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Fractional Inequalities Associated With a Generalized Mittag-Leffler Function and Applications

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Abstract. Mittag-Leffler function is very useful in the theory of fractional differential equations. Ostrowski's inequality is very important in numerical computations and error analysis of numerical quadrature rules. In this paper, Ostrowski's inequality via generalized Mittag-Leffler function is established. In application point of view, bounds of fractional Hadamard's inequalities are straightforward consequences of these inequalities. The presented results are also contained in particular several fractional inequalities and have connection with some known and already published results.

1. Introduction

Exponential function plays a vital role in the theory of integer order differential equations. A Swedish mathematician Mittag-Leffler introduced a generalization of exponential function denoted by $E_\alpha(z)$ which is well known as the Mittag-Leffler function and is defined in [13]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}; z \in \mathbb{C},$$

where $\Gamma(.)$ is the gamma function and $\alpha \in \mathbb{C}, \Re(\alpha) > 0$. It plays an important role in the theory of fractional differential equations. It arises naturally in the solution of fractional differential equations and produces several well known classical functions by taking particular values of parameters α and β. The Mittag-Leffler functions are used in many areas of science and engineering especially in the investigations of the fractional generalization of kinetic equation and in the study of complex systems. Due to its importance, Mittag-Leffler function is generalized by many mathematicians: for example Wiman [23], Prabhakar [15], Shukla and Prajapati [20], Salim [18], Salim and Faraj [19], Rahman et al. [17].

Recently, Andrić et al. [1] defined the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(.;p)$ as follows:

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Definition 1.1. *Let $\mu, \alpha, l, \gamma, c \in C$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is defined as*

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{n,k}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{1}$$

Here β_p is the generalized beta function defined as

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt$$

and $(c)_{n,k}$ is the Pochhammer k-symbol defined as $(c)_{n,k} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

In [1], properties of the generalized Mittag-Leffler function are discussed in which derivative property of the generalized Mittag-Leffler function is given as follows.

Theorem 1.2. *If $m \in N, \omega, \mu, \alpha, l, \gamma, c \in C, \Re(\mu), \Re(\alpha), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$ and $0 < k < \delta + \Re(\mu)$, then*

$$\begin{aligned}
&\left(\frac{d}{dt}\right)^m E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) \\
&= \frac{(c)_{mk}}{(l)_{m\delta}} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + (n+m)k, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c+mk)_{nk}}{\Gamma(\mu(n+m)+\alpha)} \frac{(n+1)_m t^n}{(l+m\delta)_{n\delta}}; \\
&\left(\frac{d}{dt}\right)^m [t^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)] = t^{\alpha-m-1} E^{\gamma,\delta,k,c}_{\mu,\alpha-m,l}(\omega t^{\mu};p) \quad \Re(\alpha) > m.
\end{aligned} \tag{2}$$

For more information on both special functions and such functions, see, *e.g.*, [4, 8, 11, 12, 21]. The corresponding left and right sided generalized fractional integrals $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}$ are defined as follows.

Definition 1.3. *[1] Let $\omega, \mu, \alpha, l, \gamma, c \in C$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integrals $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f$ are defined as*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p) f(t) dt \tag{3}$$

and

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f\right)(x;p) = \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(t-x)^{\mu};p) f(t) dt. \tag{4}$$

From extended generalized fractional integrals, we have

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} 1\right)(x;p) \\
&= \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(w(x-t)^{\mu};p) dt \\
&= \int_a^x (x-t)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\gamma+nk, c-\gamma)}{\mathrm{B}(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{w^n (x-t)^{\mu n}}{(l)_{n\delta}} dt \\
&= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\gamma+nk, c-\gamma)}{\mathrm{B}(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{w^n}{(l)_{n\delta}} \int_a^x (x-t)^{\mu n+\alpha-1} dt \\
&= (x-a)^{\alpha} \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\gamma+nk, c-\gamma)}{\mathrm{B}(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{w^n}{(l)_{n\delta}} (x-a)^{\mu n} \frac{1}{\mu n+\alpha}.
\end{aligned}$$

Hence,

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}1\right)(x;p) = (x-a)^{\alpha}\, E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(w(x-a)^{\mu};p)$$

and similarly

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-}1\right)(x;p) = (b-x)^{\beta}\, E^{\gamma,\delta,k,c}_{\mu,\beta+1,l}(w(b-x)^{\mu};p).$$

We use the following notations in our results

$$C_{\alpha,a^+}(x;p) = \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}1\right)(x;p) \tag{5}$$

and

$$C_{\beta,b^-}(x;p) = \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-}1\right)(x;p). \tag{6}$$

Recently, we have published a version of the Hadamard's inequality by using this generalized form of fractional integral operator [9].

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for extended generalized fractional integral holds:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) C_{\alpha,a^+}(b;p) &\leq \frac{\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',a^+}f\right)(b;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',b^-}f\right)(a;p)}{2} \\ &\leq \left(\frac{f(a)+f(b)}{2}\right) C_{\alpha,b^-}(a;p), \end{aligned} \tag{7}$$

where $\omega' = \frac{\omega}{(b-a)^{\mu}}$.

Inequality (7) is a compact generalization of the Hadamard's inequality which produces a variety of fractional Hadamard's inequalities by setting particular values of parameters involved in the generalized Mittag-Leffler function utilized in corresponding fractional integral operators. Recently, we have published another compact version of the Hadamard's inequality which is stated in the following theorem [7].

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for extended generalized fractional integral holds*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) C_{\alpha,\frac{a+b}{2}^+}(b;p) &\leq \frac{\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',\frac{a+b}{2}^+}f\right)(b;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',\frac{a+b}{2}^-}f\right)(a;p)}{2} \\ &\leq \left(\frac{f(a)+f(b)}{2}\right) C_{\alpha,\frac{a+b}{2}^-}(a;p), \end{aligned} \tag{8}$$

where $\omega' = \frac{2^{\mu}\omega}{(b-a)^{\mu}}$.

We are interested to associate the Mittag-Leffler functions with the well known Ostrowski's inequality. By means of straightforward natural inequalities, some new fractional versions of Ostrowski's inequality are produced. Further, these versions are used to find the estimations of the fractional versions of Hadamard's inequality. In the following we state the Ostrowski's inequality which is proved by Ostrowski [14] in 1938.

Theorem 1.6. *Let $f : I \longrightarrow \mathbb{R}$ where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o, the interior of I and $a, b \in I^o$, $a < b$. If $|f'(t)| \leq M$ for all $t \in [a,b]$ and for all $x \in [a,b]$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M. \tag{9}$$

It is always in the focus of researchers especially working in the field of mathematical analysis. Many authors are continuously working on inequality (9) and have produced very interesting results. Recently in [2, 3, 5, 6, 16], some new Ostrowski's type inequalities are established.

In the section ahead, Ostrowski's inequalities via Mittag-Leffler function have been established. The established results are may be useful in solving fractional mathematical models and in the study of fractional integral operators and their applications. Also these results provide the error bounds of some fractional Hadamard's inequalities which are presented in Section 3.

2. Main Results

First of all, we give the following fractional Ostrowski's type inequality containing generalized Mittag-Leffler function.

Theorem 2.1. *Let $f : I \longrightarrow \mathbb{R}$ where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o, the interior of I and $a, b \in I^o$, $a < b$. If f is integrable function and $|f'(t)| \le M$ for all $t \in [a, b]$, then for $\alpha, \beta \ge 1$, the following inequality for fractional integrals (3) and (4) holds*

$$\begin{aligned}&\Big|f(x)((b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)+(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p))\\&-\left(\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}f\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}f\right)(x;p)\right)\Big|\\&\le M\Big((b-x)^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)+(x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\\&-\left(C_{\beta,b^-}(x;p)+C_{\alpha,a^+}(x;p)\right)\Big).\end{aligned} \tag{10}$$

Proof. Let $x \in [a, b]$, $t \in [a, x]$, and $\alpha \ge 1$. Then the following inequality for Mittag-Leffler function holds true

$$(x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p) \le (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p). \tag{11}$$

From (11) and given condition of boundedness of f' one can have the following integral inequalities

$$\begin{aligned}&\int_a^x (M-f'(t))(x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)dt\\&\le (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\int_a^x (M-f'(t))dt\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\int_a^x (M+f'(t))(x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)dt\\&\le (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\int_a^x (M+f'(t))dt.\end{aligned} \tag{13}$$

First we consider the inequality (12) as follows

$$\begin{aligned}&M\int_a^x (x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)dt\\&-\int_a^x (x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)f'(t)dt\\&\le (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\int_a^x (M-f'(t))dt.\end{aligned} \tag{14}$$

Therefore (14) takes the following form after integrating by parts and using derivative property (2) and a simple computation:

$$(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(x)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^{+}}f\right)(x;p) \tag{15}$$
$$\leq M\left((x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)-C_{\alpha,a^{+}}(x;p)\right).$$

Similarly working on same lines from (13), one can get

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^{+}}f\right)(x;p)-(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(x) \tag{16}$$
$$\leq M\left((x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)-C_{\alpha,a^{+}}(x;p)\right).$$

From (15) and (16), the following modulus inequality holds

$$\left|(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(x)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^{+}}f\right)(x;p)\right| \tag{17}$$
$$\leq M\left((x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)-C_{\alpha,a^{+}}(x;p)\right).$$

Now on the other hand, we let $x\in[a,b]$, $t\in[x,b]$ and $\beta\geq 1$. Then the following inequality holds for Mittag-Leffler function

$$(t-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p)\leq(b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p). \tag{18}$$

From (18) and the condition of boundedness of f', one can have the following integral inequalities

$$\int_x^b(M-f'(t))(t-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p)dt \tag{19}$$
$$\leq(b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)\int_x^b(M-f'(t))dt$$

and

$$\int_x^b(M+f'(t))(t-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p)dt \tag{20}$$
$$\leq(b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)\int_x^b(M+f'(t))dt.$$

Following the same procedure as we did for (12) and (13), one can get from (19) and (20) the following modulus inequality

$$\left|(b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)f(x)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^{-}}f\right)(x;p)\right| \tag{21}$$
$$\leq M\left((b-x)^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)-C_{\beta,b^{-}}(x;p)\right).$$

Inequalities (17) and (21) constitute (10) which are required. □

In the following, we give direct consequences of above proved theorem.

Corollary 2.2. *If we put $\alpha = \beta$ in (10), then we get the following fractional integral inequality*

$$\begin{aligned}
&\Big| f(x)((b-x)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^{\mu};p) + (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)) \\
&-\left(\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,b^-}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}f\right)(x;p)\right)\Big| \\
&\leq M\Big((b-x)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^{\mu};p) + (x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p) \\
&-\left((C_{\alpha,b^-})(x;p) + C_{\alpha,a^+}(x;p)\right)\Big).
\end{aligned} \tag{22}$$

Remark 2.3. *(i) If we take $\omega = p = 0$ in (10), then we get [5, Theorem 1.2].*
(ii) If we take $\alpha = \beta = 1$ and $\omega = p = 0$ in (10), then we get Ostrowski's inequality (9).

The next result is a general form of fractional Ostrowski's inequality containing generalized Mittag-Leffler function.

Theorem 2.4. *Let $f : I \longrightarrow \mathbb{R}$ where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o, the interior of I and $a, b \in I^o$, $a < b$. If f is integrable function and $m < f'(t) \leq M$ for all $t \in [a, b]$, then for $\alpha, \beta \geq 1$, the following inequality for fractional integrals (3) and (4) holds*

$$\begin{aligned}
&\left((x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p) - (b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)\right)f(x) \\
&-\left(\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}f\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}f\right)(x;p)\right) \\
&\leq M\left((x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p) - C_{\alpha,a^+}(x;p)\right) \\
&-m\left((b-x)^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p) - C_{\beta,b^-}(x;p)\right)
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\left((b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p) - (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\right)f(x) \\
&+\left(\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,a^+}f\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,b^-}f\right)(x;p)\right) \\
&\leq M\left((b-x)^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p) - C_{\beta,b^-}(x;p)\right) \\
&-m\left((x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p) - C_{\alpha,a^+}(x;p)\right).
\end{aligned} \tag{24}$$

Proof. Proof is similar as the proof of Theorem 2.1, just after comparing conditions on derivative of f, so we left it for the reader. □

Some comments on above result are given as follows:

Remark 2.5. *(i) If we take $\omega = p = 0$ in (23) and (24), then we get [5, Theorem 1.3].*
(ii) If we take $m = -M$ in Theorem 2.4, then with some rearrangements we get Theorem 2.1.

In the following, we have established a result related to fractional Ostrowski's inequality containing generalized Mittag-Leffler function.

Theorem 2.6. *Let $f : I \longrightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o, the interior of I and $a, b \in I^o$, $a < b$. If f is integrable function and $|f'(t)| \leq M$ for all $t \in [a, b]$, then for $\alpha, \beta \geq 1$, the following inequality* *for fractional integrals (3) and (4) holds*

$$\begin{aligned}
&\Big|((b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)f(b)+(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(a)) \\
&-\left(\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,x^{+}}f\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^{-}}f\right)(a;p)\right)\Big| \\
&\leq M\Big((b-x)^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)+(x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p) \\
&-\left(\left(C_{\beta,x^{+}}\right)(b;p)+\left(C_{\alpha,x^{-}}\right)(a;p)\right)\Big).
\end{aligned} \tag{25}$$

Proof. Let $x \in [a,b]$, $t \in [a,x]$ and $\alpha \geq 1$. Then the following inequality holds true for Mittag-Leffler function

$$(t-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p) \leq (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p). \tag{26}$$

From (11) and given condition of boundedness on f' one can have the following integral inequalities

$$\begin{aligned}
&\int_a^x (M-f'(t))(t-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)dt \\
&\leq (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\int_a^x (M-f'(t))dt
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\int_a^x (M+f'(t))(t-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)dt \\
&\leq (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\int_a^x (M+f'(t))dt.
\end{aligned} \tag{28}$$

First we consider the inequality (27) as follows

$$\begin{aligned}
&M\int_a^x (t-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)dt \\
&-\int_a^x (t-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)f'(t)dt \\
&\leq (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\int_a^x (M-f'(t))dt
\end{aligned} \tag{29}$$

Therefore (29) takes the following form after integrating by parts and using derivative property (2) and a simple computation

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^{-}}f\right)(a;p)-(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(a) \\
&\leq M\Big((x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)-\left(C_{\alpha,x^{-}}\right)(a;p)\Big).
\end{aligned} \tag{30}$$

Similarly working on same lines from (28), one can get

$$\begin{aligned}
&(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(a)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^{-}}f\right)(a;p) \\
&\leq M\Big((x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)-\left(C_{\alpha,x^{-}}\right)(a;p)\Big).
\end{aligned} \tag{31}$$

From (15) and (31,) the following modulus inequality holds

$$\begin{aligned}&\left|(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(a)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}f\right)(a;p)\right|\\&\leq M\left((x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)-\left(C_{\alpha,x^-}\right)(a;p)\right).\end{aligned}\tag{32}$$

Now on the other hand we let $x\in[a,b]$, $t\in[x,b]$ and $\beta\geq 1$. Then the following inequality holds for Mittag-Leffler function

$$(b-t)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p)\leq(b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p).\tag{33}$$

From (18) and given condition of boundedness of f', one can have the following integral inequalities

$$\begin{aligned}&\int_x^b(M-f'(t))(b-t)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p)dt\\&\leq(b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)\int_x^b(M-f'(t))dt\end{aligned}\tag{34}$$

and

$$\begin{aligned}&\int_x^b(M+f'(t))(b-t)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p)dt\\&\leq(b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)\int_x^b(M+f'(t))dt.\end{aligned}\tag{35}$$

Following the same procedure as we did for (27) and (28), one can get from (34) and (35) the following modulus inequality

$$\begin{aligned}&\left|(b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)f(b)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta-1,l,\omega,x^+}f\right)(b;p)\right|\\&\leq M\left((b-x)^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)-\left(C_{\beta,x^+}\right)(b;p)\right).\end{aligned}\tag{36}$$

Inequalities (32) and (36) constitute (25) which are required. □

Some direct consequences are given below.

Corollary 2.7. *If we put $\alpha=\beta$ in (25), then we get the following fractional integral inequality*

$$\begin{aligned}&\Big|((b-x)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^{\mu};p)f(b)+(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(a))\\&-\left(\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^+}f\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha-1,l,\omega,x^-}f\right)(a;p)\right)\Big|\\&\leq M\Big((b-x)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^{\mu};p)+(x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\\&-\left(\left(C_{\alpha,x^+}\right)(b;p)+\left(C_{\alpha,x^-}\right)(a;p)\right)\Big).\end{aligned}$$

Remark 2.8. *(i) If we take $\omega=p=0$, then we get [5, Theorem 1.4].*
(ii) A more general form of Theorem 2.6 like Theorem 2.4 holds which we leave for the reader.

3. Applications

In this section, we just describe some applications of Theorem 2.1 and Theorem 2.6. By applying Theorem 2.1 at end points of the interval $[a, b]$ and adding the resulting inequalities one obtain the error bounds of compact form of the fractional Hadamard's inequality stated in Theorem 1.4. While by applying Theorem 2.6 at the mid point of $[a, b]$, one can obtain error bounds of other compact version of fractional Hadamard's inequality stated in Theorem 1.5. These results actually provide a variety of error bounds of the Hadamard's inequalities in fractional calculus point of view by setting convenient values to the parameters involved in the generalized fractional integral operator. Of course, such inequalities may be useful in the theory of fractional differential equations.

Concluding Remarks

We have investigated much general fractional integrals inequalities. By taking parameters particular values, quite interesting results can be obtained. For example taking $p = 0$, fractional integrals inequalities for fractional integrals defined by Salim and Faraj in [19], taking $l = \delta = 1$, fractional integrals inequalities for fractional integrals defined by Rahman et al. in [17], taking $p = 0$ and $l = \delta = 1$, fractional integrals inequalities for fractional integrals defined by Shukla and Prajapati in [20] and see also [22], taking $p = 0$ and $l = \delta = k = 1$, fractional integrals inequalities for fractional integrals defined by Prabhakar in [15], taking $p = \omega = 0$ fractional integral inequalities for Riemann-Liouville fractional integrals.

Conflict of interest

Authors have no any conflict of interest.

References

[1] M. Andrić, G. Farid and J. Pečarić, A further extension of Mittag-Leffler function, Fract. Calc. Appl. Anal., 21(5) (2018), 1377–1395.
[2] S. S. Dragomir, Th. M. Rassias (Eds.), Ostrowski-type inequalities and applications in numerical integration, Kluwer academic publishers, Dordrecht, Boston, London 2002.
[3] S. S. Dragomir and S. Wang, An inequality of Ostrowski-Gruss type and its applications to the estimation of error bounds for some special means and some numerical quadrature rules, Comput. Math. Appl., 33 (1997), 15–20.
[4] G. E. Andrews, R. Askey, R. Roy, Special functions. Encyclopedia of Mathematics and its Applications, 71, Cambridge University Press, 1999.
[5] G. Farid, Some new Ostrowski type inequalities via fractional integrals, Int. J. Anal. Appl., 14(1) (2017), 64–68.
[6] G. Farid, Straightforward proofs of Ostrowski inequality and some related results, Int. J. Anal., 2016 (2016), 5 pages.
[7] G. Farid, K. A. Khan N. Latif, A. Ur. Rehman and S. Mehmood, General fractional integral inequalities for convex and m-convex functions via an extended generalized Mittag-Lefller function, J. Inequal. Appl., (2018), 2018:243.
[8] G. Gasper, Basic Hypergeometric Series, Cambridge Univ. Press, 2004.
[9] S. M. Kang, G. Farid, W. Nazeer, B. Tariq, Hadamard and Fejér-Hadamard inequalities for extended generalized fractional integrals involving special functions, J. Inequal. Appl., (2018), 2018:119.
[10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, 204, Elsevier, New York-London, 2006.
[11] F. Klein, Vorlesungen über die hypergeometrische Funktion. Grundlehren der Mathematischen Wissenschaften, 39, Springer-Verlag, 1981.
[12] G. V. Milovanović and M. Th. Rassias (eds.), Analytic Number Theory, Approximation Theory and Special Functions, Springer, 2014.
[13] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, C.R. Acad. Sci. Paris, 137 (1903), 554–558.
[14] A. Ostrowski, Über die Absolutabweichung einer dierentierbaren Funktion von ihren Integralmittelwert, Comment. Math. Helv., 10 (1938), 226–227.
[15] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., 19 (1971), 7–15.
[16] X. Qiaoling, Z. Jian, and L. Wenjun, A new generalization of Ostrowski-type inequality involving functions of two independent variables, Comput. Math. Appl., 60 (2010), 2219–2224.
[17] G. Rahman, D. Baleanu, M. A. Qurashi, S. D. Purohit, S. Mubeen and M. Arshad, The extended Mittag-Leffler function via fractional calculus, J. Nonlinear Sci. Appl., 10 (2017), 4244–4253.
[18] T. O. Salim, Some properties relating to the generalized Mittag-Leffler function, Adv. Appl. Math. Anal., 4 (2009), 21–30.
[19] T. O. Salim and A. W. Faraj, A Generalization of Mittaf-Leffler function and integral operator associated with integral calculus, J. Frac. Calc. Appl., 3(5) (2012), 1–13.

[20] A. K. Shukla and J. C. Prajapati, On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl., 336 (2007), 797–811.
[21] L. J. Slater, Generalized Hypergeometric Functions, Cambridge Univ. Press, 2008.
[22] H. M. Srivastava and Z. Tomovski, Fractional calculus with an integral operator containing generalized Mittag-Leffler function in the kernal, Appl. Math. Comput., 211(1) (2009), 198–210.
[23] A. Wiman, Uber den fundamental satz in der theori der functionen, Acta Math., 29 (1905), 191–201.