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A Refined Bound for the Z_1-Spectral Radius of Tensors

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Abstract. A refined upper bound for the Z_1-spectral radius of tensors is given, which needs less computations than that presented by Wang et al. in [Applied Mathematics and Computation, 329 (2018) 266-277]. Numerical experiments involving Uniform distribution, Gaussian distribution, Poisson distribution and Binomial distribution are given to show the effectiveness of the proposed bound.

1. Introduction

The Z_1-eigenvalue of tensors and its corresponding eigenvectors are useful for computing the limiting probability distribution in high order Markov chain [1, 10] and the PageRank vector in multilinear PageRank models [7, 11], and also have applications in image matching [5], best rank-one approximation of tensors[14, 17], and hypergraph theory [2, 8].

Definition 1.1. *[1] A real number $\lambda \in \mathbb{R}^n$ and a non-zero real vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ are called a Z_1-eigenvalue and a Z_1-eigenvector of an order m dimension n real tensor $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{R}^{[m,n]}$ ($\mathbb{R}^{[m,n]}$ denotes the set of the order m dimension n tensors over real numbers $\mathbb{R}$) if*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x},\ \|\mathbf{x}\|_1 = \sum_{k=1}^{n} |x_k| = 1, \tag{1}$$

where $\mathcal{A}\mathbf{x}^{m-1}$ is a vector with its i-th component being

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m},\ i \in [n] := \{1, \ldots, n\}.$$

Furthermore, the Z_1-spectral radius of $\mathcal{A}$ is denoted by

$$\rho_{z_1}(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma_1(\mathcal{A})\},$$

where $\sigma_1(\mathcal{A})$ is the set of all Z_1-eigenvalues of $\mathcal{A}$.

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There are a variety of results on the Z_1-eigenvalues and its corresponding Z_1-eigenvectors, such as, algorithms for computing Z_1-eigenvalues and its corresponding Z_1-eigenvectors [3], bounds for the Z_1-spectral radius [9, 12, 16], and the uniqueness conditions for the positive Z_1-eigenvector for nonnegative tensors [1, 4, 7, 10, 11].

Very recently, Wang et al. [16] provided an upper bound for the Z_1-spectral radius of tensors as follows.

Theorem 1.2. *[16, Theorem 2.5] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. Then*

$$\rho_{z_1}(\mathcal{A}) \leq \min\left\{C_1(\mathcal{A}), (R(\mathcal{A}))^{\frac{1}{m-1}}\left(\min_{t\in[m]\setminus\{1\}} C_t(\mathcal{A})\right)^{\frac{m-2}{m-1}}\right\}, \tag{2}$$

where $R(\mathcal{A}) := \max_{i\in[n]}\left\{r_i(\mathcal{A}) := \sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\cdots i_m}|\right\}$, *and*

$$C_t(\mathcal{A}) := \max_{i_s\in[n], s\in[m]\setminus\{t\}} \sum_{i_t=1}^{n} |a_{i_1 i_2\cdots i_t\cdots i_m}|, t \in [m].$$

As said in [16], if $m = 2$, then the bound (2) reduces to the well-known Frobenius's bound [6] for the spectral radius $\rho(A)$ of a matrix A, i.e.,

$$\rho(A) \leq \min\{C_1(A), C_2(A)\},$$

where $C_1(A)$ and $C_2(A)$ are the maximum column sum and row sum of $\mathcal{A}$, respectively.

Although the bound (2) depends only on the entries of a given tensor $\mathcal{A}$, unlike matrices case it involves the term $R(\mathcal{A})$, and thus needs extra computations. In this paper, we give a refinement bound for the Z_1-spectral radius of tensors:

$$\rho_{z_1}(\mathcal{A}) \leq \min_{t\in[m]} C_t(\mathcal{A}),$$

which has nothing to do with $R(\mathcal{A})$ like matrices case, and prove that the new bound is better than that in Theorem 1.2 ([16, Theorem 2.5]).

2. Main results

Let

$$[n]^{m-1} = \left\{(i_2, i_3, \ldots, i_m) : i_j \in [n], j = 2, 3, \ldots, m\right\}.$$

Obviously, $[n]^1 = [n]$.

Theorem 2.1. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. Then*

$$\rho_{z_1}(\mathcal{A}) \leq \min_{t\in[m]} C_t(\mathcal{A}). \tag{3}$$

Proof. Suppose that a nonzero vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ with

$$||\mathbf{x}||_1 = \sum_{k=1}^{n} |x_k| = 1$$

such that $\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}$. We next consider the following two cases $t = 1$ and $t = 2, \ldots, m$.

Case I: $t = 1$. From (1) we get

$$\lambda x_{i_1} = \sum_{i_2,\ldots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \ i_1 \in [n].$$

Taking modulus in the above equation and using the triangle inequality give

$$\begin{aligned}
|\lambda| = |\lambda| \sum_{i_1=1}^{n} |x_{i_1}| &\leq \sum_{i_1,i_2,\ldots,i_m=1}^{n} |a_{i_1 i_2 \cdots i_m}||x_{i_2}| \cdots |x_{i_m}| \\
&= \sum_{i_2,\ldots,i_m=1}^{n} \left(|x_{i_2}| \cdots |x_{i_m}| \sum_{i_1=1}^{n} |a_{i_1 i_2 \cdots i_m}| \right) \\
&\leq \left(\sum_{i_2,\ldots,i_m=1}^{n} |x_{i_2}| \cdots |x_{i_m}| \right) \max_{i_s \in [n], s \in [m] \setminus \{1\}} \sum_{i_1=1}^{n} |a_{i_1 i_2 \cdots i_m}| \\
&= C_1(\mathcal{A}),
\end{aligned}$$

where the last equality holds because

$$\sum_{i_2,\ldots,i_m=1}^{n} |x_{i_2}| \cdots |x_{i_m}| = \prod_{k=2,3,\ldots,m} \left(\sum_{i_k=1}^{n} |x_{i_k}| \right) = 1.$$

Thus, $\rho_{z_1}(\mathcal{A}) \leq C_1(\mathcal{A})$.

Case II: $t = 2, \ldots, m$. Let $|x_k| = \max_{i \in [n]} |x_i|$. Then $|x_k| \neq 0$. From the k-th equality of (1) we get

$$\lambda x_k = \sum_{(i_2,\ldots,i_m) \in [n]^{m-1}} a_{k i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality give

$$\begin{aligned}
|\lambda||x_k| &\leq \sum_{(i_2,\ldots,i_m) \in [n]^{m-1}} |a_{k i_2 \cdots i_m}||x_{i_2}| \cdots |x_{i_m}| \\
&= \sum_{i_p=1}^{n} \left(\left(\sum_{(i'_2,\ldots,i'_{m-1}) \in [n]^{m-2}} |a_{k i'_2 \cdots i_p \cdots i'_{m-1}}| \prod_{\substack{s=2,\\ s \neq p}}^{m-1} |x'_{i_s}| \right) |x_{i_p}| \right) \\
&\leq \left(\max_{i_p \in [n]} \left(\sum_{(i'_2,\ldots,i'_{m-1}) \in [n]^{m-2}} |a_{k i'_2 \cdots i_p \cdots i'_{m-1}}| \prod_{\substack{s=2,\\ s \neq p}}^{m-1} |x'_{i_s}| \right) \right) \sum_{i_p=1}^{n} |x_{i_p}| \\
&= \max_{i_p \in [n]} \left(\sum_{(i'_2,\ldots,i'_{m-1}) \in [n]^{m-2}} |a_{k i'_2 \cdots i_p \cdots i'_{m-1}}| \prod_{\substack{s=2,\\ s \neq p}}^{m-1} |x'_{i_s}| \right)
\end{aligned}$$

$$
\begin{aligned}
&= \max_{i_p\in[n]}\left(\sum_{i_q=1}^{n}\left(\sum_{(i''_2,\dots,i''_{m-2})\in[n]^{m-3}}|a_{ki''_2\cdots i_p\cdots i_q\cdots i''_{m-2}}|\prod_{\substack{s=2,\\ s\neq p,q}}^{m-2}|x''_{i_s}|\right)|x_{i_q}|\right)\\
&\leq \max_{i_p\in[n]}\left(\left(\max_{i_q\in[n]}\left(\sum_{(i''_2,\dots,i''_{m-2})\in[n]^{m-3}}|a_{ki''_2\cdots i_p\cdots i_q\cdots i''_{m-2}}|\prod_{\substack{s=2,\\ s\neq p,q}}^{m-2}|x''_{i_s}|\right)\right)\sum_{i_q=1}^{n}|x_{i_q}|\right)\\
&= \max_{i_p\in[n]}\left(\max_{i_q\in[n]}\left(\sum_{(i''_2,\dots,i''_{m-2})\in[n]^{m-3}}|a_{ki''_2\cdots i_p\cdots i_q\cdots i''_{m-2}}|\prod_{\substack{s=2,\\ s\neq p,q}}^{m-2}|x''_{i_s}|\right)\right)\\
&= \max_{(i_p,i_q)\in[n]^2}\left(\sum_{(i''_2,\dots,i''_{m-2})\in[n]^{m-3}}|a_{ki''_2\cdots i_p\cdots i_q\cdots i''_{m-2}}|\prod_{\substack{s=2,\\ s\neq p,q}}^{m-2}|x''_{i_s}|\right)\\
&\ \vdots\\
&= \max_{(i^*_2,\dots,i^*_{m-1})\in[n]^{m-2}}\left(\sum_{i_t=1}^{n}|a_{ki^*_2\cdots i_t\cdots i^*_{m-1}}||x_{i_t}|\right)\\
&\leq \left(\max_{(i^*_2,\dots,i^*_{m-1})\in[n]^{m-2}}\left(\sum_{i_t=1}^{n}|a_{ki^*_2\cdots i_t\cdots i^*_{m-1}}|\right)\right)|x_k|
\end{aligned}
$$

Dividing $|x_k| \neq 0$ on both sides yields

$$
\begin{aligned}
|\lambda| &\leq \max_{(i^*_2,\dots,i^*_{m-1})\in[n]^{m-2}}\left(\sum_{i_t=1}^{n}|a_{ki^*_2\cdots i_t\cdots i^*_{m-1}}|\right)\\
&\leq \max_{i_1\in[n]}\max_{(i^*_2,\dots,i^*_{m-1})\in[n]^{m-2}}\left(\sum_{i_t=1}^{n}|a_{i_1i^*_2\cdots i_t\cdots i^*_{m-1}}|\right)\\
&= \max_{i_s\in[n],s\in[m]\setminus\{t\}}\left(\sum_{i_t=1}^{n}|a_{i_1i_2\cdots i_t\cdots i_m}|\right).
\end{aligned}
$$

Apparently, the inequality above holds for any $t = 2,\dots,m$, and hence

$$
|\lambda| \leq \min_{t\in[m]\setminus\{1\}}\max_{i_s\in[n],s\in[m]\setminus\{t\}}\left(\sum_{i_t=1}^{n}|a_{i_1i_2\cdots i_t\cdots i_m}|\right) = \min_{t\in[m]\setminus\{1\}} C_t(\mathcal{A}),
$$

consequently,

$$
\rho_{z_1}(\mathcal{A}) \leq \min_{t\in[m]\setminus\{1\}} C_t(\mathcal{A}).
$$

The conclusion follows from Case I and Case II. □

If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a nonnegative tensor, then the bound (3) reduces to

$$
\rho_{z_1}(\mathcal{A}) \leq \min_{t\in[m]}\max_{i_s\in[n],s\in[m]\setminus\{t\}}\sum_{i_t=1}^{n}a_{i_1i_2\cdots i_t\cdots i_m},
$$

which is the exact upper bound in Corollary 3.6 of [9] for the weakly symmetric nonnegative irreducible tensor case. Apparently, the bound (3) needs less computations than the bound (2) because the latter has to compute $R(\mathcal{A})$. Next, we establish a comparison result to show that the bound (3) is less than or equal to the bound (2).

Theorem 2.2. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. Then*

$$\min_{t\in[m]} C_t(\mathcal{A}) \le \min\left\{C_1(\mathcal{A}), (R(\mathcal{A}))^{\frac{1}{m-1}}\left(\min_{t\in[m]\setminus\{1\}} C_t(\mathcal{A})\right)^{\frac{m-2}{m-1}}\right\},$$

where $R(\mathcal{A})$ and $C_t(\mathcal{A})$, $t \in [m]$ are defined as in Theorem 1.2.

Proof. Note that for any $t = 2, 3, \ldots, m$,

$$\max_{i_s\in[n], s\in[m]\setminus\{t\}} \sum_{i_t=1}^{n} |a_{i_1 i_2 \cdots i_t \cdots i_m}| \le \max_{i\in[n]} r_i(\mathcal{A}).$$

Hence, $\min_{t\in[m]} C_t(\mathcal{A}) \le \min_{t\in[m]\setminus\{1\}} C_t(\mathcal{A}) \le R(\mathcal{A})$. Furthermore, from $\min_{t\in[m]} C_t(\mathcal{A}) \le C_1(\mathcal{A})$, we have

$$\begin{aligned}\min_{t\in[m]} C_t(\mathcal{A}) &= \min\{C_1(\mathcal{A}), \min_{t\in[m]} C_t(\mathcal{A})\}\\ &\le \min\left\{C_1(\mathcal{A}), \left(\min_{t\in[m]} C_t(\mathcal{A})\right)^{\frac{1}{m-1}}\left(\min_{t\in[m]\setminus\{1\}} C_t(\mathcal{A})\right)^{\frac{m-2}{m-1}}\right\}\\ &\le \min\left\{C_1(\mathcal{A}), (R(\mathcal{A}))^{\frac{1}{m-1}}\left(\min_{t\in[m]\setminus\{1\}} C_t(\mathcal{A})\right)^{\frac{m-2}{m-1}}\right\}.\end{aligned}$$

The proof is complete. □

Remark here that besides the bound (2) in Theorem 1.2 ([16, Theorem 2.5]), there are another bounds for the Z_1-spectral radius, for instance, in 2015, Li et al. [9, Theorem 2.1] derived the following upper bound about the Z_1-spectral radius of $\mathcal{A}$:

$$\rho_{z_1}(\mathcal{A}) \le \min_{k\in[m]} \max_{i_k\in[n]} \sum_{\substack{i_s\in[n],\\ s\in[m]\setminus\{k\}}} |a_{i_1 \cdots i_k \cdots i_m}|.$$

As stated in [16, Remark 3],

$$\min\left\{C_1(\mathcal{A}), (R(\mathcal{A}))^{\frac{1}{m-1}}\left(\min_{t\in[m]\setminus\{1\}} C_t(\mathcal{A})\right)^{\frac{m-2}{m-1}}\right\} \le \min_{k\in[m]} \max_{i_k\in[n]} \sum_{\substack{i_s\in[n],\\ s\in[m]\setminus\{k\}}} |a_{i_1 \cdots i_k \cdots i_m}|.$$

Hence,

$$\min_{t\in[m]} C_t(\mathcal{A}) \le \min_{k\in[m]} \max_{i_k\in[n]} \sum_{\substack{i_s\in[n],\\ s\in[m]\setminus\{k\}}} |a_{i_1 \cdots i_k \cdots i_m}|.$$

This implies that the bound in Theorem 2.1 is better than that in [9, Theorem 2.1].

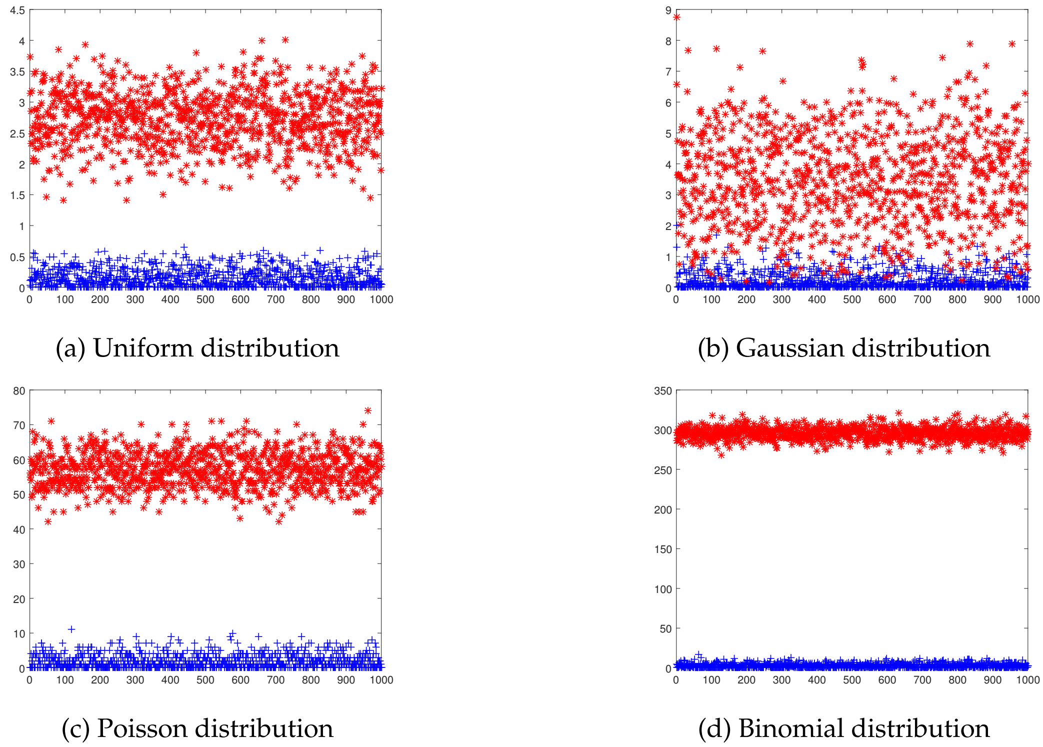

Figure 1: The bound differences for four distributions entries.

Example 2.3. *Consider* 4×10^3 *order* 4 *dimensional* 2 *tensors generated by the way from [16], i.e., tensors are implemented randomly with four different distributions (Uniform distribution, Gaussian distribution, Poisson distribution and binomial distribution) entries. In uniform distribution case, all entries are in the range of* $[0, 1]$*. In gaussian distribution case, the parameters* μ *and* σ *are generated randomly in the range of* $[0, 1]$*. For convenience, all the entries of tensor* $\mathcal{A}$ *are shifted to be positive. In poisson distribution case, the parameter* λ *is set to be 10. In binomial distribution case, the number of entries is set to be 100. And the probability of success for each trial p is set to be 0.5.*

The differences of the bounds in Theorem 1.2, Theorem 2.1 and [9, Theorem 2.1] are drawn in Figure 1, where the star symbol in red color '$*$' means the upper bound in [9, Theorem 2.1] minus the upper bound in Theorem 2.1, and the cross symbol in blue color $+$ means the upper bound in Theorem 1.2 minus the upper bound in Theorem 2.1. From all sub-figures it is easy to see that there are no '$*$' and '$+$' below zero. This means that the upper bound in Theorem 2.1 is better than that in Theorem 1.2 and [9, Theorem 2.1].

3. Conclusions

In this paper, we give a new upper bound for the Z_1-spectral radius for tensors, and it needs less computations, and is sharper than that in [16].

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