



## Hartman–Wintner and Lyapunov-Type Inequalities for High Order Fractional Boundary Value Problems

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**Abstract.** In this paper, we obtain Hartman-Wintner and Lyapunov-type inequalities for the three-point fractional boundary value problem of the fractional Liouville-Caputo differential equation of order  $\alpha \in (2, 3]$ . The results presented in this work are sharper than the existing results in the literature. As an application of the results, the fractional Sturm–Liouville eigenvalue problems have also been presented. Moreover, we examine the nonexistence of the nontrivial solution of the fractional boundary value problem.

### 1. Introduction

The celebrated Lyapunov inequality [9] asserts that if  $p \in C([a, b], \mathbb{R})$  and  $y(x)$  is a nontrivial solution of the boundary value problem

$$\begin{aligned} y''(x) + p(x)y(x) &= 0, \quad x \in [a, b] \\ y(a) = y(b) &= 0, \end{aligned} \tag{1}$$

then the following integral inequality holds;

$$\int_a^b |p(s)| ds > \frac{4}{b-a}. \tag{2}$$

The constant 4 is sharp in the sense that it cannot be changed by a bigger number. There have been many extensions and generalizations of Lyapunov-type inequality over the last decades. This inequality and its generalizations and extensions are very useful in various mathematical problems involving oscillation and eigenvalue problems [5]. An interesting and important extension of this inequality has been established by Hartman and Wintner [5]. The authors proved that if  $y$  is a nontrivial solution of the boundary value problem (1), then the following inequality holds;

$$\int_a^b (b-s)(s-a)p_+(s) ds > b-a, \quad s \in (a, b),$$

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where  $p_+$  is the nonnegative part of the continuous function  $p$ . An extension of Lyapunov-type inequality to the third order linear differential equation has been obtained by Parhi and Panigrahi [12]. The authors showed that if  $y(x)$  is a nontrivial solution of the following third order differential equation

$$y'''(x) + p(x)y(x) = 0, \quad x \in [a, c], \tag{3}$$

$$y(a) = y(b) = y(c) = 0, \quad a < b < c, \tag{4}$$

then

$$\int_a^b |p(s)| ds > \frac{4}{(b-a)^2}. \tag{5}$$

The inequality (5) was also derived for the third order differential equation (3) subject to the boundary conditions  $y(a) = y(c) = y''(d) = 0$  for some  $d \in (a, c)$ . However, the boundary condition (4) implies that for some  $d \in (a, c)$ ,  $y''(d) = 0$ . Recently, Lyapunov-type inequality (5) has been improved by Dhar and Kong [2]. They established Lyapunov-type inequalities for the third order linear differential equation (3) with boundary condition (4) as follows (Corollary 2.1 [2])

$$\int_a^c p_+(s) ds > \frac{8}{(c-a)^2},$$

$$\int_a^c p_-(s) ds > \frac{8}{(c-a)^2},$$

$$\int_a^b p_-(s) ds + \int_b^c p_+(s) ds > \frac{8}{(c-a)^2}.$$

As a result,

$$\int_a^c |p(s)| ds > \frac{8}{(c-a)^2}. \tag{6}$$

The inequality (6) is obviously sharper than (3) and, to the best of our knowledge, the constant 8 is the best constant in the literature.

More recently, Lyapunov-type inequalities have been studied and extended for the fractional boundary value problems. Ferreira [3] proved that if the fractional boundary value problem of the Riemann-Liouville differential equation has a nontrivial solution

$$D_a^\alpha u(x) + p(x)y(x) = 0, \quad x \in [a, b], \quad \alpha \in (1, 2), \tag{7}$$

$$u(a) = u(b) = 0, \tag{8}$$

where  $p$  is a continuous function on  $[a, b]$ , then the following inequality is satisfied

$$\int_a^b |p(s)| ds > \Gamma(\alpha) \left( \frac{4}{b-a} \right)^{\alpha-1}. \tag{9}$$

The author changed the Riemann-Liouville derivative [8] by the Liouville-Caputo derivative  ${}^C D_a^\alpha$  and investigated the interval of the nonexistence of real zeros of some Mittag-Leffler functions [4]. Some generalizations and applications of Lyapunov-type inequalities for the fractional boundary value problems with different boundary conditions have been studied in the literature. The problem (7) subject to the fractional boundary condition in [14], a Robin boundary condition in [6], a mixed boundary condition in [7] has been considered and a class of fractional boundary value problems has been studied in [11], [15].

Jleli and Samet [7] considered the fractional boundary value problem

$${}^C D_a^\alpha u(x) + p(x)y(x) = 0, \quad x \in [a, b], \quad \alpha \in (1, 2]$$

subject to the mixed boundary condition

$$u'(a) = y(b) = 0, \tag{10}$$

and obtained the following Lyapunov-type inequality

$$\int_a^b (b-s)^{\alpha-1} |p(s)| ds \geq \Gamma(\alpha). \tag{11}$$

Very recently, Cabrera et al [1] established new Hartman-Wintner-type inequalities for a class of fractional boundary value problem involving nonlocal boundary conditions

$$\begin{aligned} D_a^\alpha y(x) + p(x)y(x) &= 0, \quad x \in [a, b], \\ y(a) = y'(a) &= 0, \quad y'(b) = \gamma y(c), \end{aligned} \tag{12}$$

where  $D_a^\alpha$  is the Riemann-Liouville fractional derivative [8] of order  $\alpha \in (2, 3]$  and  $p \in C([a, b], \mathbf{R})$  with  $a < c < b$  and  $0 \leq \gamma(c-a)^{(\alpha-1)} < (\alpha-1)(b-a)^{\alpha-2}$ . They proved the following Hartman-Wintner-type inequality if the problem (12) has a nontrivial solution;

$$\int_a^b (b-s)^{\alpha-2} (s-a) |p(s)| ds \geq K\Gamma(\alpha), \tag{13}$$

where

$$K = \left( 1 + \frac{\gamma(b-a)^{\alpha-1}}{(\alpha-a)(b-a)^{\alpha-2} - \gamma(c-a)^{\alpha-1}} \right)^{-1}.$$

A similar work has been presented by Ma [10] for the fractional boundary value problem of order  $\alpha \in (2, 3]$ :

$$\begin{aligned} {}^C D_a^\alpha y(x) + p(x)y(x) &= 0, \quad x \in [a, b], \\ y(a) = y''(a) &= y(b) = 0. \end{aligned} \tag{14}$$

The author has demonstrated that if the problem (14) has a positive solution, then the following Lyapunov-type inequality holds:

$$\int_a^b |p(s)| ds > \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}. \tag{15}$$

The results related to Hartman-Wintner-type and Lyapunov-type inequalities are very limited in the literature for the high order fractional boundary problems. Fill this gap, Hartman-Wintner-type inequalities will be established for the following boundary fractional boundary value problem involving the Liouville-Caputo fractional derivative of order  $\alpha \in (2, 3]$  in this paper;

$$\begin{aligned} {}^C D_a^\alpha u(x) + p(x)u(x) &= 0, \quad a \leq x \leq c, \alpha \in (2, 3], \\ u(a) = u(b) &= u(c) = 0, \quad a < b < c, \end{aligned} \tag{16}$$

where  ${}^C D_0^\alpha$  is the Liouville-Caputo derivative [8] and  $p \in C([a, c], \mathbf{R})$ .

The best method used so far for finding Lyapunov-type inequality for the fractional differential equations is to convert the fractional boundary value problems to an equivalent integral equation and then finding the maximum of the corresponding Green's function. However, finding the maximum value of the Green's function for the fractional differential equation based on Liouville-Caputo derivative is not easy task. As an alternative, we follow a little different approach and achieve a similar maximum value for Green's function on an appropriate interval. To the best of our knowledge, this is the first result on Hartman-Wintner-type

and Lyapunov-type inequalities for the three-point boundary value problem of the high order fractional differential equation.

The inequalities obtained here are sharper than the Hartman-Wintner-type inequality (13) for the fractional boundary value problem (16) and consequently we provide the best constant for Lyapunov-type inequalities in the literature for the high order fractional boundary value problem. For the convenience of the discussion, we first introduce some definitions and lemmas that will be needed in this work. For detailed explanations for the fractional differential settings, we refer the reader to the books [8] and [13].

**Definition 1.1.** (Riemann-Liouville [8], [13]) Let  $\alpha \in (n, n + 1]$ ,  $n \in \mathbf{N}$  and  $u \in C([a, b])$ ,  $a < b$ . The Riemann-Liouville fractional derivative of order  $\alpha$  of the function  $u$  defined as

$$D_a^\alpha u(x) = \frac{1}{\Gamma(n + 1 - \alpha)} \frac{d^n}{dx^n} \int_a^x (x - t)^{n-\alpha} f(t) dt. \tag{17}$$

**Definition 1.2.** (Liouville-Caputo [8], [13]) Let  $\alpha \in (n, n + 1]$ ,  $n \in \mathbf{N}$  and  $u \in C^{n+1}(a, b)$ ,  $a < b$ . The Liouville-Caputo fractional derivative of order  $\alpha$  of the function  $u$  defined by

$${}^C D_a^\alpha u(x) = \frac{1}{\Gamma(n + 1 - \alpha)} \int_a^x (x - t)^{n-\alpha} f^{(n+1)}(t) dt. \tag{18}$$

**Lemma 1.3.** (Ferreira [4], Lemma 1.)  $y \in C[a, b]$  is a solution of the following fractional boundary value problem

$$\begin{aligned} {}^C D_a^\beta y(x) + p(x)y(x) &= 0, \quad a \leq x \leq b, \quad \beta \in (1, 2], \\ y(a) = y(b) &= 0, \end{aligned}$$

if and only if  $y$  satisfies the integral equation

$$y(x) = \int_a^b G(x, s)p(s)y(s) ds,$$

where

$$G(x, s) = \frac{1}{\Gamma(\beta)} \begin{cases} \frac{x-a}{b-a}(b-s)^{\beta-1} - (x-s)^{\beta-1} & a \leq s \leq x \leq b, \\ \frac{x-a}{b-a}(b-s)^{\beta-1} & a \leq x \leq s \leq b. \end{cases} \tag{19}$$

## 2. Main Results

In this section, we will present the necessary condition for the existence of solutions of the fractional differential equation (16) by establishing Lyapunov-type inequalities for the problem (16). To establish these inequalities, we first give an equivalent integral equation for the problem (16) and then we obtain Hartman-Wintner-type inequalities for the problem (16).

**Lemma 2.1.** Let  $\alpha = \beta + 1 \in (2, 3]$  with  $\beta \in (1, 2]$ . Then  $u \in C^3[a, c]$  is a solution of the fractional boundary value problem (16) if and only if  $u$  is a solution of the following integral equation

$$u(x) = \int_{x_1}^{x_2} \left( \int_b^x G(\eta, s) d\eta \right) p(s)u(s) ds, \quad x_1 \in [a, b], x_2 \in [b, c], \tag{20}$$

where the Green's function  $G(t, s)$  given by

$$G(x, s) = \frac{1}{\Gamma(\alpha - 1)} \begin{cases} \frac{x-x_1}{x_2-x_1}(x_2-s)^{\alpha-2} - (x-s)^{\alpha-2} & x_1 \leq s \leq t \leq x_2, \\ \frac{x-x_1}{x_2-x_1}(x_2-s)^{\alpha-2} & x_1 \leq t \leq s \leq x_2. \end{cases} \tag{21}$$

*Proof.* Since  $u(x)$  is continuous on  $[a, c]$  and  $u(a) = u(b)$ , there exists  $x_1 \in (a, b)$  so that  $u'(x_1) = 0$ . Similarly, there exists  $x_2 \in (b, c)$  such that  $u'(x_2) = 0$ . Let  $v(x) := u'(x)$  for  $x \in [a, c]$ . Then we have that  ${}^C D_a^\beta v(x) = {}^C D_a^\alpha u(x)$  since  ${}^C D_a^\beta u'(x) = {}^C D_a^\alpha u(x)$ . Lemma 1.3 implies that  $v$  is a solution of the following fractional boundary value problem

$$\begin{aligned} {}^C D_a^\beta v(x) + z(x) &= 0, & x_1 \leq x \leq x_2, \\ v(x_1) = v(x_2) &= 0, \end{aligned} \tag{22}$$

if and only if  $v$  solves the integral equation

$$v(x) = \int_{x_1}^{x_2} G(x, s)z(s) ds. \tag{23}$$

where the Green’s function  $G(x, s)$  given by (19).

Observe that the solution  $u(x)$  of the fractional boundary value problem (16) with  $v(x) = u'(x)$  satisfies the problem (22) when  $z(x)$  is replaced by  $p(x)u(x)$ . Thus, we have by the help of the equation (23)

$$u'(x) = \int_{x_1}^{x_2} G(x, s)p(s)u(s) ds.$$

Thus, we obtain by Fubini’s theorem and the observation that  $\beta = \alpha - 1$ ;

$$u(t) = \int_b^t \int_{t_1}^{t_2} G(\eta, s)p(s)u(s) ds d\eta = \int_{t_1}^{t_2} \left( \int_b^t G(\eta, s) d\eta \right) p(s)u(s) ds, \tag{24}$$

which is the desired result.  $\square$

We have for  $s \in [t_1, t_2]$

$$\begin{aligned} \int_{t_1}^{t_2} G(x, s) dx &= \int_{t_1}^s G(x, s) dx + \int_s^{t_2} G(x, s) dx \\ &= \frac{1}{\Gamma(\beta)} \left[ \frac{(t_2 - s)^{\beta-1}}{t_2 - t_1} \int_{t_1}^s (x - t_1) dx + \int_s^{t_2} \left( \frac{x - t_1}{t_2 - t_1} (t_2 - s)^{\beta-1} - (t - s)^{\beta-1} \right) dx \right] \\ &= \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} \left( \frac{t_2 - t_1}{2} - \frac{t_2 - s}{\beta} \right). \end{aligned}$$

Note that when  $1 < \beta \leq 2$ , we have  $\frac{t_2 - t_1}{2} \leq \frac{t_2 - t_1}{\beta}$ . Hence we find the following bound

$$\int_{t_1}^{t_2} G(x, s) dx \leq \frac{(s - t_1)(t_2 - s)^{\beta-1}}{\beta\Gamma(\beta)} \leq \frac{(s - a)(c - s)^{\beta-1}}{\beta\Gamma(\beta)} = \frac{(s - a)(c - s)^{\alpha-2}}{(\alpha - 1)\Gamma(\alpha - 1)}. \tag{25}$$

We are now ready to present our main result. Hartman–Wintner-type inequalities for the fractional differential equation (16) has been obtained in the following theorem. From now on, we assume  $-\infty < a < b < c < \infty$ .

**Theorem 2.2.** *If the fractional boundary value problem (16) satisfies  $u(x) \not\equiv 0$  for  $x \in (a, b) \cup (b, c)$ , then one of the following Hartman-Wintner-type inequalities holds:*

- (A)  $\int_a^c (s - a)(c - s)^{\alpha-2} p_-(s) ds > \Gamma(\alpha - 1)(\alpha - 1)$ ,
- (B)  $\int_a^c (s - a)(c - s)^{\alpha-2} p_+(s) ds > \Gamma(\alpha - 1)(\alpha - 1)$ ,
- (C)  $\int_a^b (s - a)(c - s)^{\alpha-2} p_-(s) ds + \int_b^c (s - a)(c - s)^{\alpha-2} p_+(s) ds > \Gamma(\alpha - 1)(\alpha - 1)$ ,

where  $p_-(s) := \max\{-p(s), 0\}$  and  $p_+(s) := \max\{p(s), 0\}$ .

*Proof.* We follow the idea of the paper [2]. As in the proof of Lemma 2.1, there are  $x_1 \in (a, b)$  and  $x_2 \in (b, c)$  such that  $u'(x_1) = 0$  and  $u'(x_2) = 0$ . Next, we assume, without loss of generality, one of the following conditions on  $u(x)$  holds:

- (1)  $u(x) > 0$  on  $(a, c) \setminus \{b\}$  and  $u(x_1) \geq u(x_2)$ ,
- (2)  $u(x) > 0$  on  $(a, c) \setminus \{b\}$  and  $u(x_1) < u(x_2)$ ,
- (3)  $u(x) > 0$  on  $(a, b)$ ,  $u(x) < 0$  on  $(b, c)$ , and  $u(x_1) \geq -u(x_2)$ ,
- (4)  $u(x) > 0$  on  $(a, b)$ ,  $u(x) < 0$  on  $(b, c)$ , and  $u(x_1) < -u(x_2)$ .

Let  $M = \max\{|u(x_1)|, |u(x_2)|\}$ . We proceed the proof by examining the cases:

- (1) Since  $u(x_1) \geq u(x_2)$  and  $u(x) > 0$ , we get  $u(x_1) = M$ . Now, replacing  $x$  by  $x_1$  in the equation (24) gives that

$$M = \int_{x_1}^{x_2} \left( \int_b^{x_1} G(\eta, s) d\eta \right) p(s) u(s) ds = \int_{x_1}^{x_2} \left( \int_{x_1}^b G(\eta, s) d\eta \right) (-p(s)) u(s) ds.$$

We observe that  $G(\eta, s) \geq 0$  for  $(\eta, s) \in [x_1, x_2] \times [x_1, x_2]$ ,  $u(x) \in [0, M]$  and  $u(x)$  is not constant on  $[x_1, x_2]$ . Since  $-p(x) \leq p_-(x)$ , we get

$$M < M \int_{x_1}^{x_2} \left( \int_{x_1}^{x_2} G(\eta, s) d\eta \right) p_-(s) ds.$$

The inequality (25) leads to find that

$$\begin{aligned} \beta \Gamma(\beta) &< \int_{x_1}^{x_2} (s-a)(c-s)^{\beta-1} p_-(s) ds \leq \int_a^c (s-a)(c-s)^{\beta-1} p_-(s) ds \\ &= \int_a^c (s-a)(c-s)^{\alpha-2} p_-(s) ds. \end{aligned}$$

This shows that (A) is satisfied when the condition (1) is true.

- (2) We have  $u(x_2) = M$  since  $u(x_2) > u(x_1)$  and  $u(x) > 0$  in this case. Substituting  $x = x_2$  in the equation (24), we find that

$$M = \int_{x_1}^{x_2} \left( \int_b^{x_1} G(\eta, s) d\eta \right) p(s) u(s) ds.$$

We again observe that  $G(\eta, s) \geq 0$  for  $(\eta, s) \in [x_1, x_2] \times [x_1, x_2]$ ,  $u(x) \in [0, M]$  and  $u(x)$  is not constant on  $[x_1, x_2]$ . Since  $p(x) \leq p_+(x)$ , we have

$$M < M \int_{x_1}^{x_2} \left( \int_{x_1}^{x_2} G(\eta, s) d\eta \right) p_+(s) ds.$$

The inequality (25) yields

$$\begin{aligned} \beta \Gamma(\beta) &< \int_{x_1}^{x_2} (s-a)(c-s)^{\beta-1} p_+(s) ds \leq \int_a^c (s-a)(c-s)^{\beta-1} p_+(s) ds \\ &= \int_a^c (s-a)(c-s)^{\alpha-2} p_+(s) ds. \end{aligned}$$

This shows the case (B) is satisfied when the condition (2) is satisfied.

(3) We have  $u(x_1) = M$  in this case. Replacing  $x$  with  $x_1$  in (24) reveals that

$$\begin{aligned} M &= \int_{x_1}^{x_2} \left( \int_b^{x_1} G(\eta, s) d\eta \right) p(s) u(s) ds = \int_{x_1}^{x_2} \left( - \int_{x_1}^b G(\eta, s) d\eta \right) p(s) u(s) ds \\ &= - \int_{x_1}^b \int_{x_1}^{x_2} G(\eta, s) p(s) u(s) ds d\eta \\ &= \int_{x_1}^b \left( \int_{x_1}^b G(\eta, s) (-p(s)) u(s) ds + \int_b^{x_2} G(\eta, s) p(s) (-u(s)) ds \right) d\eta. \end{aligned}$$

Since  $u(x) > 0$  on  $(a, b)$  and  $u(x) < 0$  on  $(b, c)$ , we have

$$\begin{aligned} \beta \Gamma(\beta) &< \int_{x_1}^b (s-a)(c-s)^{\beta-1} p_-(s) ds + \int_b^{x_2} (s-a)(c-s)^{\beta-1} p_+(s) ds \\ &\leq \int_a^b (s-a)(c-s)^{\beta-1} p_-(s) ds + \int_b^c (s-a)(c-s)^{\beta-1} p_+(s) ds \\ &= \int_a^b (s-a)(c-s)^{\alpha-2} p_-(s) ds + \int_b^c (s-a)(c-s)^{\alpha-2} p_+(s) ds. \end{aligned}$$

This shows the case (C) holds true when the condition (3) is satisfied.

(4) Similar to the condition (3), we can prove the assertion (C) when the condition (4) is true.

Therefore, the proof is completed.  $\square$

We present some consequences of Theorem 2.2. The first consequence is the following Lyapunov-type inequalities.

**Corollary 2.3.** *If the problem (16) has a nontrivial solution, then one of the following Lyapunov-type inequalities holds:*

1.  $\int_a^c p_-(s) ds > \frac{\Gamma(\alpha-1)(\alpha-1)(\alpha-1)^{\alpha-1}}{(c-a)^{\alpha-1}(\alpha-2)^{\alpha-2}},$
2.  $\int_a^c p_+(s) ds > \frac{\Gamma(\alpha-1)(\alpha-1)(\alpha-1)^{\alpha-1}}{(c-a)^{\alpha-1}(\alpha-2)^{\alpha-2}},$
3.  $\int_a^b p_-(s) ds + \int_b^c p_+(s) ds > \frac{\Gamma(\alpha-1)(\alpha-1)(\alpha-1)^{\alpha-1}}{(c-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}.$

Consequently,

$$\int_a^c |p(s)| ds > \frac{\Gamma(\alpha-1)(\alpha-1)(\alpha-1)^{\alpha-1}}{(c-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}.$$

*Proof.* We will prove the conclusion 1. The conclusion 2. and conclusion 3. can be proved by the same argument. Let us define the function  $f(x)$  for  $\alpha \in (2, 3]$  as follows;

$$f(x) = (c-x)^{\alpha-2}(x-a), \quad x \in [a, c].$$

Differentiating the function  $f$  gives that

$$f'(x) = (c-x)^{\alpha-3}((\alpha-2)a + c + (1-\alpha)x), \quad x \in (a, c).$$

Therefore,  $f'(x) = 0$  if and only if  $x = x_0 := \frac{c + (\alpha - 2)a}{\alpha - 1}$ . We see that  $f$  has a maximum value at  $x = x_0$ . Thus, we have

$$\max_{a \leq x \leq c} f(x) = f(x_0) = (\alpha - 2)^{\alpha-2} \left( \frac{b-a}{\alpha-1} \right)^{\alpha-1}. \tag{26}$$

We rewrite the result (A) of Theorem 2.2 by the help of (26) as

$$(\alpha - 1)\Gamma(\alpha - 1) < \int_a^c f(x)p_-(x) dx \leq f(x_0) \int_a^c p_-(x) dx = (\alpha - 2)^{\alpha-2} \left( \frac{b-a}{\alpha-1} \right)^{\alpha-1} \int_a^c p_-(x) dx,$$

which reveals the desired result after a little algebraic manipulation.  $\square$

**Remark 2.4.** If we let  $\alpha = 3$ , then the fractional boundary value problem (16) turns into the third order linear differential equation (3), and we recover the result (6) presented in the paper [2]. The constant in the inequalities provided in this paper bigger than the constants in (13) and (15), so we have improved the inequalities in the literature.

Next, as a consequence of Theorem 2.2, we find a lower bound for the eigenvalues of the fractional boundary value problem.

**Corollary 2.5.** If  $\lambda$  is an eigenvalue of the following fractional boundary value problem

$$\begin{aligned} {}^c D_a^\alpha u(x) + \lambda u(x) &= 0, x \in [a, c], \alpha \in (2, 3], \\ u(a) = u(b) = u(c) &= 0, \quad a < b < c, \end{aligned} \tag{27}$$

then we have

$$|\lambda| > \frac{\alpha(\alpha - 1)^2 \Gamma(\alpha - 1)}{(c - a)^\alpha}.$$

*Proof.* If  $\lambda$  is an eigenvalue of the fractional boundary value problem (27), then the problem (27) has a nontrivial solution. Appealing Theorem 2.2, we get

$$\int_a^c |\lambda|(s - a)(c - s)^{\alpha-2} ds > (\alpha - 1)\Gamma(\alpha - 1).$$

We calculate the integral

$$\int_a^c (s - a)(c - s)^{\alpha-2} ds = \frac{(c - a)^\alpha}{\alpha(\alpha - 1)}.$$

Thus we find

$$|\lambda| > \frac{\alpha(\alpha - 1)^2 \Gamma(\alpha - 1)}{(c - a)^\alpha},$$

which concludes the proof.  $\square$

The following result shows nonexistence of nontrivial solutions of the fractional boundary value problem (16).

**Corollary 2.6.** If the following three integral inequalities hold:

$$(i) \int_a^c p_-(s) ds \leq \frac{\Gamma(\alpha - 1)(\alpha - 1)(\alpha - 1)^{\alpha-1}}{(c - a)^{\alpha-1}(\alpha - 2)^{\alpha-2}},$$



$$(ii) \int_a^c p_+(s) ds \leq \frac{\Gamma(\alpha - 1)(\alpha - 1)(\alpha - 1)^{\alpha-1}}{(c - a)^{\alpha-1}(\alpha - 2)^{\alpha-2}},$$

$$(iii) \int_a^b p_-(s) ds + \int_b^c p_+(s) ds \leq \frac{\Gamma(\alpha - 1)(\alpha - 1)(\alpha - 1)^{\alpha-1}}{(c - a)^{\alpha-1}(\alpha - 2)^{\alpha-2}},$$

then the fractional boundary value problem (16) does not have any nontrivial solution.

*Proof.* Assume, for a moment, that the fractional boundary value problem (16) has a nontrivial solution  $u(x)$ . Since  $u(a) = u(b) = 0$  and  $u(b) = u(c) = 0$ , there exist  $x_1 \in [a, b)$  and  $y_1 \in (b, c]$  so that  $u(x_1) = u(y_1) = 0$  and  $u(x) \neq 0$  for  $x \in (x_1, b) \cup (b, y_1)$ . Now, we have that  $u(x_1) = u(b) = u(y_1) = 0$ . By appealing to the conclusion 1. ( conclusion 2. and conclusion 3., respectively) of Corollary 2.3, we get

$$\begin{aligned} \int_a^c p_-(s) ds &\geq \int_{x_1}^{y_1} p_-(s) ds > \frac{\Gamma(\alpha - 1)(\alpha - 1)(\alpha - 1)^{\alpha-1}}{(y_1 - x_1)^{\alpha-1}(\alpha - 2)^{\alpha-2}} \\ &\geq \frac{\Gamma(\alpha - 1)(\alpha - 1)(\alpha - 1)^{\alpha-1}}{(c - a)^{\alpha-1}(\alpha - 2)^{\alpha-2}}. \end{aligned}$$

However, this contradicts the assumption (i) (the assumption (ii) and (iii), respectively) above. Thus, under the assumptions (i),(ii) and (iii), the fractional boundary value problem (16) has only the zero solution.  $\square$

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