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Univalence Criteria for a General Integral Operator

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Abstract. For some classes of analytic functions f and g in the open unit disk $\mathbb{U}$, we consider the general integral operator $\mathcal{M}_n$, that was introduced in a recent work [2] and we obtain new conditions of univalence for this integral operator. The key tools in the proofs of our results are the Pascu's and the Pescar's univalence criteria. Some corollaries of the main results are also considered. Relevant connections of the results presented here with various other known results are briefly indicated.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} :| z |< 1\}$$

and satisfy the following usual normalization conditions:

$$f(0) = f^{'}(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

In [24] Silverman define the class $\mathcal{G}_b$. Precisely, for $0 < b \leq 1$ he considered the class

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < b \left| \frac{zf'(z)}{f(z)} \right|, \quad z \in \mathbb{U} \right\}. \tag{2}$$

For some interesting investigations regarding sufficient conditions for univalence of various families of integral operators see the work by (for example) Breaz et al.[7], Deniz et al. [8], Frasin [9], Stanciu et al. [25] and Oprea et al [14]. We consider the integral operators

$$\mathcal{M}_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} (g_i{}'(t))^{\beta_i} \left(\frac{g_i(t)}{t} \right)^{\gamma_i} \right] \mathrm{d}t \right\}^{\frac{1}{\delta}}, \tag{3}$$

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where f_i, g_i are analytic in $\mathbb{U}$, and $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ for all $i = \overline{1,n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\text{Re}\delta > 0$.

Remark 1.1. *The integral operator $\mathcal{M}_n$ defined by (3), introduced by Bărbatu and Breaz in the paper [2], is a general integral operator of Pfaltzgraff, Kim-Merkes and Ovesea types which extends also the other operators as follows:*

i) For $n = 1$, $\delta = 1$, $\alpha_i - 1 \equiv \alpha_i$ and $\beta_1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [11]

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt.$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [23]

$$\mathcal{G}_\alpha(z) = \int_0^z (f'(t))^\alpha dt.$$

iii) For $\alpha_i - 1 \equiv \alpha_i$ and $\beta_i = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [5]

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [18].

iv) For $\alpha_i - 1 = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [6]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left[f_i'(t)\right]^{\alpha_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [22] .

v) For $\alpha_i - 1 \equiv \alpha_i$ and $\gamma_i = 0$ we obtain the integral operator which was defined and studied by Pescar in [20]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} (f_i'(t))^{\beta_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Frasin in [10] and by Ovesea in [15].

vi) For $\alpha_i - 1 \equiv \alpha_i$ and $\gamma_i = 0$ we obtain the integral operator which was studied by Ularu in [26]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} (g_i'(t))^{\beta_i} dt\right]^{\frac{1}{\delta}}.$$

Thus, the integral operator $\mathcal{M}_n$, introduced here by the formula (3), can be considered as an extension and a generalization of these operators above mentioned.

In the present paper, we derive the univalence conditions for the integral operator $\mathcal{M}_n$, when $g_i \in G_{b_i}$ and $f_i \in \mathcal{A}$ for all $i = \overline{1,n}$.

The following univalence conditions were derived by Pascu [13], [14] and Pescar in [16]. These are extensions of some very well-known and important univalence criteria for analytic functions defined in the open unit disk $\mathbb{U}$ that have been obtained by Ahlfors [1] and Becker [4] and by Becker (see [3]).

Theorem 1.2. *(Pascu [16]) Let $f \in \mathcal{A}$ and $\gamma \in \mathbb{C}$. If $Re\gamma > 0$ and*

$$\frac{1 - |z|^{2Re\gamma}}{Re\gamma} \left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t) dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Theorem 1.3. *(Pascu [17]) Let $\delta \in \mathbb{C}$ with $Re\delta > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2Re\delta}}{Re\delta}\left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then, for any complex γ with $Re\gamma \geq Re\delta$, the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Theorem 1.4. *(Pescar [19]) Let γ be complex number, $Re\gamma > 0$ and c a complex number, $|c| \leq 1$, $c \neq -1$, and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + ...$ If*

$$\left|c\,|z|^{2\gamma} + \left(1-|z|^{2\gamma}\right)\frac{zf''(z))}{\gamma f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma [12].

Lemma 1.5. *(General Schwarz Lemma [12]) Let f be the function regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R, R > 0\}$ with $\left|f(z)\right| < M$ for a fixed number $\mathrm{M} > 0$ fixed. If $f(z)$ has one zero with multiplicity order bigger than a positive integer m for $z = 0$, then*

$$\left|f(z)\right| \leq \frac{M}{R^m}z^m, \quad z \in \mathbb{U}_R.$$

The equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

2. The main univalence criteria

Our main results give sufficient conditions for the general integral operator $\mathcal{M}_n$ defined by (3) to be univalent in the open disk $\mathbb{U}$.

Theorem 2.1. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$, with*

$$c \geq \sum_{i=1}^{n}\left[|\alpha_i - 1| + (2b_i+1)\left|\beta_i\right| + \left|\gamma_i\right|\right]. \tag{4}$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < 1, \quad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1, \tag{5}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{M}_n$, defined by (3) is in the class $\mathcal{S}$.

Proof. We define the function

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} (g_i'(t))^{\beta_i} \left(\frac{g_i(t)}{t}\right)^{\gamma_i} \mathrm{dt},$$

$g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ so, that obviously

$$H_n'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} (g_i'(z))^{\beta_i} \left(\frac{g_i(z)}{z}\right)^{\gamma_i}.$$

The function H_n is regular in $\mathbb{U}$ and satisfy the following usual normalization conditions

$$H_n(0) = H_n^{'}(0) - 1 = 0$$

and

$$\frac{zH_n''(z)}{H_n'(z)} = \sum_{i=1}^{n} \left[(\alpha_i - 1)\left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} + \gamma_i \left(\frac{zg_i'(z)}{g_i(z)} - 1\right)\right] =$$

$$= \sum_{i=1}^{n} \left[(\alpha_i - 1)\left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \beta_i \left(\frac{zg_i''(z)}{g_i'(z)} - \frac{zg_i'(z)}{g_i(z)} + 1\right) + \beta_i \left(\frac{zg_i'(z)}{g_i(z)} - 1\right) + \gamma_i \left(\frac{zg_i'(z)}{g_i(z)} - 1\right)\right]. \tag{6}$$

Since $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$ for all $i = \overline{1,n}$ from (2), (5) and (6), we obtain

$$\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \sum_{i=1}^{n} \left[|\alpha_i - 1|\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \left|\beta_i\right|\left|\frac{zg_i''(z)}{g_i'(z)} - \frac{zg_i'(z)}{g_i(z)} + 1\right| + \left|\beta_i\right|\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + \left|\gamma_i\right|\left|\frac{zg_i'(z)}{g_i(z)} - 1\right|\right] \leq$$

$$\leq \sum_{i=1}^{n} \left[|\alpha_i - 1|\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \left|\beta_i\right| b_i \left|\frac{zg_i'(z)}{g_i(z)}\right| + \left|\beta_i\right|\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + \left|\gamma_i\right|\left|\frac{zg_i'(z)}{g_i(z)} - 1\right|\right] \leq$$

$$\leq \sum_{i=1}^{n} \left[|\alpha_i - 1|\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \left|\beta_i\right| b_i \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + \left|\beta_i\right| b_i + \left|\beta_i\right|\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + \left|\gamma_i\right|\left|\frac{zg_i'(z)}{g_i(z)} - 1\right|\right] \leq$$

$$\leq \sum_{i=1}^{n} \left[|\alpha_i - 1|\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \left(\left|\beta_i\right| b_i + \left|\beta_i\right| + \left|\gamma_i\right|\right)\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + \left|\beta_i\right| b_i\right] \leq$$

$$\leq \sum_{i=1}^{n} \left[|\alpha_i - 1| + \left|\beta_i\right| b_i + \left|\beta_i\right| + \left|\gamma_i\right| + \left|\beta_i\right| b_i\right] \leq \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1)\left|\beta_i\right| + \left|\gamma_i\right|\right], \tag{7}$$

which readily shows that

$$\frac{1 - |z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \frac{1 - |z|^{2c}}{c}\left(\sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1)\left|\beta_i\right| + \left|\gamma_i\right|\right]\right) \leq$$

$$\leq \frac{1}{c}\left(\sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1)\left|\beta_i\right| + \left|\gamma_i\right|\right]\right) \leq 1. \tag{8}$$

By Theorem 1.2 it results that the integral operator $\mathcal{M}_n$ given by (3) is in the class $\mathcal{S}$. □

Theorem 2.2. *Let $\alpha_i, \beta_i, \gamma_i$ be complex numbers, $M_i \geq 1$, $N_i \geq 1$ are real numbers, for all $i = \overline{1,n}$ and $\gamma \in \mathbb{C}$ with $c = Re\gamma$*

$$c \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| (2M_i + 1) + \left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right| \right) (2N_i + 1) + b_i \left|\beta_i\right| \right]. \tag{9}$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ satisfy

$$\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| < 1, \quad \left| f_i(z) \right| \leq M_i, \quad \left| g_i(z) \right| \leq N_i, \tag{10}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then for any complex number δ with $Re\delta \geq Re\gamma$, the integral operator $\mathcal{M}_n$, given by (3) is in the class $\mathcal{S}$.

Proof. From the proof of Theorem 2.1, we have:

$$\left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \sum_{i=1}^{n} \left[|\alpha_i - 1| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right| \right) \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| + b_i \left|\beta_i\right| \right].$$

Thus, we obtain

$$\frac{1-|z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + b_i \left|\beta_i\right| + \left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right| \right) \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \right] \leq$$

$$\leq \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| \left(\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} \right| \left| \frac{f_i(z)}{z} \right| + 1 \right) + \left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right| \right) \left(\left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} \right| \left| \frac{g_i(z)}{z} \right| + 1 \right) + b_i \left|\beta_i\right| \right]. \tag{11}$$

Since $\left|f_i(z)\right| \leq M_i$, $\left|g_i(z)\right| \leq N_i$, $z \in \mathbb{U}$, $i = \overline{1,n}$ and for each f_i, g_i satisfy conditions (10), then applying General Schwarz Lemma, we obtain $\left|f_i(z)\right| \leq M_i |z|$, $\left|g_i(z)\right| \leq N_i |z|$ for all $z \in \mathbb{U}$, $i = \overline{1,n}$. Using these inequalities from (11) we have

$$\frac{1-|z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| \left(\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| M_i + M_i + 1 \right) \right] +$$

$$+ \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[\left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right| \right) \left(\left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| N_i + N_i + 1 \right) + b_i \left|\beta_i\right| \right] \leq$$

$$\leq \frac{1}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| (2M_i + 1) + \left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right| \right) (2N_i + 1) + b_i \left|\beta_i\right| \right],$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and from the hypothesis, we get

$$\frac{1-|z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq 1, \qquad z \in \mathbb{U}.$$

Applying Theorem 1.3. for the function H_n, we prove that $\mathcal{M}_n$ is in the class $\mathcal{S}$. □

Theorem 2.3. *Let $\alpha_i, \beta_i, \gamma_i$ be complex numbers and $\delta \in \mathbb{C}$ with*

$$Re\delta \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1) \left|\beta_i\right| + \left|\gamma_i\right| \right], \tag{12}$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1) \left|\beta_i\right| + \left|\gamma_i\right| \right]. \tag{13}$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ *and*

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| < 1, \tag{14}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the integral operator* $\mathcal{M}_n$, *given by (3) is in the class* $\mathcal{S}$.

Proof. From (7), we deduce that

$$\left| c\,|z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{zH_n''(z)}{\delta H_n'(z)} \right| \leq |c| + \left| \frac{1 - |z|^{2\delta}}{\delta} \right| \left| \frac{zH_n''(z)}{\delta H_n'(z)} \right| \leq |c| + \left| \frac{1 - |z|^{2\delta}}{\delta} \right| \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1) \left|\beta_i\right| + \left|\gamma_i\right| \right] \leq$$

$$\leq |c| + \frac{1}{|\delta|} \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1) \left|\beta_i\right| + \left|\gamma_i\right| \right] \leq |c| + \frac{1}{Re\delta} \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1) \left|\beta_i\right| + \left|\gamma_i\right| \right] \leq 1.$$

Finally, by applying Theorem 1.4, we conclude that $\mathcal{M}_n$ defined by (3) is in the class $\mathcal{S}$. □

Theorem 2.4. *Let* $\alpha_i, \beta_i, \gamma_i$ *be complex numbers,* $M_i \geq 1$, $N_i \geq 1$ *are real numbers and* $\delta \in \mathbb{C}$ *with*

$$Re\delta \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| (2M_i + 1) + \left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right|\right) (2N_i + 1) + b_i \left|\beta_i\right| \right], \tag{15}$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[|\alpha_i - 1| (2M_i + 1) + \left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right|\right) (2N_i + 1) + b_i \left|\beta_i\right| \right]. \tag{16}$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ *satisfy*

$$\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| < 1, \tag{17}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the integral operator* $\mathcal{M}_n$, *given by (3) is in the class* $\mathcal{S}$.

Proof. From the proof of Theorem 2.3, we have

$$\left| c\,|z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{zH_n''(z)}{\delta H_n'(z)} \right| \leq |c| + \frac{1}{Re\delta} \sum_{i=1}^{n} \left[|\alpha_i - 1| (2M_i + 1) + \left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right|\right) (2N_i + 1) + b_i \left|\beta_i\right| \right],$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and from the hypothesis, we get

$$\left| c\,|z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{zH_n''(z)}{\delta H_n'(z)} \right| \leq 1.$$

Applying Theorem 1.4 for the function H_n, we prove that $\mathcal{M}_n$ is in the class $\mathcal{S}$. □

3. Corollaries and consequences

First of all, upon setting $\delta = 1$ in Theorem 2.1, we immediately arrive at the following corollary:

Corollary 3.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, with*

$$c \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1) \left|\beta_i\right| + \left|\gamma_i\right| \right].$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{M}_n^*$, defined by

$$\mathcal{M}_n^*(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} (g_i{}'(t))^{\beta_i} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{18}$$

is in the class $\mathcal{S}$.

Letting $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 3.2. *Let $\gamma, \alpha_i, \beta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, with*

$$c \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1) \left|\beta_i\right| \right].$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{F}_n$, defined by

$$\mathcal{F}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} (g_i{}'(t))^{\beta_i} \right] dt \tag{19}$$

is in the class $\mathcal{S}$.

Remark 3.3. *The integral operator given by (19) is a known result proven in [26].*

Letting $\delta = 1$ and $\beta_i = 0$ in Theorem 2.1, we have the following corollary:

Corollary 3.4. *Let $\gamma, \alpha_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, with*

$$c \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + \left|\gamma_i\right| \right].$$

If $f_i, g_i \in \mathcal{A}$ and

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{G}_n$, defined by

$$\mathcal{G}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{20}$$

is in the class $\mathcal{S}$.

Remark 3.5. *The integral operator given by (20) is another known result proven in [13].*

Putting $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 3.6. *Let $\gamma, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, with*

$$c \geq \sum_{i=1}^{n} \left[(2b_i + 1) \left|\beta_i\right| + \left|\gamma_i\right| \right].$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$ and

$$\left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{I}_n$, defined by

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left[(g_i{}'(t))^{\beta_i} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{21}$$

is in the class $\mathcal{S}$.

Remark 3.7. *The integral operator from (21) was proven in [20].*

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_1 - 1 = \beta_1 = \gamma_1$ in Theorem 2.1, we obtain the next corollary:

Corollary 3.8. *Let α be complex number, $Re\alpha > 0$, with*

$$Re\alpha \geq |\alpha - 1| (2b + 3).$$

If $g \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ and

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < 1, \quad \left| \frac{z g'(z)}{g(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, defined by

$$\mathcal{M}(z) = \left\{ \alpha \int_0^z \left[f(t) g'(t) \frac{g(t))}{t} \right]^{\alpha - 1} dt \right\}^{\frac{1}{\alpha}}, \tag{22}$$

is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$, $\alpha_1 - 1 = \beta_1 = \gamma_1$ and $b_1 = b$ in Theorem 2.2, we obtain the following corollary:

Corollary 3.9. *Let α be complex number, $M \geq 1$, $N \geq 1$, $Re\alpha > 0$ with*

$$Re\alpha \geq |\alpha - 1| (2M + 2bN + 4N + 2b + 3).$$

If $g \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ satisfy

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| < 1, \quad \left|f(z)\right| \leq M, \quad \left|g(z)\right| \leq N$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, given by (22) is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$, $\alpha_1 - 1 = \beta_1 = \gamma_1$ and $b_1 = b$ in Theorem 2.3, we obtain the following corollary:

Corollary 3.10. *Let $\alpha \in \mathbb{C}^*$ with*

$$Re\alpha \geq |\alpha - 1| (2b + 3),$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\alpha} |\alpha - 1| (2b + 3).$$

If $g \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad \left| \frac{zg'(z)}{g(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, given by (22) is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$, $\alpha_1 - 1 = \beta_1 = \gamma_1$, $b_1 = b$ in Theorem 2.4, we obtain the following corollary:

Corollary 3.11. *Let $\alpha \in \mathbb{C}^*$, $M \geq 1$, $N \geq 1$ with*

$$Re\alpha \geq |\alpha - 1| (2M + 2bN + 4N + 2b + 3),$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\alpha} |\alpha - 1| (2M + 2bN + 4N + 2b + 3).$$

If $g \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ satisfy

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| < 1$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, given by (22) is in the class $\mathcal{S}$.

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