Filomat 34:6 (2020), 1853–1862
https://doi.org/10.2298/FIL2006853M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An Improvement of Recent Results in Controlled Metric Type Spaces

Nabil Mlaiki[a], Hassen Aydi[b,c,e,g], Nizar Souayah[d], Thabet Abdeljawad[a,e,f]

[a]*Department of Mathematics and General Sciences, Prince Sultan University, P. O. Box 66833, 11586 Riyadh, Saudi Arabia*
[b]*Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[c]*Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[d]*Department of Natural Sciences, Community College Al-Riyadh, King Saud University, Saudi Arabia, 11586*
[e]*Department of Medical Research, China Medical University, Taichung 40402, Taiwan*
[f]*Department of Computer Science and Information Engineering, Asia University, Taichung 40402, Taiwan*
[g]*Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia*

Abstract. Using control functions introduced by Sintunavarat et al. (2011), we present and improve some recent fixed point results in the class of controlled metric type spaces. We also illustrate the presented results by some nontrivial examples.

1. Introduction

One of the main applications of fixed point theory is to solve integral and differential equations by successive approximation methods, for example see [2, 3, 13, 24, 25, 28, 30, 34]. The Banach contraction principle was generalized many times to extend its application, see [4, 5, 31, 33, 36, 37]. As an example of these generalizations, the concept of $b-$ metric spaces was introduced in [18, 20], which is a generalization of the regular metric space, see [6–8, 12, 14–16, 22, 32, 38, 39]. A strong b-metric has been introduced by Kirk and Shahzad [26]. Recently, the authors in [21] introduced a type of extended $b-$metric spaces by replacing the constant s by a function $\theta(x, y)$ depending on the parameters of the left hand-side of the triangle inequality (see also [1, 9–11, 17, 23]).

Definition 1.1. *[21] Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$ be a given function. An extended b-metric is a function $d : X \times X \to [0, \infty)$ such that for all $\eta, \xi, u \in X$,*

1. $d(\eta, \xi) = 0 \iff \eta = \xi$;
2. $d(\eta, \xi) = d(\xi, \eta)$;
3. $d(\eta, \xi) \leq \theta(\eta, \xi)[d(\eta, u) + d(u, \xi)]$.

One of generalizations of $b-$metric spaces has been provided by Mlaiki et al. [29] who introduced the concept of controlled metric type spaces by employing a control function function $\delta : X \times X \to [1, \infty)$ to act separately on each term in the right-hand side of the triangle inequality.

2010 *Mathematics Subject Classification*. 47H10; 54H25; 46J10
Keywords. Fixed point, controlled metric type space, extended $b-$metric space
Received: 18 January 2019; Accepted: 20 January 2020
Communicated by Naseer Shahzad
Corresponding authors: Hassen Aydi (hassen.aydi@tdtu.edu.vn) and Thabet Abdeljawad (tabdeljawad@psu.edu.sa).
Email addresses: nmlaiki@psu.edu.sa (Nabil Mlaiki), hassen.aydi@tdtu.edu.vn (Hassen Aydi), nsouayah@ksu.edu.sa (Nizar Souayah), tabdeljawad@psu.edu.sa (Thabet Abdeljawad)

Definition 1.2. *[29] Given a nonempty set X and* $\delta : X \times X \to [1, \infty)$. *The function* $\omega : X \times X \to [0, \infty)$ *is called a controlled metric type if*
(d1) $\omega(\eta, \xi) = 0$ *if and only if* $\eta = \xi$*;*
(d2) $\omega(\eta, \xi) = \omega(\xi, \eta)$*;*
(d3) $\omega(\eta, \xi) \leq \delta(\eta, u)\omega(\eta, u) + \delta(u, \xi)\omega(u, \xi)$,
for all $\eta, \xi, u \in X$. *The pair* (X, ω) *is called a controlled metric type space.*

Remark 1.3. *Note that every b−metric space is a controlled metric type space, which is not in general an extended b-metric space when taking the same function, that is, in the case* $\theta = \delta$.

Example 1.4. *[29] Choose* $X = \{1, 2, \cdots\}$. *Take* $\omega : X \times X \to [0, \infty)$ *as*

$$\omega(\eta, \xi) = \begin{cases} 0, & \iff \eta = \xi \\ \frac{1}{\eta}, & \text{if } \eta \text{ is even and } \xi \text{ is odd} \\ \frac{1}{\xi}, & \text{if } \eta \text{ is odd and } \xi \text{ is even} \\ 1, & \text{otherwise.} \end{cases}$$

Consider $\delta : X \times X \to [1, \infty)$ *as*

$$\delta(\eta, \xi) = \begin{cases} \eta, & \text{if } \eta \text{ is even and } \xi \text{ is odd} \\ \xi, & \text{if } \eta \text{ is odd and } \xi \text{ is even} \\ 1, & \text{otherwise.} \end{cases}$$

ω *is a controlled metric type. On the other hand, for an integer* $p \geq 2$,

$$\omega(2p+1, 4p+1) = 1 > \frac{1}{p} = \delta(2p+1, 4p+1)[\omega(2p+1, 2p) + \omega(2p, 4p+1)],$$

that is, ω *is not an extended b-metric when considering the same function* $\delta = \theta$.

The topological concepts as Cauchyness, convergence, completeness, continuity on controlled metric type spaces are given as follows:

Definition 1.5. *Let* (X, ω) *be a controlled metric type space and* $\{\xi_n\}_{n\geq 0}$ *be a sequence in X.*
(1) We say that $\{\xi_n\}$ *is convergent to some* ξ *in X, if for every* $\epsilon > 0$, *there exists* $N = N(\epsilon) \in \mathbb{N}$ *such that* $d(\xi_n, \xi) < \epsilon$ *for each* $n \geq N$. *This is written as* $\lim_{n\to\infty} \xi_n = \xi$.
(2) We say that the sequence $\{\xi_n\}$ *is Cauchy, if for every* $\epsilon > 0$, *there is some* $N = N(\epsilon) \in \mathbb{N}$ *so that* $\omega(\xi_m, \xi_n) < \epsilon$ *for all* $m, n \geq N$.
(3) (X, ω) *is said complete if every Cauchy sequence is convergent.*

Definition 1.6. *Let* (X, ω) *be a controlled metric type space. Let* $\eta \in X$ *and* $\alpha > 0$.
(i) The open ball $B(\eta, \alpha)$ *is*

$$B(\eta, \alpha) = \{\xi \in X, \omega(\eta, \xi) < \alpha\}.$$

(ii) The self-mapping T on X is called continuous at $\eta \in X$ *if for each* $r > 0$, *there exists* $\beta > 0$ *such that* $T(B(\eta, \beta)) \subseteq B(T\eta, r)$.

Clearly, if T is continuous at ξ, then $\xi_n \to \xi$ implies that $T\xi_n \to T\xi$ at the limit $n \to \infty$.

Now, let (X, ω) be a controlled metric type space and $f : X \to X$ be a given mapping. Throughout this paper, unless otherwise specified, we will use the following notations:

$$\mathrm{A} = \{\lambda : X \to (0, 1),\ \lambda(f\eta) \leq \lambda(\eta) \ \text{ for each } \ \eta \in X\}.$$

and

$$\mathrm{B} = \{\lambda : X \to (0, 1/2),\ \lambda(f\eta) \le \lambda(\eta) \ \text{ for each } \ \eta \in X\}.$$

The above classes of control functions have been initiated by Sintunavarat et al. [35] (with a little bit modification, that is, the ranges do not contain 0). See also the recent paper of Kumrod and Sintunavarat [27]. In this paper, we provide some fixed point results for nonlinear Banach, Kannan and Chatterjea type contractive mappings by utilizing the above control functions. We also present a fixed point result involving cyclical orbital contractions. Moreover, we present some examples.

2. Main results

Throughout the paper, (X, ω) is a complete controlled metric type space by the function $\delta : X \times X \to [1, \infty)$. Our first main result corresponds to a nonlinear Banach type result on controlled metric type spaces.

Theorem 2.1. *Let $T: X \to X$ be a mapping such that*

$$\omega(Tx, Ty) \le \lambda(x)\omega(x, y), \tag{1}$$

for all $x, y \in X$, where $\lambda \in \mathrm{A}$. For $\xi_0 \in X$, take $\xi_n = T^n \xi_0$. Suppose that

$$\sup_{m \ge 1} \lim_{i \to \infty} \frac{\delta(\xi_{i+1}, \xi_{i+2})}{\delta(\xi_i, \xi_{i+1})} \delta(\xi_{i+1}, \xi_m) < \frac{1}{\lambda(\xi_0)}. \tag{2}$$

Also, assume that for every $\xi \in X$, we have

$$\lim_{n \to \infty} \delta(\xi_n, \xi) \ \text{ and } \ \lim_{n \to \infty} \delta(\xi, \xi_n) \ \text{ exist and are finite.} \tag{3}$$

Then T has a unique fixed point.

Proof. Consider the sequence $\{\xi_n = T^n \xi_0\}$. By using (1), we get

$$\omega(\xi_n, \xi_{n+1}) \le \lambda(\xi_{n-1})\omega(\xi_{n-1}, \xi_n) \quad \text{for all } \ n \ge 1.$$

Since $\lambda \in \mathrm{A}$, we have

$$\omega(\xi_n, \xi_{n+1}) \le \lambda(\xi_0)\omega(\xi_{n-1}, \xi_n) \quad \text{for all } \ n \ge 1.$$

By induction,

$$\omega(\xi_n, \xi_{n+1}) \le [\lambda(\xi_0)]^n\, \omega(\xi_0, \xi_1) \quad \text{for all } \ n \ge 0. \tag{4}$$

Choose $k =: \lambda(\xi_0) \in (0, 1)$. For all natural numbers $n < m$, as in [29], we have

$$\begin{aligned}
\omega(\xi_n, \xi_m) &\le \delta(\xi_n, \xi_{n+1})\omega(\xi_n, \xi_{n+1}) + \delta(\xi_{n+1}, \xi_m)\omega(\xi_{n+1}, \xi_m) \\
&\le \delta(\xi_n, \xi_{n+1})k^n \omega(\xi_0, \xi_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^{i} \delta(\xi_j, \xi_m) \right) \delta(\xi_i, \xi_{i+1}) k^i \omega(\xi_0, \xi_1) \\
&\le \delta(\xi_n, \xi_{n+1})k^n \omega(\xi_0, \xi_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^{i} \delta(\xi_j, \xi_m) \right) \delta(\xi_i, \xi_{i+1}) k^i \omega(\xi_0, \xi_1).
\end{aligned}$$

Let

$$S_p = \sum_{i=0}^{p} \left(\prod_{j=0}^{i} \delta(\xi_j, \xi_m) \right) \delta(\xi_i, \xi_{i+1}) k^i.$$

Hence, we have

$$\omega(\xi_n, \xi_m) \leq \omega(\xi_0, \xi_1) \left[k^n \delta(\xi_n, \xi_{n+1}) + (S_{m-1} - S_n)\right]. \tag{5}$$

Condition (2), by using the ration test, guarantees that $\lim_{n\to\infty} S_n$ exists and hence the real sequence $\{S_n\}$ is Cauchy. Finally, if we take the limit in the inequality (5) as $n, m \to \infty$, we deduce that

$$\lim_{n,m\to\infty} \omega(\xi_n, \xi_m) = 0, \tag{6}$$

that is, $\{\xi_n\}$ is a Cauchy sequence in the complete controlled metric type space (X, ω), so $\{\xi_n\}$ converges to some $u \in X$. We shall show that u is a fixed point of T. The triangle inequality yields that

$$\omega(u, \xi_{n+1}) \leq \delta(u, \xi_n)\omega(u, \xi_n) + \delta(\xi_n, \xi_{n+1})\omega(\xi_n, \xi_{n+1}).$$

Using (2), (3) and (6), we deduce that

$$\lim_{n\to\infty} \omega(u, \xi_{n+1}) = 0. \tag{7}$$

Using again the triangle inequality and (1),

$$\begin{aligned}
\omega(u, Tu) &\leq \delta(u, \xi_{n+1})\omega(u, \xi_{n+1}) + \delta(\xi_{n+1}, Tu)\omega(\xi_{n+1}, Tu)\\
&\leq \delta(u, \xi_{n+1})\omega(u, \xi_{n+1}) + \lambda(\xi_n)\delta(\xi_{n+1}, Tu)\omega(\xi_n, u)\\
&\leq \delta(u, \xi_{n+1})\omega(u, \xi_{n+1}) + \lambda(\xi_0)\delta(\xi_{n+1}, Tu)\omega(\xi_n, u).
\end{aligned}$$

Taking the limit as $n \to \infty$ and taking (3) and (7) into view, we deduce that $\omega(u, Tu) = 0$, that is, $Tu = u$. Finally, assume that T has two fixed points, say u and v. Thus,

$$\omega(u, v) = \omega(Tu, Tv) \leq \lambda(u)\omega(u, v),$$

which holds unless $\omega(u, v) = 0$, so $u = v$. Hence T has a unique fixed point. □

We illustrate Theorem 2.1 by the following example.

Example 2.2. *Let $X = [0, 1]$. Consider the controlled metric type d defined as*

$$\omega(x, y) = |x - y|^2,$$

where $\delta(x, y) = x + y + 1$ for $x, y \in X$. Take $Tx = \frac{x^2}{4}$. Choose $\lambda : X \to [0, 1)$ as $\lambda(x) = \frac{x+1}{4}$. Then $\lambda \in$ A. Take $\xi_0 = 0$, so (2) is satisfied. Let $x, y \in X$, then

$$\begin{aligned}
\omega(Tx, Ty) =& \frac{(x^2 - y^2)^2}{16} = \frac{1}{16}|x - y|^2(x + y)^2\\
&\leq \frac{1}{4}|x - y|^2\\
&\leq \frac{x + 1}{4}|x - y|^2\\
&= \lambda(x)\omega(x, y),
\end{aligned}$$

that is, (1) holds. All hypotheses of Theorem 2.1 hold, and so the mapping T has a unique fixed point, which is $u = 0$.

In the following theorem, we propose a fixed point result using the nonlinear Kannan type contraction via the auxiliary function $\lambda \in$ B. It is an answer to an open question in [29].

Theorem 2.3. *Let $T\colon X \to X$ be such that*

$$\omega(Tx, Ty) \leq \lambda(x)[\omega(x, Tx) + \omega(y, Ty)], \tag{8}$$

for all $x, y \in X$, where $\lambda \in \mathrm{B}$. For $\xi_0 \in X$, take $\xi_n = T^n\xi_0$. Suppose that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\delta(\xi_{i+1},\xi_{i+2})}{\delta(\xi_i,\xi_{i+1})}\delta(\xi_{i+1},\xi_m) < \frac{1-\lambda(\xi_0)}{\lambda(\xi_0)}. \tag{9}$$

Also, assume that for every $\xi \in X$, we have

$$\lim_{n\to\infty}\delta(\xi,\xi_n) \ \text{ exists, is finite and } \ \lim_{n\to\infty}\delta(\xi_n,\xi) < \frac{1}{\lambda(\xi_0)}. \tag{10}$$

Then T has a unique fixed point.

Proof. Consider the sequence $\{\xi_n = T\xi_{n-1}\}$ in X satisfying the hypotheses (9) and (10). From (8), we obtain

$$\begin{aligned}\omega(\xi_n,\xi_{n+1}) &= \omega(T\xi_{n-1},T\xi_n)\\ &\leq \lambda(\xi_{n-1})[\omega(\xi_{n-1},T\xi_{n-1}) + \omega(\xi_n,T\xi_n)]\\ &\leq \lambda(\xi_0)[\omega(\xi_{n-1},\xi_n) + \omega(\xi_n,\xi_{n+1})].\end{aligned}$$

Consider $a = \lambda(\xi_0)$. Then $\omega(\xi_n,\xi_{n+1}) \leq \frac{a}{1-a}\omega(\xi_{n-1},\xi_n)$. By induction, we get

$$\omega(\xi_n,\xi_{n+1}) \leq (\frac{a}{1-a})^n\omega(\xi_1,\xi_0), \ \ \forall n \geq 0. \tag{11}$$

Now, let us prove that $\{\xi_n\}$ is a Cauchy sequence. Using the triangle inequality, for all $n, m \in \mathbb{N}$ we obtain

$$\omega(\xi_n,\xi_m) \leq \delta(\xi_n,\xi_{n+1})\omega(\xi_n,\xi_{n+1}) + \delta(\xi_{n+1},\xi_m)\omega(\xi_{n+1},\xi_m).$$

Similar to the proof of Theorem 2.1, we get

$$\begin{aligned}\omega(\xi_n,\xi_m) &\leq \delta(\xi_n,\xi_{n+1})\omega(\xi_n,\xi_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\delta(\xi_j,\xi_m)\right)\delta(\xi_i,\xi_{i+1})\omega(\xi_i,\xi_{i+1})\\ &\quad + \prod_{k=n+1}^{m-1}\delta(\xi_k,\xi_m)\omega(\xi_{m-1},\xi_m)\\ &\leq \delta(\xi_n,\xi_{n+1})(\frac{a}{1-a})^n\omega(\xi_0,\xi_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\delta(\xi_j,\xi_m)\right)\delta(\xi_i,\xi_{i+1})(\frac{a}{1-a})^i\omega(\xi_0,\xi_1)\\ &\quad + \prod_{i=n+1}^{m-1}\delta(\xi_i,\xi_m)(\frac{a}{1-a})^{m-1}\omega(\xi_0,\xi_1).\end{aligned}$$

Since $0 \leq a < \frac{1}{2}$, we have $\frac{a}{1-a} \in (0,1)$. As in the proof of Theorem 2.1, we deduce that $\{\xi_n\}$ is a Cauchy sequence in the complete controlled metric space (X,ω). So there exists $u \in X$ as limit of $\{\xi_n\}$ in (X,ω). Assume that $Tu \neq u$. We have

$$\begin{aligned}0 < \omega(u,Tu) &\leq \delta(u,\xi_{n+1})\omega(u,\xi_{n+1}) + \delta(\xi_{n+1},Tu)\omega(\xi_{n+1},Tu)\\ &\leq \delta(u,\xi_{n+1})\omega(u,\xi_{n+1}) + \delta(\xi_{n+1},Tu)\lambda(\xi_n)[\omega(\xi_n,\xi_{n+1}) + \omega(u,Tu)]\\ &\leq \delta(u,\xi_{n+1})\omega(u,\xi_{n+1}) + \delta(\xi_{n+1},Tu)\lambda(\xi_0)[\omega(\xi_n,\xi_{n+1}) + \omega(u,Tu)].\end{aligned} \tag{12}$$

Passing to the limit in the both sides of (12) and making use of the condition (10), we deduce that $0 < \omega(u, Tu) < \omega(u, Tu)$, which is a contradiction. Hence $Tu = u$. To prove the uniqueness of the fixed point u, suppose that T has another fixed point v. Then

$$\begin{aligned}\omega(u, v) = \omega(Tu, Tv) &\leq \lambda(u)[\omega(u, Tu) + \omega(v, Tv)] \\ &= \lambda(u)[\omega(u, u) + \omega(v, v)] = 0.\end{aligned}$$

Therefore, $u = v$ and so T has a unique fixed point.
□

Theorem 2.3 is illustrated by the following example.

Example 2.4. *Consider $X = \{0, 1, 2\}$. Take the controlled metric type d defined as*

$$\omega(0, 1) = \frac{1}{2}, \quad \omega(0, 2) = \frac{11}{20}, \quad \omega(1, 2) = \frac{3}{20}.$$

Take $\delta : X \times X \to [1, \infty)$ to be symmetric and be defined by

$$\delta(0, 0) = \delta(1, 1) = \delta(2, 2) = \delta(1, 2) = 1, \quad \delta(0, 2) = 2, \quad \delta(0, 1) = \frac{3}{2}.$$

Given $T : X \to X$ as

$$T0 = 2 \quad \text{and} \quad T1 = T2 = 1.$$

Consider $\lambda : X \to [0, \frac{1}{2})$ as $\lambda(0) = \frac{99}{200}$ and $\lambda(1) = \frac{3}{10}$ and $\lambda(2) = \frac{49}{100}$. Then $\lambda \in \mathrm{B}$. Take $\xi_0 = 0$, so (9) is satisfied.

On the other hand, it is easy that (8) holds. All hypotheses of Theorem 2.3 hold, and so the mapping T has a unique fixed point, which is $u = 1$.

Now, we again give a response to an open question in [29], which is the study of a nonlinear Chatterjea type contraction via an auxiliary function $\lambda \in \mathrm{B}$.

Theorem 2.5. *Let $T \colon X \to X$ be such that*

$$\omega(Tx, Ty) \leq \lambda(x)[\omega(x, Ty) + \omega(y, Tx)], \tag{13}$$

for all $x, y \in X$, where $\lambda \in \mathrm{B}$. For $\xi_0 \in X$, take $\xi_n = T^n \xi_0$. Suppose that

$$\sup_{i \geq 1} \delta(\xi_{i-1}, \xi_i) = \beta \quad \text{exists and is finite,} \tag{14}$$

$$0 < \lambda(\xi_0) < \frac{1}{2\beta}, \tag{15}$$

and

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\delta(\xi_{i+1}, \xi_{i+2})}{\delta(\xi_i, \xi_{i+1})} \delta(\xi_{i+1}, \xi_m) < \frac{\beta \lambda(\xi_0)}{1 - \beta \lambda(\xi_0)}. \tag{16}$$

Also, assume that d is continuous with respect to its first variable, and for every $\xi \in X$, we have

$$\lim_{n \to \infty} \delta(\xi, \xi_n) \quad \text{exists, is finite and} \quad \lim_{n \to \infty} \delta(\xi_n, \xi) < \frac{1}{\lambda(\xi_0)}. \tag{17}$$

Then T has a unique fixed point.

Proof. Consider the sequence $\{\xi_n = T\xi_{n-1}\}$ in X satisfying the hypotheses (14), (15), (16) and (17). From (13) and (14), we obtain

$$\begin{aligned}\omega(\xi_n, \xi_{n+1}) &= \omega(T\xi_{n-1}, T\xi_n)\\ &\leq \lambda(\xi_{n-1})[\omega(\xi_{n-1}, T\xi_n) + \omega(\xi_n, T\xi_{n-1})]\\ &= \lambda(\xi_{n-1})\omega(\xi_{n-1}, \xi_{n+1})\\ &\leq \lambda(\xi_0)[\delta(\xi_{n-1}, \xi_n)\omega(\xi_{n-1}, \xi_n) + \delta(\xi_n, \xi_{n+1})\omega(\xi_n, \xi_{n+1})]\\ &\leq \beta\lambda(\xi_0)[\omega(\xi_{n-1}, \xi_n) + \omega(\xi_n, \xi_{n+1})].\end{aligned}$$

Consider $b = \frac{\beta\lambda(\xi_0}{1-\beta\lambda(\xi_0)}$. By (15), we have $b \in (0,1)$. Then $\omega(\xi_n, \xi_{n+1}) \leq b\omega(\xi_{n-1}, \xi_n)$. By induction, we get

$$\omega(\xi_n, \xi_{n+1}) \leq b^n\omega(\xi_0, \xi_1), \quad \forall n \geq 0. \tag{18}$$

Now, let us prove that $\{\xi_n\}$ is a Cauchy sequence. Using the triangle inequality, for all $n, m \in \mathbb{N}$ we obtain

$$\omega(\xi_n, \xi_m) \leq \delta(\xi_n, \xi_{n+1})\omega(\xi_n, \xi_{n+1}) + \delta(\xi_{n+1}, \xi_m)\omega(\xi_{n+1}, \xi_m).$$

Similar to the proof of Theorem 2.1, we get

$$\begin{aligned}\omega(\xi_n, \xi_m) &\leq \delta(\xi_n, \xi_{n+1})\omega(\xi_n, \xi_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\delta(\xi_j, \xi_m)\right)\delta(\xi_i, \xi_{i+1})\omega(\xi_i, \xi_{i+1})\\ &\quad + \prod_{k=n+1}^{m-1}\delta(\xi_k, \xi_m)\omega(\xi_{m-1}, \xi_m)\\ &\leq \delta(\xi_n, \xi_{n+1})(b^n\omega(\xi_0, \xi_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\delta(\xi_j, \xi_m)\right)\delta(\xi_i, \xi_{i+1})b^i\omega(\xi_0, \xi_1)\\ &\quad + \prod_{i=n+1}^{m-1}\delta(\xi_i, \xi_m)b^{m-1}\omega(\xi_0, \xi_1).\end{aligned}$$

As in the proof of Theorem 2.1, we deduce that $\{\xi_n\}$ is a Cauchy sequence in the complete controlled metric space (X, ω). So there exists $u \in X$ as limit of $\{\xi_n\}$ in (X, ω). Assume that $Tu \neq u$. We have

$$\begin{aligned}0 < \omega(u, Tu) &\leq \delta(u, \xi_{n+1})\omega(u, \xi_{n+1}) + \delta(\xi_{n+1}, Tu)\omega(\xi_{n+1}, Tu)\\ &\leq \delta(u, \xi_{n+1})\omega(u, \xi_{n+1}) + \delta(\xi_{n+1}, Tu)\lambda(\xi_n)[\omega(\xi_n, Tu) + \omega(u, \xi_{n+1})]\\ &\leq \delta(u, \xi_{n+1})\omega(u, \xi_{n+1}) + \delta(\xi_{n+1}, Tu)\lambda(\xi_0)[\omega(\xi_n, Tu) + \omega(u, \xi_{n+1})].\end{aligned} \tag{19}$$

Passing to the limit in the both sides of (19) and making use of the condition (17) and the continuity of d with respect to its first variable, we deduce that $0 < \omega(u, Tu) < \omega(u, Tu)$, which is a contradiction. Hence $Tu = u$.

To prove the uniqueness of the fixed point u, suppose that T has another fixed point v. Then

$$\begin{aligned}\omega(u, v) = \omega(Tu, Tv) &\leq \lambda(u)[\omega(u, Tv) + \omega(v, Tu)]\\ &= \lambda(u)[\omega(u, u) + \omega(v, v)] = 0.\end{aligned}$$

Therefore, $u = v$ and T has a unique fixed point.

□

Now, we introduce cyclical orbital contractions in the class of controlled metric type spaces.

Definition 2.6. *Let U and v be two non-empty subsets of a controlled metric type space (X, ω). Let $T : U \cup V \to U \cup V$ be a cyclic mapping (that is, $T(U) \subseteq V$ and $TV \subseteq U$) such that for some $x \in U$, there exists $k_x \in (0, 1)$ such that*

$$\omega(T^{2n}x, Ty) \leq k_x \omega(T^{2n-1}x, y), \tag{20}$$

where $n = 1, 2, \cdots$ and $y \in U$. Then T is called a controlled cyclic orbital contraction mapping.

In closing, we prove the following result.

Theorem 2.7. *Let U and V be two non-empty closed subsets of a complete controlled metric type space (X, ω). Let $T\colon X \to X$ be a controlled cyclic orbital contraction mapping For $\xi_0 \in U$, take $\xi_n = T^n \xi_0$. Suppose that*

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\delta(\xi_{i+1}, \xi_{i+2})}{\delta(\xi_i, \xi_{i+1})} \delta(\xi_{i+1}, \xi_m) < \frac{1}{k_{\xi_0}}. \tag{21}$$

Also, assume that for every $\xi \in X$, we have

$$\lim_{n \to \infty} \delta(\xi_n, \xi) \text{ and } \lim_{n \to \infty} \delta(\xi, \xi_n) \text{ exist and are finite.} \tag{22}$$

Then $U \cap V$ is non-empty and T has a unique fixed point in $U \cap V$.

Proof. Suppose there exists x (say ξ_0) in U satisfying (20). Define the iterative sequence $\{\xi_n = T^n \xi_0\}$. Since $\xi_0 \in U$ and T is cyclic, we have

$$\xi_{2n} \in U \quad \text{and} \quad \xi_{2n+1} \in V, \quad \text{for all } n \geq 0. \tag{23}$$

By using (20), we get

$$\omega(T^2 x, Tx) \leq k_x \omega(Tx, x).$$

Again

$$\omega(T^3 x, T^2 x) = \omega(T^2 x, T(T^2 x)) \leq k_x \omega(Tx, T^2 x) \leq (k_x)^2 \omega(Tx, x).$$

By induction, we obtain that

$$\omega(\xi_n, \xi_{n+1}) \leq [k_x]^n \, \omega(\xi_0, \xi_1) \quad \text{for all } n \geq 0. \tag{24}$$

Similarly to the proof of Theorem 2.1, we can easily deduce that

$$\lim_{n,m \to \infty} \omega(\xi_n, \xi_m) = 0, \tag{25}$$

that is, $\{\xi_n\}$ is a Cauchy sequence in the complete controlled metric type space (X, ω), so $\{\xi_n\}$ converges to some $u \in X$. Since $\{T^{2n}x\}$ is in U and U is closed, the limit u is in S_1. Similarly, $\{T^{2n-1}x\}$ is in the closed subset V, so $u \in V$. That is, $u \in U \cap V$, hence $U \cap V$ is not empty. Let us prove that u is a fixed point of T. We have

$$\omega(u, \xi_{n+1}) \leq \delta(u, \xi_n) \omega(u, \xi_n) + \delta(\xi_n, \xi_{n+1}) \omega(\xi_n, \xi_{n+1}).$$

Using (21), (22) and (25), we get that

$$\lim_{n \to \infty} \omega(u, \xi_{n+1}) = 0. \tag{26}$$

Using the triangle inequality and (20),

$$\begin{aligned}
\omega(u, Tu) &\leq \delta(u, T^{2n}x) \omega(u, T^{2n}x) + \delta(T^{2n}x, Tu) \omega(T^{2n}x, Tu) \\
&\leq \delta(u, T^{2n}x) \omega(u, T^{2n}x) + k_x \delta(T^{2n}x, Tu) \omega(T^{2n-1}x, u) \\
&= \delta(u, \xi_{n+1}) \omega(u, \xi_{n+1}) + k_x \delta(\xi_{n+1}, Tu) \omega(\xi_{2n-1}, u).
\end{aligned}$$

Taking the limit as $n \to \infty$ and by using (22) and (26), we deduce that $\omega(u, Tu) = 0$, that is, $Tu = u$. Finally, assume that T has two fixed points, say u and v (they are in U). Thus,

$$\omega(u, v) = \omega(Tu, Tv) = \omega(T^{2n}u, Tv) \leq k_u \omega(T^{2n-1}u, v) = k_u \omega(u, v),$$

which holds unless $\omega(u, v) = 0$, so $u = v$. Hence T has a unique fixed point. $\square$

The following example illustrates Theorem 2.7.

Example 2.8. *Let $X = U \cup V$ where $U = [\frac{1}{4}, \frac{1}{2}]$ and $V = [\frac{1}{2}, 1]$. Consider the controlled metric type d defined as*

$$\omega(x, y) = |x - y|^2,$$

where $\delta(x, y) = x + y + 1$ for $x, y \in X$. Take $Tx = \frac{1}{2}$ if $x \in U$ and $Tx = \frac{x}{2}$ if $x \in V \setminus \{\frac{1}{2}\}$. For $x \in U$, take $k_x = \frac{x+1}{2}$. Note that for all $x, y \in U$ and $n \geq 1$, we have

$$Ty = \frac{1}{2}, \quad T^{2n-1}x = \frac{1}{2} \quad and \quad T^{2n}x = \frac{1}{2}.$$

We deduce that (20) holds. It is not difficult to see that T satisfies all the hypotheses of Theorem 2.7. Therefore, T has a unique fixed point, which is $u = \frac{1}{2} \in U \cap V$.

Perspectives

It is an open question to treat the cases of Hardy-Rogers contraction, Ćirić contraction and Suzuki contraction.

Acknowledgements

The first and last authors would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Authors' contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

Conflicts of interest

The authors declare no conflict of interest.

Funding

This research received no external funding.

References

[1] T. Abdeljawad, R.P. Agarwal, E. Karapınar, P.S. Kumari, Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space, Symmetry, 11 (2019), 686.
[2] T. Abdeljawad, F. Jarad, D. Baleanu, On the existence and the uniqueness theorem for fractional differential equations with bounded delay within Caputo derivatives, Science in China, Mathematics, 51 (10) (2008), 1775-1786.
[3] T. Abdeljawad, D. Baleanu, F. Jarad, Existence and uniqueness theorem for a class of delay differential equations with left and right Caputo fractional derivatives, J. Math. Phys. 49 (8), (2008).
[4] T. Abdeljawad, Meir-Keeler contractive fixed and common fixed point theorems, Fixed Point Theory Appl. 2013, 2013:19.
[5] K. Abodayeh, N. Mlaiki, T. Abdeljawad, W. Shatanawi, Relations between partial metric spaces and M-metric spaces, Caristi Kirk's Theorem in M-metric type spaces, Journal of Mathematical Analysis, 7 (3) (2016), 1-12.
[6] T. Abdeljawad, K. Abodayeh, N. Mlaiki, On fixed point generalizations to partial b-metric spaces, Journal Comput. Anal. Appl. 19 (5) (2015), 883-891.
[7] H. Afshari, M. Atapour, H. Aydi, Generalized $\alpha - \psi$-Geraghty multivalued mappings on b-metric spaces endowed with a graph, TWMS Journal of Applied and Engineering Mathematics, 7 (2) (2017), 248-260.

[8] N. Alharbi, H. Aydi, A. Felhi, C. Ozel, S. Sahmim, α-contractive mappings on rectangular b-metric spaces and an application to integral equations, Journal of Mathematical Analysis, 9 (3) (2018), 47-60.

[9] B. Alqahtani, E. Karapinar, A. Ozturk On $(\alpha,\psi)-K$-contractions in the extended b-metric space, Filomat, 32 (15) (2018), 5337-5345.

[10] B. Alqahtani, A. Fulga, E. Karapinar, Common fixed point results on extended b-metric space, Journal of Inequalities and Applications, 2018, 2018:158.

[11] B. Alqahtani, A. Fulga, E. Karapinar, Non-unique fixed point results in extended b-metric space, Mathematics, 6 (5) (2018), 68.

[12] E. Ameer, M. Arshad, W. Shatanawi, Common fixed point results for generalized $\delta_* - \psi-$contraction multivalued mappings in $b-$metric spaces, J. Fixed Point Theory Appl. Volume 19 (2017), Issue 4, pp 3069-3086.

[13] E. Ameer, H. Aydi, M. Arshad, H. Alsamir, M.S. Noorani, Hybrid multivalued type contraction mappings in δ_K-complete partial b-metric spaces and applications, Symmetry 2019, 11(1), 86.

[14] H. Aydi, E. Karapinar, M.F. Bota, S. Mitrović, A fixed point theorem for set-valued quasi-contractions in b-metric spaces, Fixed Point Theory Appl. 2012, 2012:88.

[15] H. Aydi, M.F. Bota, E. Karapinar, S. Moradi, A common fixed point for weak ϕ-contractions on b-metric spaces, Fixed Point Theory, 13 (2) (2012), 337-346.

[16] H. Aydi, R. Banković, I. Mitrović, M. Nazam, Nemytzki-Edelstein-Meir-Keeler type results in b-metric spaces, Discrete Dynamics in Nature and Society, Volume 2018, Article ID 4745764, 7 pages.

[17] H. Aydi, A. Felhi, T. Kamran, E. Karapinar, M.U. Ali, On nonlinear contractions in new extended b-metric spaces, Application and Applied Mathematics: An International Journal (AAM) 14 (1) (2019), 537 -547.

[18] I.A. Bakhtin, The contraction mapping principle in almost metric spaces, Funct. Anal. 30 (1989), 26-37.

[19] T.L. Hicks, B.E. Rhoedes, A Banach type fixed point theorem, Math. Japan. 24 (1979), 327-330.

[20] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostra. 1 (1993), 5-11.

[21] T. Kamran, M. Samreen, Q. UL Ain, A Generalization of b-metric space and some fixed point theorems, Mathematics, 5 (19) (2017), 1-7.

[22] E. Karapınar, S. Czerwik, H. Aydi, (α,ψ)-Meir-Keeler contraction mappings in generalized b-metric spaces, Journal of Function spaces, Volume 2018 (2018), Article ID 3264620, 4 pages.

[23] E. Karapinar, S.K. Panda, D. Lateef, A new approach to the solution of Fredholm integral equation via fixed point on extended b-metric spaces, Symmetry, 10 (2018), 512.

[24] A.A. Kilbas, M.H. Srivastava, J.J. Trujillo, Theory and Application of Fractional Differential Equations, North Holland Mathematics Studies, 204, (2006).

[25] M.A. Khamsi, W.A. Kirk, An introduction to metric spaces and fixed point theory, John Willey and Sons, INC. 1996.

[26] W. Kirk, N. Shahzad, Fixed point theory in distance spaces, Springer, Cham, 2014, http://dx.doi.org/10.1007/978-3-319-10927-5.

[27] P. Kumrod, W. Sintunavarat, An improvement of recent results in M-metric spaces with numerical results, Journal of Mathematical Analysis, 8 (6) (2017), 202-213.

[28] H.R. Marasi, H. Piri, H. Aydi, Existence and multiplicity of solutions for nonlinear fractional differential equations, J. Nonlinear Sci. Appl. 9 (2016), 4639-4646.

[29] N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric type spaces and the related contraction principle, Mathematics, 2018, 6(10), 194.

[30] N. Mlaiki, A. Mukheimer, Y. Rohen, N. Souayah, T. Abdeljawad, Fixed point theorems for $\alpha-\psi-$contractive mapping in S_b-metric spaces, Journal of Mathematical Analysis, 8 (5) (2017), 40-46.

[31] D.K. Patel , T. Abdeljawad, D. Gopal, Common fixed points of generalized Meir-Keeler contractions, Fixed Point Theory Appl. 2013, 2013:260.

[32] J.R. Roshan, V. Parvaneh, Sh. Sedghi, N. Shobkolaei, W. Shatanawi, Common fixed points of almost generalized $(\psi,\varphi)_s-$contractive mappings in ordered $b-$metric spaces, Fixed Point Theory Appl. 2013, 2013:159.

[33] B. Samet, C. Vetro, P. Verto, Fixed point theorems for $\alpha-\psi-$contractive type mappings, Nonlinear Anal. 75 (2012), 2154-2165.

[34] B. Samet, H. Aydi, On some inequalities involving Caputo fractional derivatives and applications to special means of real numbers, Mathematics, 2018, 6(10), 193.

[35] W. Sintunavarat, Y. J. Cho, P. Kumam, Common fixed point theorems for c-distance in ordered cone metric spaces, Computers and Mathematics with Applications, 62 (2011), 1969–1978.

[36] N. Souayah, N. Mlaiki, M. Mrad, The G_M-Contraction Principle for Mappings on $M-$Metric Spaces Endowed With a Graph and Fixed Point Theorems, IEEE Access, 6 (2018), 25178 - 25184.

[37] N. Souayah, N. Mlaiki, A fixed point theorem in S_b-metric spaces, J. Math. Computer Sci. 16 (2016), 131-139.

[38] W. Shatanawi, Fixed and common fixed point for mapping satisfying some nonlinear contraction in b-metric spaces, Journal of Mathematical Analysis, 7 (4) (2016), 1–12.

[39] W. Shatanawi, A. Pitea, Lazovic, Contraction conditions using comparison functions on b-metric spaces, Fixed Point Theory Appl. 2014, 2014:135.