A Note on the Power Graphs of Finite Nilpotent Groups

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Abstract.

The power graph $P(G)$ of a group $G$ is the graph with vertex set $G$ and two distinct vertices are adjacent if one is a power of the other. Two finite groups are said to be conformal, if they contain the same number of elements of each order. Let $Y$ be a family of all non-isomorphic odd order finite nilpotent groups of class two or $p$-groups of class less than $p$. In this paper, we prove that the power graph of each group in $Y$ is isomorphic to the power graph of an abelian group and two groups in $Y$ have isomorphic power graphs if they are conformal. We determine the number of maximal cyclic subgroups of a generalized extraspecial $p$-group $(p$ odd) by determining the power graph of this group. We also determine the power graph of a $p$-group of order $p^4$ $(p$ odd).

1. Introduction

Given a group, there are different methods to associate a graph with the group. Recently, the power graph associated with a group has deserved a lot of attention. The term “power graph” was first considered and introduced by Kelarev and Quinn [12]. Let $G$ be a group. The undirected power graph $P(G)$ has the vertex set $G$ and two distinct vertices $x$ and $y$ are adjacent if $x = y^m$ or $y = x^m$ for some positive integer $m$. Because this paper deals only with undirected graphs, for convenience throughout we use the term “power graph” to refer to an undirected power graph defined as above, see also [1, Section 3].

Recently, a lot of interesting results on the power graphs have been obtained, see for examples [3–5, 8, 18]. A detailed list of open problems and results about power graphs can be found in [1]. Cameron and Ghosh [4] showed that for two finite abelian groups $A_1$ and $A_2$, $P(A_1) \cong P(A_2)$ if and only if $A_1 \cong A_2$. They also showed that two finite groups which have isomorphic power graphs are conformal [3, 4]. In general, converse of above result is false (see Remark 4.17). In Section 4 of this paper, we find a family of non-abelian groups in which converse holds, that is, if two finite groups are conformal, then they have isomorphic power graphs.

In [15], Mehranian, Gholami and Ashrafi gave the structure of the power graphs of cyclic groups, dicyclic groups, semidihedral groups and Mathieu group $M_{11}$ or the Janko group $J_1$. In [10], Ghorbani and Barfaraz obtained the structure of power graphs of groups of order a product of three primes. The structure of the power graphs of elementary abelian $p$-groups and dihedral groups are also known [7, 18].

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In this paper, we find the structure of the power graphs of generalized extraspecial $p$-groups and $p$-groups of order $p^k$ ($p$ odd) and as an application, we find the number of maximal cyclic subgroups (a cyclic subgroup that is not a proper subgroup of any another proper cyclic subgroup) of generalized extraspecial $p$-groups. Let $Z(G)$ denote the center of the group $G$. A finite $p$-group $G$ is called extraspecial $p$-group if $Z(G)$ and $y_2(G)$ coincide and have order $p$, where $y_2(G)$ is the commutator subgroup of $G$. If $Z(G)$ of a finite $p$-group $G$ is cyclic and $y_2(G)$ has order $p$, then $G$ is said to be a generalized extraspecial $p$-group. For more details see [19].

2. Notations and Basic Definitions

Throughout the paper all groups considered are finite and $p$ denotes a prime. Let $C(G)$ denote the set of all distinct cyclic subgroups of the group $G$. Further, let $c_p(G)$ denote the number of cyclic subgroups of order $p^k$ ($k$ is a non-negative integer) in the group $G$. Cardinality of a set $X$ is denoted by $|X|$. $o(x)$ denotes the order of the element $x$ in the group $G$ and identity element of the group $G$ is denoted by 1.

Let $Γ$ be a graph. A set of pairwise non-adjacent vertices of $Γ$ is called an independent set. The independence number of a graph $Γ$ is the cardinality of the largest independent set and is denoted by $β(Γ)$. Let $Γ_1$ and $Γ_2$ be the graphs with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ respectively. Then their union $Γ_1 ∪ Γ_2$ is the graph with vertex set $V = V_1 ∪ V_2$ and edge set $E = E_1 ∪ E_2$. The join of $Γ_1$ and $Γ_2$ is denoted by $Γ_1 + Γ_2$ and it consists of graph $Γ_1 ∪ Γ_2$ and all edges joining $V_1$ with $V_2$. For any graph $Γ$, let $∪_{Γ}^e$ denote the graph obtain by union of $s$ copies of $Γ$.

**Definition 2.1.** Let $G$ be a group. For elements $u$ and $v$ in $G$, define a relation $R$ such that $uRv$ if $u = v$ or $u = v^{-1}$. It is evident that $R$ is an equivalence relation.

Let $[u]$ denote the equivalence class containing $u ∈ G$ under the relation $R$ and let $C′(G)$ denote the set of all equivalence classes $G/R$. Following [9], write

$$C′(G) = \{[u] \mid u ∈ G\} = \{[u_{00}], [u_{11}], \ldots, [u_{11}], \ldots, [u_{21}], \ldots, [u_{m1}], \ldots, [u_{mn}]\},$$

where $[u_{00}] = \{1\}$ and $[u_{ij}] = \{u_{il,1}, \ldots, u_{il,2}\}$.

**Definition 2.2.** Let $G$ be a group. For $u$ and $v$ in $G$, we say $u < v$ if one of the following holds.

(i) for some $i$ and $l$, $u = u_{il,1}, v = v_{il,2}$, and $l_1 < l_2$.

(ii) $\langle u \rangle \not\subseteq \langle v \rangle$.

Define $u ≤ v$ if $u < v$ or $u = v$.

**Definition 2.3.** [9] An ordered pair $(S, ≤_S)$, where $S$ is a finite set, is said to be a partially ordered set or poset if the binary relation $≤_S$ is reflexive, antisymmetric and transitive. For $u, v ∈ S$, if $u ≤ v$ or $v ≤ u$, then $u$ and $v$ are said to be comparable otherwise $u$ and $v$ are incomparable.

**Definition 2.4.** [9] Let $(S, ≤_S)$ be a poset. Then the comparability graph of $S$ is the graph with vertex set $S$, where two distinct elements are joined if they are comparable and it is denoted by $T_S$.

Let $G$ be a group. It is immediate from Definition 2.2, $(G, ≤)$ is a poset. For rest of this paper, let us denote this poset by $L_G$. Clearly, the comparability graph of $L_G$ is the power graph of a group $G$, that is, $P(G) = T_{L_G}$ ([9, Example 1]).

**Definition 2.5.** [9] A subset $S'$ of $S$ in a poset $(S, ≤_S)$ is said to be chain, if all elements in $S'$ are pairwise comparable. A subset $W$ of $S$ is said to be homogeneous if one of the following condition holds, for any $v ∈ S \setminus W$.

- for all $u ∈ W$, $u ≤_S v$.
- for all $u ∈ W$, $v ≤_S u$.
- for all $u ∈ W$, $u$ and $v$ are incomparable.

**Definition 2.6.** [9] A chain in a poset $(S, ≤_S)$ that is also homogeneous is called a homogeneous chain.

**Remark 2.7.** [9, Example 2] Let $G$ be group. Then each element $[x] ∈ C′(G)$ is a homogeneous chain in $L_G$. 

3. Basic Results

In this section, we state some results that will be used later. Let \( G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_s} \) \( \cong \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_s \rangle \) such that \( x_i^{p_i} = 1 \) for \( i \in \{1, 2, \ldots, s\} \) and \( \alpha_1, \ldots, \alpha_s \geq 1 \). Then we have the following result.

Lemma 3.1. If \( 1 \neq g \in G \), where \( g = x_1^{p_1\beta_1} x_2^{p_2\beta_2} \cdots x_s^{p_s\beta_s} \) such that \( 0 < k_i \) and \( p \nmid \beta_i \) \( \forall i \), then there are \( p^{s-1} \) cyclic subgroups of order \( o(g) \) containing \( \langle g \rangle \). Further, if for some \( i = i_w, k_i = 0, \beta_i \neq 0 \), then there doesn’t exist any cyclic subgroup of order \( o(g)p \) containing \( \langle g \rangle \).

Proof. Let \( g \in G \) such that \( g = x_1^{p_1\beta_1} x_2^{p_2\beta_2} \cdots x_s^{p_s\beta_s} \), where \( p \nmid \beta_i \). First, we count the number of elements \( h \in G \) such that \( h^p = g \). Consider \( h = x_1^{r_1} x_2^{r_2} \cdots x_s^{r_s} \). Now, \( h^p = g \) implies \( x_1^{pr_1} x_2^{pr_2} \cdots x_s^{pr_s} = x_1^{p_1\beta_1} x_2^{p_2\beta_2} \cdots x_s^{p_s\beta_s} \). So \( p^k\beta_i = pr_i \mod p \forall i \in \{1, 2, \ldots, s\} \). For fixed \( i \), latter equation has integer solution \( r_i \) if and only if \( p \nmid \beta_i \). Thus, if for some \( i = i_w, k_i = 0 \) and \( \beta_i \neq 0 \), then there doesn’t exist any \( h \in G \) such that \( h^p = g \).

Now, assume \( k_i > 0 \), \( \forall i \). So, if \( p^k\beta_i \equiv pr_i \mod p \), then \( p^{i-1}\beta_i \equiv r_i \mod p^{i-1} \). Thus, the latter equation has \( p \) distinct solutions for each \( i \) and that are \( r_i = p^{i-1}\beta_i + kp^{i-1} \), where \( 0 \leq k \leq p - 1 \). Thus, for given \( g = x_1^{p_1\beta_1} x_2^{p_2\beta_2} \cdots x_s^{p_s\beta_s} \), where \( p \nmid \beta_i \) and \( k_i > 0 \), there are \( p^s \) elements \( h \in G \) such that \( h^p = g \) and \( o(h) = o(g)p \).

Now, let \( h \) be a cyclic subgroup of order \( o(g)p \) such that \( g \subset \langle h \rangle \) and \( h^p = g \). Suppose \( w \in \langle h \rangle \) such that \( w^p = g \), then \( w = h^t \) and \( h^p = w^p = g \). This implies that \( rp \equiv p \mod o(h) \). Thus, \( r = 1 + k \frac{p\beta_i}{p} \), where \( 1 \leq k \leq p \). Hence that, there are \( \frac{p^s - 1}{p - 1} \) cyclic subgroups of order \( o(g)p \) containing \( g \) for \( k_i > 0 \) \( \forall i \). This completes the proof. \( \Box \)

Lemma 3.2. Let \( G \) be a finite abelian group such that \( G \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \). Then the number of elements of order \( p^t \) in \( G \) is

\[
\begin{cases} 
1, & t = 0 \\
p^{n+1} - 1, & t = 1 \\
p^{n+1} - p^{n+1} - 1 & 2 \leq t \leq m.
\end{cases}
\]

Proof. Let \( a_1, a_2, \ldots, a_n, a_{n+1} \) be the generators of \( G \) such that \( a_i^{p^n} = 1, a_i^p = 1 \), for \( i = 2, 3, \ldots, n + 1 \). Then each element of \( G \) is uniquely written as \( \prod_{i=1}^{n+1} a_i^{\beta_i} \), where \( 0 \leq \beta_i < p^n, 0 \leq \beta_i < p \) for \( i = 2, 3, \ldots, n + 1 \).

Take \( g = \prod_{i=1}^{n+1} a_i^{\beta_i} \). Now, for \( 1 \leq t \leq m \),

\[
g^p = \left( \prod_{i=1}^{n+1} a_i^{\beta_i^p} \right)^{p^t} = a_1^{\beta_1 p^t}.
\]

Thus, the number of the elements \( g \in G \) such that \( g^p = 1 \) is \( p^{n+1} - 1 \). Hence, the number of elements of order \( p^t \) of \( G \) is \( p^{n+1} - 1 \), for \( t = 1 \) and \( p^{n+1} - p^{n+1} - 1 \), for \( 2 \leq t \leq m \). This completes the proof. \( \Box \)

Corollary 3.3. Let \( G \) be a finite abelian group such that \( G \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \). Then

\[
c_t(G) = \begin{cases} 
1, & t = 0 \\
p^{n+1} - 1, & t = 1 \\
p^t - 2 & 2 \leq t \leq m.
\end{cases}
\]
Proof. The number of elements of order $p^j$ is equal to $c_1(G)\phi(p^j)$. Thus, the result follows from the Lemma 3.2. \qed

**Theorem 3.4.** [14] Let $A \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times \cdots \times \mathbb{Z}_{p^n}$. Then $\mathcal{P}(A)$ is isomorphic to

$$K_1 + \bigcup_{i=1}^{\ell} \left( K_{\phi(p)} + \bigcup_{j=1}^{\eta-1} \left( K_{\phi(p^j)} + \bigcup_{k=1}^{\mu-1} \left( \cdots + \bigcup_{l=1}^{\nu-1} \left( K_{\phi(p^{\nu})} \cdot \cdots \right) \right) \right) \right),$$

where $\ell = \frac{p^{\nu}-1}{p-1}$.

Let $G$ be a finite group. Recall that a cyclic subgroup of $G$ that is not a proper subgroup of any other proper cyclic subgroup of $G$ is called a maximal cyclic subgroup of $G$. Let $\mathcal{M}_G$ denote the set of all maximal cyclic subgroups of $G$.

**Theorem 3.5.** [13, Corollary 2.14] Let $G$ be a $p$-group. Then $\beta(\mathcal{P}(G)) = |\mathcal{M}_G|$.

Following [16], two finite groups are said to be conformal if they have same number of elements of each order.

**Theorem 3.6.** [16, Page 107] Two finite abelian groups are isomorphic if and only if they are conformal.

4. Power Graph of a Nilpotent Group

In this section, we use Baer’s trick to prove Theorem 4.1 and 4.2.

Let $G$ be a group. Then we may define a binary operation $\triangleleft$ on $G$ by $x \triangledown y = w(x, y)$ where $w$ is some fixed word in $x$ and $y$. If the set $G$, with the binary operation $\triangleleft$, define a group, then we say $w$ to be a group-word for $G$, and we write the corresponding group by $G_w$, that is, as a set $G_w = G$ and operation of $G_w$ is $\triangleleft$.

Let $(H, \cdot)$ be an odd order nilpotent group of class two. Then we can define a group-word $w$ as follows: for $x, y \in H$, $w(x, y) := xy[x, y]^p$ (by $xy$ we mean $x \cdot y$). If $\gamma_2(H)$ the commutator subgroup of $H$, has finite exponent $m$ and $n = \frac{m-1}{2}$, then corresponding group $H_w$ is an abelian group. Indeed, $x \triangledown y = xy[x, y]^{1/2} = yx[y, x]^{1/2} = yx[y, x]^{1/2} = y \circ x$ (for more details see [11, p. 142]). This $H_w$ is the corresponding abelian group to $H$. It is easy to observe that $H$ and $H_w$ are conformal.

**Theorem 4.1.** Let $H$ be an odd order nilpotent group of class two. Then $\mathcal{P}(H) \cong \mathcal{P}(H_w)$.

Proof. The powers of elements in $H$ and $H_w$ are same. Thus, $\mathcal{P}(H) \cong \mathcal{P}(H_w)$. This completes the proof. \qed

Above result is false for an even ordered group. For example, $D_8$ the dihedral group of order 16, is a nilpotent group of class two but $\mathcal{P}(D_8)$ is not isomorphic to the power graph of any abelian group [17, Theorem 15].

**Theorem 4.2.** Let $H^1$ and $H^2$ be two odd order nilpotent group of class two. If $H^1$ and $H^2$ are conformal, then $\mathcal{P}(H^1) \cong \mathcal{P}(H^2)$.

Proof. By Theorem 4.1, $\mathcal{P}(H^1) \cong \mathcal{P}(H^2)$ and $\mathcal{P}(H^2) \cong \mathcal{P}(H^2)$. Also $H^i$ is conformal to $H^i$, $i = 1$ or 2. Hence, $H^1_w$ and $H^2_w$ are conformal. So, by Theorem 3.6, $H^1_w \cong H^2_w$. Thus, $\mathcal{P}(H^1_w) \cong \mathcal{P}(H^2_w)$. Hence, $\mathcal{P}(H^1) \cong \mathcal{P}(H^2)$. This completes the proof. \qed

Two finite groups with isomorphic power graphs are conformal and two finite abelian groups have isomorphic power graphs if and only if they are isomorphic (see [3, 4]). Thus, we can easily deduce the following corollaries.

**Corollary 4.3.** The power graphs of two odd order nilpotent groups of class at most two are isomorphic if and only if they are conformal.
Corollary 4.4. The number of non-isomorphic power graphs for the nilpotent groups of class at most two and order \( n \) (\( n \) is odd) is equal to the number of non-isomorphic abelian groups of order \( n \).

For finite \( p \)-groups, Theorems 4.1, 4.2 can be generalized for groups of larger class. If the class of a finite \( p \)-group \( G \) is less than \( p \), then there exists a group-word \( w \) such that \( G_w \) is an abelian group [6, p. 446, Theorem 4.8]. In fact, following [6], group-word \( w \) which makes \( G_w \) abelian, can be obtained from Lazard’s inversion of the Baker-Campbell-Hausdorff formula

\[
x \circ y = xy[x, y]^{-1/2}[[x, y], x]^{1/12}[[x, y], y]^{-1/12} \cdots
\]

Thus, in similar manner as above, we can easily deduce the following result.

Theorem 4.5. Let \( X \) be a class of all finite non-isomorphic \( p \)-groups of class less than \( p \). Then for \( G \in X \), \( \mathcal{P}(G) \equiv \mathcal{P}(G_w) \), where \( G_w \) is the corresponding abelian group to \( G \) and two groups in \( X \) have isomorphic power graphs if they are conformal.

Proposition 4.6. Let \( G \) be \( p \)-group of class less than \( p \) with \( |G| = p^{n+1+m} \) such that

\[
G = \langle x_1, x_2, x_3, \ldots, x_s | x_1^{p^s} = x_2^{p^s} = \cdots = x_s^{p^s} = 1, R \rangle,
\]

where \( R \) is a set of commutator relations. Then the corresponding abelian group \( G_w \) is given as

\[
G_w \equiv \mathbb{Z}_{p^1} \times \cdots \times \mathbb{Z}_{p^r}.
\]

Proof. Let \( K = \langle x_1, x_2, x_3, \cdots, x_s | x_1^{p^s} = \cdots = x_s^{p^s} = 1, x_i x_j = x_j x_i \rangle \) for \( i, j \in \{1, \cdots, s\} \). Clearly, \( G_w = \langle x_1, x_2, x_3, \cdots, x_s \rangle \) and \( G_w = G \) as a set. Since powers of each element in \( G \) and \( G_w \) are same, so \( x_1^{p^s} = x_2^{p^s} = \cdots = x_s^{p^s} = 1 \) in \( G_w \). Also, \( x_i x_j = x_j x_i \) for all \( i, j \). Thus, the generators of \( G_w \) satisfy the relations of \( K \), so by Von Dyck’s Theorem [19, Page 51], there is a surjective homomorphism \( \phi : K \rightarrow G_w \) with \( x_i \rightarrow x_i \) for all \( i \in \{1, \cdots, s\} \). Moreover, \( |G_w| = |G| \). So, \( |G_w| = |K| \). Thus, \( G_w \equiv K \). This completes the proof. \( \Box \)

4.1. Power Graph of a Generalized Extraspecial \( p \)-Group, \( p \) Odd

In this subsection, we find the structure of power graph of a generalized extraspecial \( p \)-group \( G \) (\( p \) odd) and as a consequence, we also find the cardinality of the set \( \mathcal{M}_G \).

Let \( G \) be a generalized extraspecial \( p \)-group of order \( p^{2n+m} \), \( m \geq 1 \) and \( p \) odd (for \( m = 1 \), \( G \) will be extraspecial \( p \)-group). Then \( G \) has generators \( a_1, a_2, \cdots, a_{2n}, b \) which satisfy the following conditions:

\[
Z(G) = \langle b \rangle, \quad b^{p^m} = 1, \quad a_i^p = 1 \text{ for } i \in \{2, \cdots, 2n\}
\]

\[
[a_{2i-1}, a_{2j}] = b^{p^{m-1}}, \quad i, j \in \{1, 2, \cdots, n\}
\]

\[
[a_{2i-1}, a_j] = 1, \quad j \neq 2i
\]

\[
[a_{2i}, a_{2j}] = 1, \quad k \neq 2i - 1,
\]

and either \( a_i^p = 1 \) (in this case, \( G \) is called generalized extraspecial \( p \)-group of exponent \( p^m \)) or \( a_i^p = b \) (in this case, \( G \) is called generalized extraspecial \( p \)-group of exponent \( p^{m+1} \)). For more details see [19].

Proposition 4.7.

1. If \( G \) is a generalized extraspecial \( p \)-group of order \( p^{2n+m} \) with exponent \( p^m \) and \( p \) odd, then \( \mathcal{P}(G) \equiv \mathcal{P}(A) \), where \( A \equiv \mathbb{Z}_{p^1} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \). (\( 2n \) factors)

2. If \( G \) is a generalized extraspecial \( p \)-group of order \( p^{2n+m} \) with exponent \( p^{m+1} \) and \( p \) odd, then \( \mathcal{P}(G) \equiv \mathcal{P}(A) \), where \( A \equiv \mathbb{Z}_{p^{m+1}} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \). (\( 2n - 1 \) factors)
Proof. This follows from Theorem 4.5 and Proposition 4.6. □

Now the problem reduces to the problem of determining the power graph of the abelian group $E ≡ ℤ_{p^n} × ℤ_{p^n} × \cdots \times ℤ_{p^n} \equiv (x_1) × (x_2) × \cdots × (x_n)$, where $o(x_1) = p^n$ and $o(x_i) = p$, for $2 \leq i \leq n$ and $n > 1$.

Theorem 4.8. The power graph $P(E)$ is isomorphic to the graph

$$K_1 + [Γ_1 \cup (K_{φ(p)}) + [Γ_2 \cup (K_{φ(p^2)}) + [Γ_3 \cup (K_{φ(p^3)}) + \cdots + [Γ_{m-1} \cup (K_{φ(p^{m-1})}) + [Γ_m \cup K_{φ(p^n)})]] \cdots])],$$

where $Γ_j = \bigcup_{l=1}^{p^n-1} K_{φ(p^j)}$, for $j ∈ \{2, 3, \ldots, m\}$ and $Γ_1 = \bigcup_{l=1}^{p^n-1} K_{φ(p)}$.

Proof. Let us identify $E$ with $(x_1) × (x_2) × \cdots × (x_n)$, where $o(x_1) = p^n$ and $o(x_i) = p$, for $2 \leq i \leq n$. Then by Corollary 3.3, $E$ has $p^{n-1}$ cyclic subgroups of order $p$ and these cyclic subgroups are given as:

$$(x_1^{p^{n-1}}), (x_2^{p^{n-1}}), (x_3^{p^{n-1}}), \ldots, (x_{n-1}^{p^{n-1}}), (x_n^{p^{n-1}}),$$

where $α_i ∈ \{1, 2, \ldots, p\}$ for $1 \leq i \leq n-1$. For $m = 1$, these are the only non-trivial cyclic subgroups of $E$. Assume $m ≥ 2$.

By Lemma 3.1, except the cyclic subgroup $(x_1^{p^{n-1}})$, none of the other cyclic subgroups of order $p$ are contained in cyclic subgroups of a higher order. Moreover, cyclic subgroup $(x_1^{p^{n-1}})$ of order $p^{n-1}$, $t > 1$ is contained in $p^{n-1}$ cyclic subgroups of order $p^t$. Since, the number of all cyclic subgroups of order $p^t$, $t > 1$ in the group $E$ is $p^{n-1}$ (Corollary 3.3), the cyclic subgroup $(x^{p^{n-1}})$ of order $p^{n-1}$ is contained in all cyclic subgroups of order $p^t$.

Recall that $C'(E) = \{ [x] | (x) ∈ C(G) \}$, where $[x] = \{ y ∈ G | (y) = (x) \}$. Thus, the set $C'(E)$ has $p^{n-1}$ equivalence classes of cardinality $φ(p^t)$ for $1 < t ≤ m$, $p^{n-1}$ equivalence classes of cardinality $φ(p)$ and one equivalence class of cardinality one.

Following (1), we write $C'(E) = \{ [V_{i0}], [V_{i}] | i ∈ \{1, \cdots, m\} \text{ and } 1 ≤ t ≤ \frac{p^n-1}{p^i-1}, \text{ for } i=1 \text{ and } 1 ≤ t ≤ p^{n-1}, \text{ for } i > 1 \}$, where $[V_{i}]$ denotes the equivalence class of cardinality $φ(p^t)$. Moreover, $[V_{i0}] = \{1\}$ and $[V_{i}] = \{x_{i0,1}, \cdots, x_{i,φ(p^t)}\}$. By Remark 2.7, each element $[V_{i}]$ gives a chain

$$x_{i0,1} ≤ \cdots ≤ x_{i,φ(p^t)}$$

of length $φ(p^t)$ in the poset $L_E$. Clearly, the identity element of the group $E$ is comparable with every element of $E$ in $L_E$. Now, collecting all arguments, we draw the Hasse diagram of the poset $L_E$ in Figure 1.

In Figure 1, $V_{it}$ denotes the chain

$$\begin{array}{c}
\cdots \\
\vdots \\
x_{i0,2} \\
x_{i0,1} \\
\end{array}$$

$$\begin{array}{c}
x_{i,φ(p^t)} \\
x_{i,φ(p^t-1)} \\
\vdots \\
x_{i2} \\
x_{i1} \\
\end{array}$$

of length $φ(p^t)$ in the poset $L_E$. Clearly, the identity element of the group $E$ is comparable with every element of $E$ in $L_E$. Now, collecting all arguments, we draw the Hasse diagram of the poset $L_E$ in Figure 1.
We know that the comparability graph $\mathcal{T}_E$ of the poset $L_E$ is equal to the power graph of $E$. Now we deduce the $P(G)$ with the help of Figure 1. Each $[V_{ij}]$ is a chain of length $\phi(p^i)$, so the vertices corresponding to the elements of $[V_{ij}]$ give a complete graph $K_{\phi(p^i)}$ in $P(G)$. Moreover, each $[V_{ij}]$ is a homogeneous chain. Therefore if $x_{ij,m} \preceq x_{ij',m'}$ or $x_{ij',m'} \preceq x_{ij,m}$ for some $x_{ij,m} \in [V_{ij}]$ and $x_{ij',m'} \in [V_{ij'}]$, then we get $K_{\phi(p^i)} + K_{\phi(p^j)}$ in $P(G)$ corresponding the vertex subset $[V_{ij}] \cup [V_{ij'}]$, otherwise vertices corresponding to subset $[V_{ij}] \cup [V_{ij'}]$ give union of graphs $K_{\phi(p^i)}$ and $K_{\phi(p^j)}$ in $P(G)$. Now by Figure 1, we can conclude the result.

By Propositions 4.7 and Theorem 4.8, we deduce the following corollaries.

**Corollary 4.9.** Let $G$ be a generalized extraspecial $p$-group of order $p^{2n+m}$ with exponent $p^m$, $p$ odd. Then $P(G)$ is isomorphic to the graph

$$K_1 + \left[ \Gamma_1 \cup \left( K_{\phi(p^1)} + \left[ \Gamma_2 \cup \left( K_{\phi(p^2)} + \left[ \Gamma_3 \cup \left( K_{\phi(p^3)} + \left[ \cdots + \left[ \Gamma_{m-1} \cup \left( K_{\phi(p^{m-1})} + \left[ \Gamma_m \cup K_{\phi(p^{m})} \right] \right] \right] \right] \right] \right] \right] \right] \right],$$

where $\Gamma_j = \bigcup_{i=1}^{p-1} K_{\phi(p^i)}$, for $j \in \{2, 3, \cdots, m\}$ and $\Gamma_1 = \bigcup_{i=1}^{p-1} K_{\phi(p)}$.

**Corollary 4.10.** Let $G$ be a generalized extraspecial $p$-group of order $p^{2n+m}$ with exponent $p^{m+1}$, $p$ odd. Then $P(G)$ is isomorphic to the graph
where \( \Gamma_j = \bigcup_{i=1}^{p^{2n-1}} K_{(q,p^j)}, \) for \( j \in \{2, 3, \ldots, m + 1\} \) and \( \Gamma_1 = \bigcup_{i=1}^{p^{2n-1}} K_{(q,p^j)}. \)

**Theorem 4.11.** Let \( G \) be a generalized extraspecial \( p \)-group of order \( p^{2n+m} \), \( p \) odd. Then

\[
|M_c| = \begin{cases} 
\left( p^{n-1} + (b-2)(p^{n-1}) - 1 \right), & b \geq 2 \\
\left( p^{n-1} \right), & b = 1,
\end{cases}
\]

where \( a = 2n + 1, b = m, \) when exponent of \( G \) is \( p^m \) and \( a = 2n, b = m + 1, \) when exponent of \( G \) is \( p^{m+1}. \)

**Proof.** Firstly, we find the number of maximal cyclic subgroup of \( E \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p. \) By Figure 1, it is clear that for \( x \in [V_{ij}] \) \( (j > 1) \), \( \langle x \rangle \) is a maximal cyclic subgroup of \( E. \) Also for \( x \in [V_{m1}], \) \( \langle x \rangle \) is a maximal cyclic subgroup of \( E. \) Since for \( x, y \in [V_{ij}], \) \( \langle x \rangle = \langle y \rangle. \) So we need to count \( V_{ij} \) for \( j > 1 \) and \( V_{m1}. \) Thus, by Figure 5.1, we have

\[
|M_c| = \begin{cases} 
\left( p^{n-1} + (m-2)(p^{n-1}) - 1 \right), & m \geq 2 \\
\left( p^{n-1} \right), & m = 1.
\end{cases}
\]  

By Theorem 3.5 and Theorem 4.7, for generalized extraspecial \( p \)-group \( G \) of exponent \( p^m |M_c| = |M_{A_1}|, \) where \( A_1 \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p. \) and for generalized extraspecial \( p \)-group \( G \) of exponent \( p^{m+1}, |M_c| = |M_{A_2}|, \) where \( A_2 \cong \mathbb{Z}_{p^{m+1}} \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p. \)

Thus, by (2), we can complete the proof. \( \square \)

**4.2. Power Graph of a Group of Order \( p^n, p \text{ Odd} \)**

In this subsection, we find the structure of power graph of a group of order \( p^n \) \( (p \text{ odd}) \). Following [2], there are 15 groups of order \( p^4 \) up to isomorphism. We number them \( P_1 \) to \( P_{15}. \) The groups \( P_1 \) to \( P_5 \) are abelian, \( P_6 \) to \( P_{10} \) and \( P_{11} \) are of class 2, and \( P_{11} \) to \( P_{13} \) and \( P_{15} \) are of class 3. Here we list the all non-isomorphic groups of order \( p^4 \).

1. \( P_1 = \mathbb{Z}_{p^4}. \)
2. \( P_2 = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}. \)
3. \( P_3 = \mathbb{Z}_{p^4} \times \mathbb{Z}_{p}. \)
4. \( P_4 = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p}. \)
5. \( P_5 = \mathbb{Z}_{p^3} \times \mathbb{Z}_{p} \times \mathbb{Z}_{p} \times \mathbb{Z}_{p}. \)
6. \( P_6 = \langle u, v \mid u^p = v^p = 1, uv = vu, u^2w = uv^2 \rangle. \)
7. \( P_7 = \langle u, v, w \mid u^p = v^p = w^p = 1, uw = vu, u^2w = uw^2 = vw \rangle. \)
8. \( P_8 = \langle u, v \mid u^p = v^p = 1, uv = u^1v^1 \rangle. \)
9. \( P_9 = \langle u, v, w \mid u^p = v^p = w^p = 1, uv = vu, u^2w = vu^2, vw = vw \rangle. \)
10. \( P_{10} = \langle u, v, w \mid u^p = v^p = w^p = 1, uv = vu, w^2uv = uw, wvw = wv \rangle. \)
11. \( P_{11} = \langle u, v, w \mid u^p = v^p = w^p = 1, u^2v = w^1v, w^1u = w^1v, u^2w = uv \rangle. \)
12. \( P_{12} = \langle u, v, w \mid u^p = v^p = w^p = 1, uv = v^1u, w^1uv = w^1v, w^1uw = w^1v, w^2uv = uv \rangle, p > 3. \)
13. \( P_{13} = \langle u, v, w \mid u^p = v^p = w^p = 1, uv = v^1u, w^1uv = w^1v, w^1uw = w^1v, w^2uv = uv \rangle, p = 3. \)
Lemma 4.14. The following hold in groups of order $p^4$, $p > 3$.

1. $\mathcal{P}(P_6) \cong \mathcal{P}(P_2)$.
2. $\mathcal{P}(P_5) \cong \mathcal{P}(P_3)$.
3. $\mathcal{P}(P_7) \cong \mathcal{P}(P_4) \cong \mathcal{P}(P_{10}) \cong \mathcal{P}(P_{11}) \cong \mathcal{P}(P_{12}) \cong \mathcal{P}(P_{13}) \cong \mathcal{P}(P_4)$.
4. $\mathcal{P}(P_{14}) \cong \mathcal{P}(P_{15}) \cong \mathcal{P}(P_5)$.

Proof. This follows from Theorem 4.5 and Proposition 4.6. $\square$

Lemma 4.13. The following hold in groups of order $p^4$, $p = 3$.

1. $\mathcal{P}(P_6) \cong \mathcal{P}(P_2)$.
2. $\mathcal{P}(P_5) \cong \mathcal{P}(P_3)$.
3. $\mathcal{P}(P_7) \cong \mathcal{P}(P_4) \cong \mathcal{P}(P_{10}) \cong \mathcal{P}(P_{11}) \cong \mathcal{P}(P_{12}) \cong \mathcal{P}(P_{13}) \cong \mathcal{P}(P_4)$.
4. $\mathcal{P}(P_{14}) \cong \mathcal{P}(P_{15}) \cong \mathcal{P}(P_5)$.

Proof. $P_6, P_7, P_8, P_9, P_{10}, P_{14}$ are $p$-groups of class 2 and $P_2, P_3, P_4, P_5$ are abelian. Thus, by Theorem 4.1 and Proposition 4.6, we can conclude the result. $\square$

Lemma 4.14. The following hold in groups of order $p^4$, where $p$ is any prime.

1. $\mathcal{P}(P_4) = K_{p^4}$.
2. $\mathcal{P}(P_2) = K_1 + \bigcup_{i=1}^{p} K_{p^{i+1}} \cup \left( K_{p^{i+1}} + \bigcup_{i=1}^{p} K_{p^{i+1}} \right)$.
3. $\mathcal{P}(P_3) = K_1 + \bigcup_{i=1}^{p} K_{p^{i+1}}$.
4. $\mathcal{P}(P_4) = K_1 + \left[ K_{p^{i+1}} \cup \bigcup_{i=1}^{p} K_{p^{i+1}} \right]$.
5. $\mathcal{P}(P_5) = K_1 + \left[ K_{p^{i+1}} \right]$.

Proof. Since $P_1$ is a cyclic group of order $p^4$, $\mathcal{P}(P_1) = K_{p^4}$. Now, 2, 4, and 5 are determined by using Theorem 4.8 and 3 from Theorem 3.4. $\square$

Now, we find the structure of power graphs of groups $P_{11}, P_{12}, P_{13}, P_{15}$, for $p = 3$.

Lemma 4.15. For $p = 3$, the following hold:

1. $\mathcal{P}(P_{12}) = K_1 + \bigcup_{i=1}^{3} K_2 \cup \left( K_2 + \bigcup_{i=1}^{2} K_6 \right)$.
2. $\mathcal{P}(P_{13}) = K_1 + \bigcup_{i=1}^{2} K_2 \cup \left( K_2 + \bigcup_{i=1}^{1} K_6 \right)$.
3. $\mathcal{P}(P_{11}) = K_1 + \bigcup_{i=1}^{2} K_2$.
4. $\mathcal{P}(P_{15}) = K_1 + \bigcup_{i=1}^{2} K_2 \cup \left( K_2 + \bigcup_{i=1}^{1} K_6 \right)$. 
Figure 2: Hasse Diagram of $L_{p_{12}}$

Proof. Let $P_{11}, P_{12}, P_{13},$ and $P_{15}$ be the groups of order 81. Further, let $T = \langle u, v, w \mid u^9 = v^3 = w^3 = u^3w, uv = vu, v^4w = wv^3, uv = wv \rangle$, $\beta \in \{1, -1\}$. Clearly $[u, v] = 1$ and $[w^3, v] = u^3 = 1$. Thus, $u^3 \in Z(G)$. By using relations $v^w = v^u, v^u = w^v$, and $w^v = u^3v^w$, we can show that $v^w = u^3(v^w)^3$ and $w^v = u^3(v^w)^3$, where $1 \leq i \leq 9, 1 \leq j \leq 3$, and $1 \leq k \leq 3$. Thus, each element of the group $T$ can be written in the form $u^i v^j w^k$ for some $i, j, k \geq 1$. By using above relations, we can deduce that $(u^i v^j w^k)^3 = u^{3(i+j+k)} w^{3k}$. Now, for $P_{12}, \beta = 1$. Thus, $(u^i v^j w^k)^3 = u^{3(i+j+k)}$. So, $(u^i v^j w^k)^3 = 1$ for $k = 3, 1 \leq j \leq 3$, and $i \in \{3, 6, 9\}$. Therefore, $P_{12}$ has 8 elements of order 3 and 81 − 9 = 72 elements of order 9 (exponent of $P_{12}$ is 9). Hence, $P_{12}$ has 4 cyclic subgroups of order 3 and 12 cyclic subgroups of order 9.

For $\beta = -1, T = P_{13}$. Thus, $(u^i v^j w^k)^3 = u^{3(i+j+k)}w^{3k}$. In similar manner as above, we can obtain that $P_{13}$ has 13 cyclic subgroups of order 3 and 9 cyclic subgroups of order 9.

Now for $P_{11}, [u^3, v] = u^9 = 1$. By using relations $w^v = w^u, w^u = u^7v$, and $w^v = w^u$, we have $v^i w^j = u^{3(i+j)}$, where $1 \leq i \leq 9, 1 \leq j \leq 3, 1 \leq k \leq 3$. Thus, each element of the group $P_{11}$ can be written in the form $u^i v^j w^k$ for some $i, j, k \geq 1$. By using above relations, we can obtain that $(u^i v^j w^k)^3 = u^{3(i+j+k)}w^{3k}$. Using this relation, similarly as above, we can obtain that $P_{11}$ has 22 cyclic subgroups of order 3 and 6 cyclic subgroups of order 9.

Again for $P_{15}, u^3 \in Z(G)$. Using relations $v^w = v^u, v^u = u^7v^u$, and $w^v = u^3v^w$, we have $w^v = u^3(v^w)^3$. Thus, using last relation, we can deduce that $P_{15}$ has 31 cyclic subgroups of order 3 and 9 cyclic subgroups of order 9.

In all four groups, observe that the cyclic subgroup $\langle u^3 \rangle$ is contained in all cyclic subgroups of order 9. Therefore, we obtain the structure of power graph of the group $P_{12}$ and for remaining groups, power graphs can be obtained by doing similar process. Now, we find $P(P_{12})$. Since $P_{12}$ has 12 cyclic subgroups of order 9 and 4 cyclic subgroups of order 3, the set $C(P_{12})$ has 12 equivalence classes of cardinality 6 and 4 equivalence classes of cardinality 2. Following (1), we write

$C(P_{12}) = \{[V_{0a}], [V_{a}] \mid i \in \{1, 2\} \text{ and } 1 \leq t \leq 4, \text{ for } i=1 \text{ and } 1 \leq t \leq 12, \text{ for } i = 2\}$, where $[V_{a}]$ denotes the equivalence class of cardinality $\phi(3')$. Moreover, $[V_{a}] = \{1\}$ and $[V_{a}] = \{x_{t0}, x_{t0 \phi(3')}, x_{t0 \phi(3')}, x_{t0 \phi(3')}\}$.

The Hasse diagram of the poset $L_{p_{12}}$ is given in Figure 2. Since only one cyclic group of order 3 is contained in all cyclic subgroups of order 9, so only one $V_{11}$ say $V_{11}$ is connected to $V_{21}$ for all $t$ in Hasse diagram of the poset $L_{p_{12}}$.

In Figure 2, recall that $V_{it}$ denote the a chain of length $\phi(3')$ corresponding to element $[V_{it}]$ (see proof of Theorem 4.8). Thus, we get $K_{[\phi(3')]}$ in $P(P_{12})$ corresponding vertex subset $[V_{it}]$.

We know that the comparability graph $T_{L_{p_{12}}}$ of the poset $L_{p_{12}}$ is equal to the power graph of $P_{12}$. Thus, by Figure 2, we can determine that $P(P_{12}) = K_1 + \left[\cup_{i=1}^3 K_2 + \cup_{i=1}^{12} K_6\right]$. This complete the proof.

Theorem 4.16. For $p = 3$, there are 8 non-isomorphic power graphs for groups of order 81 and there are 5 non-isomorphic power graphs for groups of order $p^3, p > 3$. 
Proof. This follows from Lemmas 4.12, 4.13, 4.14, and 4.15.

Remark 4.17. For $p = 3$, $P_4$ and $P_{13}$ are conformal and their power graphs are also same and $P_3, P_{12}$ are conformal but have different power graphs.

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References

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