Filomat 34:7 (2020), 2131–2139
https://doi.org/10.2298/FIL2007131C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Dynamics of Weighted Translations Generated by Group Actions

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**Abstract.** In this paper, we consider actions of locally compact groups on measure spaces, and give a sufficient and necessary condition for weighted translations on such spaces to be chaotic. Moreover, some dynamical properties for certain cosine operator functions, generated by translations, are proved as well.

## 1. Introduction

A bounded linear operator $T$ on a separable Banach space $\mathcal{X}$ is called *hypercyclic* if there is an element $x \in \mathcal{X}$ such that the set $\{T^n x : n \in \mathbb{N}\}$ is dense in $\mathcal{X}$. Hypercyclicity arises from the invariant closed subset problem in analysis, and is related to topological dynamics. Indeed, it is well-known (cf. [18]) that topological transitivity and hypercyclicity are equivalent on $\mathcal{X}$. An operator $T$ is said to be *topologically transitive* if for any pair of non-empty open sets $U, V$ in $\mathcal{X}$, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Moreover, the operator $T$ is called *chaotic* if it is hypercyclic (transitive) and it possesses a dense set of periodic elements. Linear dynamics and hypercyclicity have been studied intensely during the last four decades. We refer to these two classic books [3, 18] on this subject.

The study of linear dynamics on the Lebesgue space $L^p(G)$ of locally compact groups $G$ was initialed by [7, 10, 11]. Since then, the theme on $L^p(G)$ attracted a lot of attention. For instance, disjoint hypercyclicity of weighted translations on $L^p(G)$ was characterized by [9, 19, 25]. Also, the existence of hypercyclic weighted translations on $L^p(G)$ was discussed in [15]. We refer to papers [13, 14] for readers interested in the hypergroup case and vector-valued version. Besides, linear dynamics for weighted translations on the Orlicz space $L^\Phi(G)$ were first investigated by [2, 12] where $\Phi$ is a Young function. Also, Abakumov and Kuznetsova in [1] focused on the density of translates in the weighted Lebesgue space $L^p_w(G)$, and observed some different phenomenon from that in [11], where $w$ is a weight on $G$. Inspired by [7, 11], the aim of this paper is to address the characterizations of linear dynamics in the wider setting, namely, group actions on measure spaces, and give some applications for quotient spaces and locally compact hypergroups. We recall the notion of group action below.

**Definition 1.1.** *Let $G$ be a locally compact group, $X$ be a locally compact Hausdorff space, and $\mu$ be a nonnegative Borel measure on $X$. A continuous function*

$$G \times X \longrightarrow X, \quad (s, x) \mapsto sx, \quad (s \in G, x \in X)$$

2010 *Mathematics Subject Classification*. Primary 54H20; Secondary 47A16
*Keywords*. chaos, topological transitivity, locally compact group, group action
Received: 26 May 2019; Accepted: 18 March 2020
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*is called an* action *of G on the measure space* $(X, \mu)$ *if*

(i) *for each* $x \in X$, $ex = x$, *where e is the identity element of G;*
(ii) *for each* $s, t \in G$ *and* $x \in X$, $s(tx) = (st)x$;
(iii) *for each* $s \in G$ *and any Borel subset* $E \subseteq X$, $sE := \{sx : x \in E\}$ *is also a Borel subset of X and* $\mu(sE) = \mu(E)$.

*In this case, simply we write* $G \curvearrowright (X, \mu)$.

In the sequel, we assume that $G$ is a locally compact group and $(X, \mu)$ is a Borel measure space. For each $1 \leq p < \infty$, the usual Lebesgue space on $(X, \mu)$ is denoted by $L^p(X, \mu)$. As a well-known fact, $L^p(X, \mu)$ is separable if $X$ is a second-countable locally compact Hausdorff space [4, Proposition 2.3]. Throughout, we assume $X$ is second-countable locally compact Hausdorff. We first introduce our setting briefly.

Let $G \curvearrowright (X, \mu)$, $a \in G$, $1 \leq p < \infty$ and $w : X \longrightarrow (0, \infty)$ be a bounded continuous function (called a *weight* on $G$). Then, the weighted translation operator $T_{a,w} : L^p(X, \mu) \longrightarrow L^p(X, \mu)$ is defined by

$$T_{a,w}(f)(x) := w(x)f(a^{-1}x),$$

where $f \in L^p(X, \mu)$ and $x \in X$. Clearly, $T_{a,w}$ is linear. Setting $\|w\|_{\sup} := \sup_{x \in X} w(x)$, we have $\|w\|_{\sup} < \infty$ and for each $f \in L^p(X, \mu)$,

$$\begin{aligned}
\|T_{a,w}f\|_p^p &= \int_X |T_{a,w}f(x)|^p \, d\mu(x) \\
&= \int_X |f(a^{-1}x)|^p w(x)^p \, d\mu(x) \\
&= \int_X |f(x)|^p w(ax)^p \, d\mu(x) \\
&\leq \|w\|_{\sup}^p \int_X |f(x)|^p \, d\mu(x) \\
&= \|w\|_{\sup}^p \, \|f\|_p^p.
\end{aligned}$$

This implies that $\|T_{a,w}\| \leq \|w\|_{\sup}$, and so, $T_{a,w}$ is a well-defined bounded operator on $L^p(X, \mu)$.

The work in [7, 11] highly relies on the property of some special elements of a group, namely, aperiodic elements. In a locally compact group $G$, an element $a \in G$ is called *periodic* (or *compact* in [20]) if the closed subgroup $G(a)$ generated by $a$ is compact. We call an element in $G$ *aperiodic* if it is not periodic. In [11, Lemma 2.1], the authors obtained an equivalent condition of aperiodicity. Here, we give the definition of aperiodic elements with respect to an action below.

**Definition 1.2.** *Let* $G \curvearrowright (X, \mu)$. *An element* $a \in G$ *is called* aperiodic *with respect to the given action if for each compact subset* $E \subseteq X$, *there exists an integer* $N > 0$ *such that for each* $n > N$, *we have* $a^n E \cap E = \emptyset$.

We note that if we consider the natural action of a locally compact group $G$ on itself, that is, $X = G$, then the above definition coincides with the concept of aperiodic elements of $G$ in [11, Lemma 2.1].

In Section 2, we will study chaotic weighted translations on $L^p(X, \mu)$. Dynamics of certain cosine operator functions will be tackled in Section 3. In Section 4, we will give group actions to some special cases including quotient spaces and locally compact hypergroups as well.

## 2. Chaotic weighted translations

In this section, we will give some sufficient and necessary conditions for weighted translation operators to be chaotic. In order to simplify the notation, we write $\kappa_t w(x) := w(t^{-1}x)$ where $t \in G$ and $x \in X$. Then, for each $f \in L^p(X,\mu)$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} T_{a,w}^m f(x) &= T_{a,w}^{m-1}\left(w\,\kappa_a f\right)(x) \\ &= T_{a,w}^{m-2}(w\,\kappa_a w\,\kappa_{a^2} f)(x) \\ &= (w\,\kappa_a w\,\kappa_{a^2} w\cdots\kappa_{a^{m-1}} w\,\kappa_{a^m} f)(x) \\ &= \prod_{s=0}^{m-1}\kappa_{a^s} w(x)\, f(a^{-m}x). \end{aligned}$$

Now, we are ready to give the main result of this section.

**Theorem 2.1.** *Let $G \curvearrowright (X,\mu)$, and $w : X \longrightarrow (0,\infty)$ be a weight. Let $1 \le p < \infty$, and $a \in G$ be an aperiodic element with respect to the given action. Let $T_{a,w}$ be the weighted translation operator on $L^p(X,\mu)$. Then the followings are equivalent.*

(i) *$T_{a,w}$ is chaotic.*

(ii) *$\mathcal{P}(T_{a,w})$ is dense in $L^p(X,\mu)$.*

(iii) *For each compact subset $E \subseteq X$ with $\mu(E) > 0$, there exists a sequence $(S_k)_{k=1}^{\infty}$ of Borel subsets of $E$ such that $\mu(E) = \lim_{k\to\infty}\mu(S_k)$ and*

$$\lim_{k\longrightarrow\infty}\left(\sum_{l=1}^{\infty}\int_{S_k}\left(\prod_{s=1}^{ln_k}\kappa_{a^{-s}} w\right)^p d\mu + \sum_{l=1}^{\infty}\int_{S_k}\left(\prod_{s=0}^{ln_k-1}\kappa_{a^s} w\right)^{-p} d\mu\right) = 0, \tag{1}$$

*for some strictly increasing sequence $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$.*

*Proof.* (ii)⇒ (iii). Suppose that $\mathcal{P}(T_{a,w})$ is dense in $L^p(X,\mu)$, $E \subseteq X$ is compact, and $\mu(E) > 0$. Since $a$ is an aperiodic element of $G$ with respect to the action $G \curvearrowright (X,\mu)$, there is $N \in \mathbb{N}$ such that for each $n > N$, $a^nE \bigcap E = \varnothing$. So, for each distinct $r,s \in \mathbb{Z}$ and $n > N$, we have $a^{rn}E \bigcap a^{sn}E = \varnothing$. Note that $\chi_E \in L^p(X,\mu)$ because $E$ is compact. Since $\mathcal{P}(T_{a,w})$ is dense in $L^p(X,\mu)$, for each $k \in \mathbb{N}$, there is an element $f_k \in \mathcal{P}(T_{a,w})$ such that $\|f_k - \chi_E\|_p < \frac{1}{4^k}$. By the definition of chaos, there exists an increasing sequence $(n_k) \subset \mathbb{N}$ with $n_k > N$ such that $T_{a,w}^{n_k} f_k = f_k$. So, for each distinct $r,s \in \mathbb{Z}$, we have $a^{rn_k}E \bigcap a^{sn_k}E = \varnothing$.

Put

$$A_k := \left\{x \in E : \left|\prod_{s=0}^{n_k-1}\kappa_{a^s} w(x) f_k(a^{-n_k}x) - 1\right| \ge \frac{1}{2^k}\right\}, \quad (k \in \mathbb{N}).$$

One can easily see that for each $x \in E$, $x \in A_k$ if and only if

$$\left|\prod_{s=0}^{ln_k-1}\kappa_{a^s} w(x) f_k(a^{-ln_k}x) - 1\right| \ge \frac{1}{2^k}$$

for all $l \in \mathbb{N}$. By Definition 1.1, $A_k$ is a Borel set. By the definition of $A_k$ above, we have

$$\begin{aligned} \frac{1}{2^{pk}}\mu(A_k) &\le \int_{A_k} |w(x)w(a^{-1}x)...w(a^{-(n_k-1)}x)f(a^{-n_k}x) - 1|^p d\mu(x) \\ &\le \int_X |T_{a,w}^{n_k} f_k(x) - \chi_E(x)|^p d\mu(x) \\ &= \|T_{a,w}^{n_k} f_k - \chi_E\|_p^p \\ &= \|f_k - \chi_E\|_p^p < \frac{1}{4^{pk}}. \end{aligned}$$

and so $\mu(A_k) < \dfrac{1}{2^{pk}}$.

Put

$$B_k := \left\{x \in E : |f_k(x) - 1| \geq \frac{1}{2^k}\right\}, \quad (k \in \mathbb{N}).$$

Then,

$$\begin{aligned}\frac{1}{2^{pk}}\mu(B_k) &\leq \int_{B_k} |f(x) - 1|^p d\mu(x) \\ &= \int_X |f_k(x) - \chi_E(x)|^p d\mu(x) \\ &= \|f_k - \chi_E\|_p^p < \frac{1}{4^{pk}},\end{aligned}$$

and so $\mu(B_k) < \dfrac{1}{2^{pk}}$.

For each $k \in \mathbb{N}$, put $S_k := E\backslash(A_k \cup B_k)$. So $\lim_{k\to\infty} \mu(S_k) = \mu(E)$. Since $\mu$ is a $G$-invariant measure, for each $l \in \mathbb{N}$, we have

$$\begin{aligned}\int_{a^{ln_k}E} |f_k(x)|^p d\mu(x) &= \int_E |f_k(a^{ln_k}x)|^p d\mu(x) \\ &= \int_E |T_{a,w}^{ln_k} f_k(a^{ln_k}x)|^p d\mu(x) \\ &\geq \int_{S_k} |T_{a,w}^{ln_k} f_k(a^{ln_k}x)|^p d\mu(x) \\ &= \int_{S_k} \left(\prod_{s=0}^{ln_k-1} \kappa_{a^s} w(a^{ln_k}x)\right)^p |f_k(x)|^p \, d\mu(x) \\ &= \int_{S_k} \left(\prod_{s=1}^{ln_k} \kappa_{a^{-s}} w(x)\right)^p |f_k(x)|^p \, d\mu(x) \\ &\geq (1 - \frac{1}{2^k})^p \int_{S_k} \left(\prod_{s=1}^{ln_k} \kappa_{a^{-s}} w(x)\right)^p d\mu(x).\end{aligned}$$

Similarly, one can obtain

$$\int_{a^{-ln_k}E} |f_k(x)|^p d\mu(x) \geq (1 - \frac{1}{2^k})^p \int_{S_k} \left(\prod_{s=0}^{ln_k-1} \kappa_{a^s} w\right)^{-p} d\mu(x).$$

By $a^{rn_k}E \bigcap a^{sn_k}E = \varnothing$ for each distinct $r, s \in \mathbb{Z}$, we have

$$\begin{aligned}\frac{1}{4^{pk}} > \|f_k - \chi_E\|_p^p &\geq \int_{X\backslash E} |f_k(x)|^p d\mu(x) \\ &\geq \sum_{l=1}^{\infty} \int_{a^{ln_k}E} |f_k(x)|^p d\mu(x) + \sum_{l=1}^{\infty} \int_{a^{-ln_k}E} |f_k(x)|^p d\mu(x) \\ &\geq (1 - \frac{1}{2^k})^p \sum_{l=1}^{\infty} \int_{S_k} \left(\prod_{s=0}^{ln_k-1} \kappa_{a^s} w\right)^{-p} d\mu(x) \\ &\quad + (1 - \frac{1}{2^k})^p \sum_{l=1}^{\infty} \int_{S_k} \left(\prod_{s=1}^{ln_k} \kappa_{a^{-s}} w(x)\right)^p d\mu(x).\end{aligned}$$

Therefore condition (iii) follows.

(iii)⇒ (i). Let $U$ and $V$ be non-empty open subsets of $L^p(X,\mu)$. We prove that $T^n_{a,w}(U)\cap V\neq\emptyset$ for some $n\in\mathbb{N}$. Given $U$ and $V$, there are continuous compact supported functions $f,g:X\to\mathbb{C}$ such that $f\in U$ and $g\in V$. Put $E:=\operatorname{supp} f\cup\operatorname{supp} g$. Then, there exist an increasing sequence $(n_k)\subseteq\mathbb{N}$ and a sequence $(S_k)$ of subsets of $E$ satisfying the conditions in (iii). Since $E$ is compact and $a$ is an aperiodic element of $G$ with respect to the given action, there exists a constant $N\in\mathbb{N}$ such that for each $n\geq N$, $a^nE\bigcap E=\emptyset$. By the relation (1), for each $l\in\mathbb{N}$,

$$\left\|\chi_{S_k}\prod_{s=1}^{ln_k}\kappa_{a^{-s}}w\right\|_p\longrightarrow 0,$$

as $k\to\infty$. But we have

$$\|T^{n_k}_{a,w}(\chi_{S_k}f)\|_p^p=\int_{S_k}\left(\prod_{s=1}^{n_k}\kappa_{a^{-s}}w\right)^p|f|^p\,d\mu\leq\|f\|^p_{\sup}\int_{S_k}\left(\prod_{s=1}^{n_k}\kappa_{a^{-s}}w\right)^p\,d\mu,$$

i.e. $\|T^{n_k}_{a,w}(\chi_{S_k}f)\|_p\longrightarrow 0$, as $k\to\infty$. Similar to the above argument, we can see that

$$\lim_{k\to\infty}\|H^{n_k}_{a,w}(\chi_{S_k}g)\|_p=0,$$

where $H_{a,w}h:=\kappa_{a^{-1}}(\frac{h}{w})$ for all $h\in L^p(X,\mu)$.

Also, for each compact supported function $h\in L^p(X,\mu)$, we have $T_{a,w}H_{a,w}h=h$. So,

$$\lim_{k\to\infty}(\chi_{S_k}f-H^{n_k}_{a,w}(g\chi_{S_k}))=f,\quad\text{and}\quad\lim_{k\to\infty}T^{n_k}_{a,w}(\chi_{S_k}f-H^{n_k}_{a,w}(g\chi_{S_k}))=g.$$

This implies that $T^{n_k}_{a,w}(U)\cap V\neq\emptyset$ for a large number $k$.

Now, we show that $\mathcal{P}(T_{a,w})$ is dense in $L^p(X,\mu)$. For this, let $U$ be a nonempty open subset of $L^p(X,\mu)$. Then there is a compact supported continuous function $f$ in $U$. Setting

$$v_k:=f\chi_{S_k}+\sum_{l=1}^{\infty}T^{ln_k}_{a,w}(f\chi_{S_k})+\sum_{l=1}^{\infty}H^{ln_k}_{a,w}(f\chi_{S_k}),\quad(k\in\mathbb{N}),$$

we have $\lim_{k\to\infty}v_k=f$ in $L^p(X,\mu)$ and $T^{n_k}_{a,w}v_k=v_k$. So, $U\cap\mathcal{P}(T_{a,w})\neq\emptyset$, which says that $\mathcal{P}(T_{a,w})$ is dense in $L^p(X,\mu)$. □

## 3. Dynamics of cosine operator functions

In this section, we will turn our attention to dynamics of cosine operator functions. Our study on cosine operator functions is motivated by [8]. However, the recent paper [8] is originally inspired by the works [6, 22, 23].

We recall that a *cosine operator function* on a Banach space $\mathcal{X}$ is a mapping $C$ from the real line into the space of bounded operators on $\mathcal{X}$ satisfying $C(0)=I$, and for each $s,t\in\mathbb{R}$, $2C(t)C(s)=C(t+s)+C(t-s)$. The latter equality is called the d'Alembert functional equation, which implies $C(t)=C(-t)$ for all $t\in\mathbb{R}$. In [6], Bonilla and Miana obtained a sufficient condition for a cosine operator function $C$ defined by

$$C(t):=\frac{1}{2}(T(t)+T(-t)),\quad(t\in\mathbb{R})$$

to be topologically transitive, where $T$ is a strongly continuous translation group on some weighted Lebesgue space $L^p(\mathbb{R})$. Also, for a Borel measure $\mu$ and $\Omega\subset\mathbb{R}^d$, Kalmes characterized in [22] transitive cosine operator functions, generated by second order partial differential operators on $L^p(\Omega,\mu)$. Moreover, Kostić showed the main structural properties of hypercyclic and chaotic integrated C-cosine functions in

[23]. In [8], we study cosine operator functions on the Lebesgue space $L^p(G)$ of locally compact groups $G$. In this section, we will extend the results of [8] to the group actions.

In the following, we will define cosine operator functions, generated by weighted translations, for our setting. Let $w$ and $w^{-1} := \frac{1}{w}$ be weights on $X$. Then, by continuity of the action, for each $b \in G$, the function $\kappa_b w$ is also a weight on $X$. In this case, with some calculation, one can see that $T_{a,w}$ is bijective and

$$T_{a,w}^{-1} = T_{a^{-1},\kappa_{a^{-1}}w^{-1}}.$$

For brevity, we put $S_{a,w} := T_{a,w}^{-1}$. Also, for each $n \in \mathbb{N}_0$, we define

$$C_{a,w}^{(n)} := \frac{1}{2}(T_{a,w}^n + S_{a,w}^n).$$

If we set $C(n) := C_{a,w}^{(n)}$ for $n \in \mathbb{N}_0$, then the sequence of operators $(C_{a,w}^{(n)})_{n\in\mathbb{N}_0}$ can be regarded as a discrete cosine operator function. We will give the sufficient conditions of topological transitivity and mixing for such cosine operator functions.

A sequence of operators $(T_n)_{n\in\mathbb{N}_0}$ on a separable Banach space $\mathcal{X}$ is called *hypercyclic* if there exists an element $x \in \mathcal{X}$ such that the orbit $\{T_n x : n \in \mathbb{N}_0\}$ is dense in $\mathcal{X}$. Also, $(T_n)_{n\in\mathbb{N}_0}$ is called *topologically transitive* if for any non-empty open subsets $U$ and $V$ of $X$, we have $T_n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$. Similarly, $(T_n)_{n\in\mathbb{N}_0}$ is said to be *topologically mixing* if $(T_n)_{n\in\mathbb{N}_0}$ satisfies $T_n(U) \cap V \neq \emptyset$ from some $n$ onwards.

**Theorem 3.1.** *Let $G \curvearrowright (X,\mu)$, and $w, w^{-1}$ be two weight functions on $X$. Let $1 \le p < \infty$, and $a \in G$ be an aperiodic element with respect to the given action. Then* (ii) *implies* (i)*.*

(i) *The sequence $(C_{a,w}^{(n)})_{n\in\mathbb{N}_0}$ is topologically transitive.*
(ii) *For each compact subset $K \subseteq X$ with $\mu(K) > 0$, there exist sequences $(D_k)$, $(E_k)$ and $(F_k)$ of Borel subsets of $X$, and a sequence $(n_k)$ of positive numbers such that for each $k$, $D_k = E_k \cup F_k$, $\mu(K) = \lim_{k\to\infty} \mu(D_k)$ and*

$$\lim_{k\longrightarrow\infty} \int_{D_k} \left(\prod_{s=1}^{n_k} \kappa_{a^{-s}} w\right)^p d\mu = \lim_{k\longrightarrow\infty} \int_{D_k} \left(\prod_{s=0}^{n_k-1} \kappa_{a^{s}} w\right)^{-p} d\mu = 0$$

*and*

$$\lim_{k\longrightarrow\infty} \int_{E_k} \left(\prod_{s=1}^{2n_k} \kappa_{a^{-s}} w\right)^p d\mu = \lim_{k\longrightarrow\infty} \int_{F_k} \left(\prod_{s=0}^{2n_k-1} \kappa_{a^{s}} w\right)^{-p} d\mu = 0.$$

*Proof.* (ii) ⇒ (i): Suppose condition (ii) holds, and $U$ and $V$ are nontrivial open subsets of $L^p(X,\mu)$. So, we can pick the compact supported non-zero functions $f \in U$ and $g \in V$. Let $K := \operatorname{supp} f \cup \operatorname{supp} g$. Since $K$ is compact, there are sequences $(D_k)$, $(E_k)$, $(F_k)$ and $(n_k)$ satisfying the condition (ii). Hence

$$\begin{aligned}\left\|T_{a,w}^{n_k}(f\chi_{D_k})\right\|_p^p &= \int_{D_k} \left(\prod_{s=0}^{n_k} \kappa_{a^{-s}} w(x)\right)^p |f(x)|^p \, d\mu(x) \\ &\le \sup_{x\in X} |f(x)|^p \int_{D_k} \left(\prod_{s=1}^{n_k} \kappa_{a^{-s}} w(x)\right)^p d\mu(x) \to 0,\end{aligned}$$

as $k \to \infty$. This implies that $\lim_{k\longrightarrow\infty} T_{a,w}^{n_k}(f\chi_{D_k}) = 0$. Similarly, we have

$$\begin{aligned}\lim_{k\longrightarrow\infty} S_{a,w}^{n_k}(f\chi_{D_k}) &= \lim_{k\longrightarrow\infty} T_{a,w}^{2n_k}(f\chi_{E_k}) = \lim_{k\longrightarrow\infty} S_{a,w}^{2n_k}(f\chi_{F_k}) = \lim_{k\longrightarrow\infty} T_{a,w}^{n_k}(g\chi_{D_k}) \\ &= \lim_{k\longrightarrow\infty} S_{a,w}^{n_k}(g\chi_{D_k}) = \lim_{k\longrightarrow\infty} T_{a,w}^{2n_k}(g\chi_{E_k}) = \lim_{k\longrightarrow\infty} S_{a,w}^{2n_k}(g\chi_{F_k}) = 0.\end{aligned}$$

Setting

$$v_k := f\chi_{D_k} + 2T_{a,w}^{n_k}(g\chi_{E_k}) + 2S_{a,w}^{n_k}(g\chi_{F_k}),$$

for all $k$, we have

$$\lim_{k\longrightarrow\infty} v_k = f \quad \text{and} \quad \lim_{k\longrightarrow\infty} C_{a,w}^{(n_k)} v_k = g.$$

This implies that there is an index $k$ with $C_{a,w}^{(n_k)} v_k \in C_{a,w}^{(n_k)}(U) \cap V$. So, condition (i) follows. □

By the above similar arguments, we have the following result for topological mixing.

**Corollary 3.2.** *Let $G \curvearrowright (X, \mu)$, and $w, w^{-1}$ be two weight functions on $X$. Let $1 \leq p < \infty$, and $a \in G$ be an aperiodic element with respect to the given action. Then,* (ii) *implies* (i)*.*

(i) *The sequence $(C_{a,w}^{(n)})_{n\in\mathbb{N}_0}$ is topologically mixing.*
(ii) *For each compact subset $K \subseteq X$ with $\mu(K) > 0$, there exists a sequence $(D_k)$ of Borel subsets of $X$ such that $\mu(K) = \lim_{k\to\infty} \mu(D_k)$ and*

$$\lim_{k\longrightarrow\infty} \int_{D_k} \left(\prod_{s=1}^{k} \kappa_{a^{-s}} w\right)^{p} d\mu = \lim_{k\longrightarrow\infty} \int_{D_k} \left(\prod_{s=0}^{k-1} \kappa_{a^{s}} w\right)^{-p} d\mu = 0.$$

## 4. Applications

In this final section, we will consider the group actions on quotient spaces and locally compact hypergroups, that is, $G \curvearrowright (X, \mu)$ when $X$ is a quotient space, or a locally compact hypergroup. In particular, one can apply the results of Sections 2 and 3 to the cases of quotient spaces and hypergroups.

### *4.1. Quotient spaces*

Let $G$ be a locally compact group and $H$ be a closed subgroup of $G$ such that $\Delta_H = \Delta_G|_H$, where $\Delta_G$ and $\Delta_H$ are modular functions of $G$ and $H$, respectively. Then, $G$ naturally acts on the quotient space $(G/H, \mu)$, where $\mu$ is the $G$-invariant regular measure on $G/H$ as in [17, Theorem 2.49].

In this case, $a \in G$ is an aperiodic element with respect to this action if, and only if, for each compact subset $E \subseteq G/H$, there is $N > 0$ such that $a^n E \cap E = \varnothing$ for all $n \geq N$. Let $E(a)$ be the subgroup generated by an element $a \in G$, and denote the closure of $E(a)$ by $G(a)$. We have the following simple observations for aperiodic elements.

**Proposition 4.1.** *Let $a \in G$ be an aperiodic element with respect to the natural action of $G$ on $G/H$. Then*

(i) *for all $x, y \in G$, there exits a constant $N > 0$ such that $xa^n y \notin H$ for all $n \geq N$;*
(ii) *$E(a) \cap H = \{e\}$.*

*Proof.* Let $a \in G$ be aperiodic with respect to the natural action of $G$ on $G/H$. Let $x, y \in G$. Then there is $N > 0$ such that for each $n \geq N$,

$$a^n\{x^{-1}H, yH\} \cap \{x^{-1}H, yH\} = \varnothing.$$

This implies that $a^n yH \neq x^{-1}H$, i.e. $xa^n y \notin H$. (ii) follows from (i) directly. □

**Corollary 4.2.** *Let $G/H$ be a discrete quotient space. Then $a \in G$ is aperiodic with respect to the natural action of $G$ on $G/H$ if and only if for all $x, y \in G$, there exits a constant $N > 0$ such that $xa^n y \notin H$ for all $n \geq N$.*

**Corollary 4.3.** *Let $H$ be an open and closed subgroup of $G$ and $a \in G$ be an aperiodic element for the action $G \curvearrowright G/H$. Then $G(a) \cap H = \{e\}$.*

### 4.2. Locally compact hypergroups

Roughly speaking, a locally compact hypergroup $K$ is a locally compact Hausdorff space such that the space $\mathcal{M}(K)$ of all its Radon measures is a Banach algebra without necessarily an action between elements of $K$. For the definition and properties of a locally compact hypergroup, we refer to the paper [21] (which is named *convo*) and the book [5]. The classical examples of hypergroups include locally compact groups, double coset spaces, polynomial hypergroups, orbit hypergroups and so on.

In this subsection, we assume that $K$ is a locally compact hypergroup with a *convolution* $* : \mathcal{M}(K) \times \mathcal{M}(K) \to \mathcal{M}(K)$, an *involution* $x \mapsto \check{x}$ on $K$, and an identity element $e$. Also, we assume that $m$ is a left Haar measure on $K$ i.e. for each $x \in K$, $\delta_x * m = m$, where $\delta_x$ is the Dirac measure at $x$. Contrary to the group case, for each $x, y \in K$, the convolution $\delta_x * \delta_y$ of two Dirac measures is not necessarily a Dirac measure. However, this is not the case for center elements of hypergroups. An element $x \in K$ is called a *center element* if for each $y \in K$, there is an element $\alpha(x, y) \in K$ such that $\delta_x * \delta_y = \delta_{\alpha(x,y)}$. The set $\mathrm{Ma}(K)$ of all center elements of $K$ is the maximal subgroup of $K$, and naturally acts on $(K, m)$ by the above mapping $(x, y) \mapsto \alpha(x, y)$ [21, 10.4]. So, the results of this paper can be applied to the action $\mathrm{Ma}(K) \curvearrowright (K, m)$.

**Example 4.4.** *Let a locally compact group $G$ be a Z-group i.e. the quotient space $G/Z(G)$ is compact, where $Z(G) := \{z \in G : \text{ for each } x \in G,\ zx = xz\}$ is the center of $G$. Setting $I := Inn(G)$ the compact group of inner automorphisms of $G$, we define the orbit space $G^I$ by*

$$G^I := \{x^I : x \in G\},$$

*where $x^I := \{g^{-1}xg : g \in G\}$. Then, $G^I$ is a hypergroup with the convolution*

$$(\delta_{x^I} * \delta_{y^I})(f) := \int_I f((\beta(x)y)^I)\, d\mu(\beta), \quad (f \in C_0(G^I))$$

*where $\mu$ is the normalized Haar measure on $I$, and the involution $(x^I)\check{} := (x^{-1})^I$; see [21, Theorem 8.3A]. Now, by [24] we have*

$$Ma(G^I) = \{z^I : z \in Z(G)\}.$$

*For this hypergroup, the above action $\alpha$ is as the following:*

$$\alpha(z^I, x^I) = (zx)^I, \quad (x \in G,\ z \in Z(G)).$$

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