in E$, we have $Ax = \sum_{U\in\mathcal{U}}\langle x, U_\psi\rangle U_\eta$ and $A^*x = \sum_{U\in\mathcal{U}}\langle x, U_\eta\rangle U_\psi$, also*

$$\begin{aligned}\langle A^*x, A^*x\rangle &= \langle \sum_{U\in\mathcal{U}}\langle x, U\eta\rangle U\psi, \sum_{U\in\mathcal{U}}\langle x, U\eta\rangle U\psi\rangle \\ &= \sum_{U\in\mathcal{U}}\langle x, U\eta\rangle\langle U\eta, x\rangle \\ &\geq c\langle K^*x, K^*x\rangle,\end{aligned} \tag{4}$$

where $c > 0$ is the lower bound for K-frame $\{U\eta \mid U \in \mathcal{U}\}$. On the other hand $R(A) = R(T_\eta{}^*)$ and so by Theorem 2.2, we have $R(K) \subset R(A)$. To complete the proof, it remains to prove that $\eta = A\psi$ and $A \in C_\psi(\mathcal{U})$. For any U and V in $\mathcal{U}$

$$\begin{aligned}\langle V\eta, AU\psi\rangle &= \langle V\eta, \sum_{U\in\mathcal{U}}\langle U\psi, W\psi\rangle W\eta\rangle \\ &= \sum_{U\in\mathcal{U}}\langle V\eta, W\eta\rangle\langle W\psi, U\psi\rangle \\ &= \langle V\psi, U\psi\rangle.\end{aligned} \tag{5}$$

This implies that $AU\psi = U\eta$, so $A\psi = \eta$. Also $AU\psi = U\eta = UA\psi$, hence $A \in C_\psi(\mathcal{U})$ and this completes the proof of this part.
($\Longleftarrow$): Suppose that there exists an operator $A \in C_\psi(\mathcal{U})$ with $R(K) \subset R(A)$ such that $\eta = A\psi$. Then for any $x \in E$, we have

$$\begin{aligned}\sum_{U\in\mathcal{U}}\langle x, U\eta\rangle\langle U\eta, x\rangle &= \sum_{U\in\mathcal{U}}\langle x, UA\psi\rangle\langle UA\psi, x\rangle \\ &= \sum_{U\in\mathcal{U}}\langle A^*x, U\psi\rangle\langle U\psi, A^*x\rangle \\ &= \langle A^*x, A^*x\rangle \\ &\leq \|A^*\|^2\|x\|^2.\end{aligned} \tag{6}$$

So $\{U\eta \mid U \in \mathcal{U}\}$ is a Bessel sequence for E. Now let T_η and T_ψ be the operators as we defined in the first part of the proof, since $\eta = A\psi$ so we have $T_\eta = T_\psi A^*$. Since $\psi \in w(\mathcal{U})$, it is easy to see that T^*_ψ is invertible and hence $R(T^*_\eta) = R(A)$. So $R(K) \subset R(T^*_\eta)$. Therefore , by using Theorem 3.2 of [8] it is concluded that $\eta \in \mathcal{U}_K(\mathcal{U})$. □

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