On \(n\)th Roots of Normal Operators

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Abstract. For \(n\)-normal operators \(A\) \([2, 4, 5]\), equivalently \(n\)-th roots of normal Hilbert space operators, both \(A\) and \(A^*\) satisfy the Bishop–Eschmeier–Putnam property \((\rho)_n\). \(A\) is decomposable and the quasi-nilpotent part \(H_n(A - \lambda)\) of \(A\) satisfies \(H_n(A - \lambda)^{-1}(0) = (A - \lambda)^{-1}(0)\) for every non-zero complex \(\lambda\). \(A\) satisfies every Weyl and Browder type theorem, and a sufficient condition for \(A\) to be normal is that either \(A\) is dominant or \(A\) is a class \(\mathcal{A}(1, 1)\) operator.

1. Introduction

Let \(B(\mathcal{H})\) denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Hilbert space \(\mathcal{H}\) into itself. Every normal operator \(A \in B(\mathcal{H})\), i.e., \(A \in B(\mathcal{H})\) such that \([A^*, A] = A^*A - AA^* = 0\), has an \(n\)th root for every positive integer \(n > 1\). Thus given a normal \(A \in B(\mathcal{H})\), there exists \(B \in B(\mathcal{H})\) such that \(B^n = A\) (and then \(\sigma(B^n) = \sigma(B)^n = \sigma(A)\)). A straightforward forward application of the Putnam-Fuglede commutativity theorem \([14, \text{Page 103}]\) applied to \([B, B^n] = 0\) then implies \([B^n, B^n] = 0\). (Conversely, \([B^n, B^n] = 0\) implies \(B^n\) is normal). Operators \(B \in B(\mathcal{H})\) satisfying \([B^n, B^n] = 0\) have been called \(n\)-normal, and a study of the spectral structure of \(n\)-normal operators, with emphasis on the properties which \(B\) inherits from its normal avatar \(B^n\), has been carried out in \([2, 4, 5]\).

Given \(A \in B(\mathcal{H})\), let \(\sigma(A) \subseteq \mathcal{L} < \frac{\pi}{2}\) denote that \(\sigma(A)\) is contained in an angle \(\mathcal{L}\), with vertex at the origin, of width less than \(\frac{\pi}{2}\). Assuming \(\sigma(B) \subseteq \mathcal{L} < \frac{\pi}{2}\) for an \(n\)-normal operator \(B \in B(\mathcal{H})\), the authors of \([2, 4, 5]\) prove that \(B\) inherits a number of properties from \(B^n\), amongst them that \(B\) satisfies Bishop-Eschmeier-Putnam property \((\rho)_n\), \(B\) is polaroid (hence also isoloid) and \(\lim_{m \to \infty} \langle x_m, y_m \rangle = 0\) for sequences \(\{x_m\}, \{y_m\} \subset \mathcal{H}\) of unit vectors such that \(\lim_{m \to \infty} ||(B - \lambda)x_m|| = 0 = \lim_{m \to \infty} ||(B - \mu)y_m||\) for distinct scalars \(\lambda, \mu \in \sigma(B)\). (All our notation is explained in the following section.) That \(B\) inherits a property from \(B^n\) in many a case has little to do with the normality of \(B^n\), but is instead a consequence of the fact that \(B^n\) has the property. Thus, if the approximate point spectrum \(\sigma_a(B^n) = \sigma_a(B^n)\) is normal (recall: \(\lambda \in \sigma_a(B^n)\) is normal if \(\lim_{m \to \infty} ||(B^n - \lambda)x_m|| = 0\) for a sequence \(\{x_m\} \subseteq \mathcal{H}\) of unit vectors implies \(\lim_{m \to \infty} ||(B^n - \lambda)^\ast x_m|| = 0\); hyponormal operators, indeed dominant operators, satisfy this property), \(\sigma(B) \subseteq \mathcal{L} < \frac{\pi}{2}\), and \(\{x_m\}, \{y_m\}\)
are sequences of unit vectors in \( H \) such that \( \lim_{m \to \infty} \| (B^n - \lambda^n)x_m \| = 0 = \lim_{m \to \infty} \| (B^n - \mu^n)y_m \| \) for some distinct \( \lambda, \mu \in \sigma_d(B) \), then

\[
\lim_{m \to \infty} \lambda^n(x_m, y_m) = \lim_{m \to \infty} \langle B^n x_m, y_m \rangle = \lim_{m \to \infty} \langle x_m, B^n y_m \rangle = \mu^n \lim_{m \to \infty} \langle x_m, y_m \rangle
\]

implies

\[
(\lambda - \mu) \lim_{m \to \infty} \langle x_m, y_m \rangle = 0 \iff \lim_{m \to \infty} \langle x_m, y_m \rangle = 0
\]

(cf. [4, Theorem 2.4]). It is well known that \( w \)-hyponormal operators satisfy property \( (\beta)_c \) ([3]). If \( B^n \in (\beta)_c \) (i.e., \( B^n \) satisfies property \( (\beta)_c \)) and \( \sigma(B) \subseteq \z < \frac{2\pi}{\nu} \), then [7, Theorem 2.9 and Corollary 2.10] imply that \( B + N \in (\beta)_c \) for every nilpotent operator \( N \) which commutes with \( B \) (cf. [5, Theorem 3.1]). Again, if \( B^n \) is polaroid and \( \sigma(B) \subseteq \z < \frac{2\pi}{\nu} \), then \( B \) is polaroid (hence also, isoloid) ([9, Theorem 4.1]). Observe that paranormal operators are polaroid. \( n \)th roots of normal operators have been studied by a large number of authors (see [18], [17], [6], [11], [13]) and there is a rich body of text available in the literature. Our starting point in this note is that an \( n \)-normal operator \( B \) considered as an \( n \)th root of a normal operator has a well defined structure ([13, Theorem 3.1]). The problem then is that of determining the “normal like” properties which \( B \) inherits. We prove in the following that the condition \( \sigma(B) \subseteq \z < \frac{2\pi}{\nu} \) may be dispensed with in many a case (though not always). Just like normal operators, \( n \)th roots \( B \) have SVEP (the single-valued extension property) everywhere, \( \sigma(B) = \sigma_d(B) \), \( B \) is polaroid (hence also, isoloid). \( B \in (\beta)_c \) (as also does \( B^* \)) and (the quasinilpotent part) \( H_0(B - \lambda) = (B - \lambda)^{-1}(0) \) at every \( \lambda \in \sigma_d(B) \) except for \( \lambda = 0 \) when we have \( H_0(B) = B^{-\pi}(0) \). Again, just as for normal operators, \( B \) satisfies various variants of the classical Weyl’s theorem \( \sigma(B) \setminus \sigma_d(B) = \Pi_0(B) \) (resp., Browder’s theorem \( \sigma(B) \setminus \sigma_d(B) = \Pi_0(B) \)). It is proved that dominant and class \( \mathcal{A}(1, 1) \) operators \( B \) are normal.

2. Notation and terminology

Given an operator \( S \in B(H) \), the point spectrum, the approximate point spectrum, the surjectivity spectrum and the spectrum of \( S \) will be denoted by \( \sigma_p(S), \sigma_a(S), \sigma_m(S) \) and \( \sigma(S) \), respectively. The isolated points of a subset \( K \) of \( C \), the set of complex numbers, will be denoted by \( \text{iso}(K) \). An operator \( X \in B(H) \) is a quasi-affinity if it is injective and has a dense range, and operators \( S, T \in B(H) \) are quasi-similar if there exist quasi-affinities \( X, Y \in B(H) \) such that \( SX = XT \) and \( YS = TY \).

\( S \in B(H) \) has SVEP, the single-valued extension property, at a point \( \lambda_0 \in C \) if for every open disc \( D \) centered at \( \lambda_0 \) the only analytic function \( f : D \to H \) satisfying \( (S - \lambda_0)f(\lambda) = 0 \) is the function \( f \equiv 0 \); \( S \) has SVEP if it has SVEP everywhere in \( C \). (Here and in the sequel, we write \( S - \lambda \) for \( S - \lambda I \).) Let, for an open subset \( \mathcal{U} \) of \( C \), \( \mathcal{E} (\mathcal{U}, H) \) (resp., \( O(\mathcal{U}, H) \)) denote the Fréchet space of all infinitely differentiable (resp., analytic) \( H \)-valued functions on \( \mathcal{U} \) endowed with the topology of uniform convergence of all derivatives (resp., topology of uniform convergence) on compact subsets of \( \mathcal{U} \). \( S \in B(H) \) satisfies property \( (\beta)_c \), \( S \in (\beta)_c \), at \( \lambda \in C \) if there exists a neighborhood \( N \) of \( \lambda \) such that for each subset \( \mathcal{U} \) of \( N \) and sequence \( \{ f_n \} \) of \( H \)-valued functions in \( \mathcal{E} (\mathcal{U}, H) \),

\[
(S - z) f_n(z) \to 0 \quad \text{in} \quad \mathcal{E} (\mathcal{U}, H) \quad \Rightarrow \quad f_n(z) \to 0 \quad \text{in} \quad \mathcal{E} (\mathcal{U}, H)
\]

(resp., \( S \) satisfies property \( (\beta) \), \( S \in (\beta) \), at \( \lambda \in C \) if there exists an \( r > 0 \) such that, for every open subset \( \mathcal{U} \) of the open disc \( \mathcal{D}(\lambda; r) \) of radius \( r \) centered at \( \lambda \) and sequence \( \{ f_n \} \) of \( H \)-valued functions in \( O(\mathcal{U}, H) \),

\[
(S - z) f_n(z) \to 0 \quad \text{in} \quad O(\mathcal{U}, H) \quad \Rightarrow \quad f_n(z) \to 0 \quad \text{in} \quad O(\mathcal{U}, H).
\]

The following implications are well known ([12], [16]):

\[
S \in (\beta)_c \quad \Rightarrow \quad S \in (\beta) \quad \Rightarrow \quad S \text{ has SVEP}; \quad S, S^* \in (\beta) \quad \Rightarrow \quad S \text{ decomposable}.
\]

The ascent \( \text{asc}(S - \lambda) \) (resp., descent \( \text{dsc}(S - \lambda) \)) of \( S \) at \( \lambda \in C \) is the least non-negative integer \( p \) such that \( (S - \lambda)^{-p}(0) = (S - \lambda)^{-(p+1)}(0) \) (resp., \( (S - \lambda)^p(H) = (S - \lambda)^{p+1}(H) \)). A point \( \lambda \in \text{iso}(S) \) (resp., \( \lambda \in \text{iso}_d(S) \))
is a pole (resp., left pole) of the resolvent of $S$ if $0 < \text{asc}(S - \lambda) = \text{dsc}(S - \lambda) < \infty$ (resp., there exists a positive integer $p$ such that $\text{asc}(S - \lambda) = p$ and $(S - \lambda)^{-1}(\mathcal{H})$ is closed) ([1]). Let

$$\Pi(S) = \{ \lambda \in \text{iso}(S) : \lambda \text{ is a pole (of the resolvent) of } S \};$$

$$\Pi^f(S) = \{ \lambda \in \text{iso}_p(S) : \lambda \text{ is a left pole (of the resolvent) of } S \}.$$ 

Then $\Pi(S) \subseteq \Pi^f(S)$, and $\Pi^f(S) = \Pi(S)$ if (and only if) $S^*$ has SVEP at points $\lambda \in \Pi^f(S)$. We say in the following that the operator $S$ is polaroid if $\{ \lambda \in \mathbb{C} : \lambda \in \text{iso}(S) \} \subseteq \Pi(S)$. Polaroid operators are isoloid (where $S$ is isoloid if $\{ \lambda \in \mathbb{C} : \lambda \in \text{iso}(S) \} \subseteq \sigma_p(S)$). Let $\sigma_x = \sigma$ or $\sigma_x$. The sets $E^+(S) = E(S)$ or $E^+(S)$ and $E_{\sigma}^+(S) = E_{\sigma}(S)$ or $E_{\sigma}^+(S)$ are then defined by

$$E^+(S) = \{ \lambda \in \text{iso}_p(S) : \lambda \in \sigma_p(S) \},$$

and

$$E_{\sigma}^+(S) = \{ \lambda \in \text{iso}_p(S) : \lambda \in \sigma_{\sigma_p}(S), \dim(S - \lambda)^{-1}(0) < \infty \}.$$ 

It is clear that

$$\Pi^f(S) \subseteq E^+(S) \text{ and } \Pi^f_0(S) \subseteq E_{\sigma}^+(S)$$

(where $\Pi^f_0(S) = \{ \lambda \in \Pi^f(S) : \dim(S - \lambda)^{-1}(0) < \infty \})$.

The quasi-nilpotent part $H_0(S)$ and the analytic core $K(S)$ of $S \in B(\mathcal{H})$ are the sets

$$H_0(S) = \left\{ x \in \mathcal{H} : \lim_{n \to \infty} \| S^n x \| = 0 \right\},$$

and

$$K(S) = \left\{ x \in \mathcal{H} : \text{there exists a sequence } \{ x_n \} \subseteq \mathcal{H} \text{ and } \delta > 0 \text{ for which } x = x_0, S x_{n+1} = x_n \text{ and } \| x_n \| \leq \delta^{n}\|x\| \text{ for all } n = 1, 2, \ldots \right\}$$

([1]). If $\lambda \in \text{iso}(S)$, then $\mathcal{H}$ has a direct sum decomposition $\mathcal{H} = H_0(S - \lambda) \oplus K(S - \lambda)$, $S - \lambda|_{H_0(S-\lambda)}$ is quasinilpotent and $S - \lambda|_{K(S-\lambda)}$ is invertible. A necessary and sufficient condition for a point $\lambda \in \text{iso}(S)$ to be a pole of $S$ is that there exist a positive integer $p$ such that $H_0(S - \lambda) = (S - \lambda)^{-p}(0)$.

In the following we shall denote the upper semi-Fredholm, the lower semi-Fredholm and the Fredholm spectrum of $S$ by $\sigma_{u}^+(S), \sigma_{l}^-(S)$ and $\sigma(S)$; $\sigma_{u}^+(S), \sigma_{l}^-(S)$ and $\sigma_{u}^-(S)$ (resp., $\sigma_{u}^+(S), \sigma_{l}^-(S)$ and $\sigma(S)$) shall denote the upper Weyl, the lower Weyl and the Weyl (resp., the upper Browder, the lower Browder and the Browder) spectrum of $S$. Additionally, we shall denote the upper $B$-Weyl, the lower $B$-Weyl and the $B$-Weyl (resp., the upper $B$-Browder, the lower $B$-Browder and the $B$-Browder) spectrum of $S$ by $\sigma_{u}^+(S), \sigma_{l}^-(S)$ and $\sigma_{u}^-(S)$ (resp., $\sigma_{u}^+(S), \sigma_{l}^-(S)$ and $\sigma_{u}^-(S)$). We refer the interested reader to the monograph ([1]) for definition, and other relevant information, on these distinguished parts of the spectrum; our interest here in these spectra is at best peripheral.

3. Results

Throughout the following, $A \in B(\mathcal{H})$ shall denote an $n$-normal operator. Considered as an $n$th root of the normal operator $A^n$, $A$ has a direct sum representation

$$A = \bigoplus_{i=0}^{\infty} A_i|_{H_i} = \bigoplus_{i=0}^{\infty} A_i, \mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i,$$

where $A_0$ is $n$-nilpotent and $A_i$, for all $i = 1, 2, \ldots$, is similar to a normal operator $N_i \in B(\mathcal{H}_i)$. Equivalently,

$$A = B_1 \oplus B_0, B_0 = A_0 \text{ and } B_1 = \bigoplus_{i=1}^{\infty} A_i,$$

where $B_0^* = 0$ and $B_1$ is quasi-similar to a normal operator $N = \bigoplus_{i=1}^{\infty} N_i \in B\left( \bigoplus_{i=1}^{\infty} \mathcal{H}_i \right)$. Quasi-similar operators preserve SV EP; hence, since the direct sum of operators has SV EP at a point if and only if the summands have SV EP at the point, $A$ and $A^*$ have SV EP (everywhere). Consequently ([1]):

$$\sigma(A) = \sigma(B_1) \cup \{0\} = \sigma(N) \cup \{0\} = \sigma_{u}(A) = \sigma_{u}(A)^r.$$
Since quasi-a are reducing): This fails for the operator $A$ (4, Remark 2.17), and a sufficient condition is that $\sigma(A) \subseteq \mathcal{Z} < \frac{2\pi}{n}$ (for then $(A - \lambda)X = 0 \implies (A^n - \lambda^n)x = 0 \implies (A^n - \lambda^n)x = 0 \iff (A^* - \lambda)x = 0$).

The polaroid property transfers to the Riesz projections $P_{\lambda}(A)$ and $P_N(A)$ corresponding to points $\lambda \in \text{iso}(B_1) = \text{iso}(N)$. Let $\Gamma$ be a positively oriented path separating $\lambda$ from $\sigma(B_1)$ and let $X, Y$ be quasi-affinities such that $B_1X = XN$ and $YB_1 = NY$. Then, for all $\mu \not\in \sigma(B_1)$,

$$P_{\lambda}(A) = \frac{1}{2\pi i} \int_{\Gamma} (\mu - B_1)^{-1}d\mu \iff YP_{\lambda}(A) = Y \left\{ \frac{1}{2\pi i} \int_{\Gamma} (\mu - B_1)^{-1}d\mu \right\} Y = P_N(\lambda)Y.$$

A similar argument proves

$$P_{\lambda}(A)X = XP_N(\lambda).$$

**Theorem 3.1.** $A$ is polaroid.

**Proof.** Continuing with the argument above, the normality of $N$ implies that the range $H_0(N - \lambda)$ of $P_N(\lambda)$ coincides with $(N - \lambda)^{-1}(0)$. Hence $(N - \lambda)P_N(\lambda) = 0$, and

$$Y(B_1 - \lambda)P_{\lambda}(A) = (N - \lambda)YP_{\lambda}(A) = (N - \lambda)P_N(\lambda)Y = 0 \implies (B_1 - \lambda)P_{\lambda}(A) = 0 \iff H_0(B_1 - \lambda) = (B_1 - \lambda)^{-1}(0).$$

Since $\lambda \in \text{iso}(B_1)$,

$$\bigoplus_{i=1}^{\infty} \mathcal{H}_i = H_0(B_1 - \lambda) \oplus K(B_1 - \lambda) = (B_1 - \lambda)^{-1}(0) \oplus K(B_1 - \lambda)$$

$$\implies \bigoplus_{i=1}^{\infty} \mathcal{H}_i = (B_1 - \lambda)^{-1}(0) \oplus (B_1 - \lambda) \bigoplus_{i=1}^{\infty} \mathcal{H}_i,$$

i.e., $\lambda$ is a (simple) pole. The $n$-nilpotent operator $B_0$ being polaroid, the direct sum $B_0 \oplus B_1$ is polaroid (since $\text{asc}(A - \lambda) \leq \text{asc}(B_0 - \lambda) \oplus \text{asc}(B_1 - \lambda)$ and $\text{dsc}(A - \lambda) \leq \text{dsc}(B_0 - \lambda) \oplus \text{dsc}(B_1 - \lambda)$ for all $\lambda$ ([20, Exercise 7, Page 293])).

Theorem 3.1 implies:

**Corollary 3.2.** $A$ is isoid (i.e., points $\lambda \in \sigma(A)$ are eigenvalues of $A$).

More is true and, indeed, Theorem 3.1 is a consequence of the following result which shows that $H_0(A - \lambda) = (A - \lambda)^{-1}(0)$ for all non-zero $\lambda \in \sigma(A)$.

**Theorem 3.3.** $H_0(A - \lambda) = (A - \lambda)^{-1}(0)$ for all non-zero $\lambda \in \sigma(A)$ and $H_0(A) = A^{-n}(0)$. In particular, $A$ is polaroid.
it follows that

\[ \lambda \in \sigma(N) \quad \text{since normal operators such that} \]

\[ \text{to prove the theorem it will suffice to prove} \]

\[ \text{implies} \]

\[ \text{for all} \ x \in H_0(B_1 - \lambda). \]

Consequently,

\[ Yx \in H_0(N - \lambda) = (N - \lambda)^{-1}(0) \implies Y(B_1 - \lambda)x = (N - \lambda)Yx = 0 \iff x \in (B_1 - \lambda)^{-1}(0), \]

and hence

\[ H_0(B_1 - \lambda) = (B_1 - \lambda)^{-1}(0) \]

for all \( \lambda \in \sigma(B_1) \). Evidently,

\[ H_0(A) = H_0(B_1 \oplus B_0) = B_1^{-1}(0) \oplus B_0^{-1}(0) \subseteq A^{-1}(0). \]

Argue now as in the proof of Theorem 3.1 to prove that \( A \) is polaroid. \( \square \)

The Riesz projection \( P_A(\lambda) \) corresponding to points \( 0 \neq \lambda \in \text{iso}(A) \) are, in general, not self-adjoint. Since \( \sigma(A) \subseteq \mathcal{L} < \frac{2\pi}{n} \) ensures \( (A - \lambda^{-1})^{-1}(0) \subseteq (A' - \lambda^{-1})^{-1}(0) \) for all \( 0 \neq \lambda \in \sigma_p(A) \), \( \sigma(A) \subseteq \mathcal{L} < \frac{2\pi}{n} \) forces \( P_A(\lambda) = P_A(\lambda)^* \) for all \( \lambda \neq 0 \).

**Corollary 3.4.** If \( \sigma(A) \subseteq \mathcal{L} < \frac{2\pi}{n} \), then the Riesz projection corresponding to non-zero \( \lambda \in \text{iso}(A) \) is self-adjoint.

**Remark 3.5.** Theorems 3.1 and 3.3 generalize corresponding results from [2], [4], [5] by removing the hypothesis that \( \sigma(A) \subseteq \mathcal{L} < \frac{2\pi}{n} \), and, in the case of Theorem 3.3, the hypothesis on the points \( \lambda \) being isolated in \( \sigma(A) \). Recall from [1, Page 336] that an operator \( S \in \mathcal{B}({\mathcal{H}}) \) is said to have property \( Q \) if \( H_0(S, \lambda) \) is closed for all \( \lambda \): Theorem 3.3 says that the nth roots \( A \) have property \( Q \). Another proof of Theorem 3.3, hence also of the fact that the operators \( A \) satisfy property \( Q \), follows from the argument below proving the subscalarity of \( A \).

Property \((\beta)_c\) (similarly \((\beta)\)) does not travel well under quasi-affinities. Thus \( CX = XB \) and \( B \in (\beta)_c \) does not imply \( C \in (\beta)_c \) (see [7, Remark 2.7] for an example). However, \( C \in (\beta)_c \) implies \( B \in (\beta)_c \) holds, as the following argument proves. If \( (f_n) \) is a sequence in \( \mathcal{E}(U, \mathcal{H}) \) such that

\[ (B - z)f_n(z) \to 0 \text{ in } \mathcal{E}(U, \mathcal{H}), \]

then

\[ X(B - z)f_n(z) = (C - z)Xf_n(z) \to 0 \text{ in } \mathcal{E}(U, \mathcal{H}). \]

Since \( C \in (\beta)_c \) and \( X \) is a quasi-affinity,

\[ Xf_n(z) \to 0 \text{ in } \mathcal{E}(U, \mathcal{H}) \implies f_n(z) \to 0 \text{ in } \mathcal{E}(U, \mathcal{H}). \]

Thus \( B \in (\beta)_c \).

**Theorem 3.6.** \( A \) and \( A^* \) satisfy property \((\beta)_c\).

**Proof.** Recall from [7, Lemma 2.2] that a direct sum of operators satisfies \((\beta)_c\) if and only if the individual operators satisfy \((\beta)_c\). The operator \( A \) being the direct sum \( B_1 \oplus B_0 \) where \( B_0, B_0' \) being nilpotent satisfy \((\beta)_c\), to prove the theorem it will suffice to prove \( B_1, B_1' \in (\beta)_c \). But this is immediate from the argument above, since normal operators \( N \) satisfy \( N, N^* \in (\beta)_c \) and since there exist quasi-affinities \( X \) and \( Y \) in \( B \left( \bigoplus_{i=1}^\infty \mathcal{H}_i \right) \) such that \( N'X = X'B_1' \) and \( NY = YB_1 \). \( \square \)

\( A \in (\beta)_c \) implies \( A \in (\beta) \), and \( A, A^* \in (\beta) \) implies \( A \) is decomposable ([16]). Hence:
**Corollary 3.7.** A is decomposable.

We consider next a sufficient condition for the operator A to be normal. However, before that we point out that the operator A satisfies almost all Weyl and Browder type theorems ([11]) satisfied by normal operators.

**Weyl’s theorem** An operator \( S \in B(H) \) satisfies

- generalized Weyl’s theorem, \( S \in gWt, \) if \( \sigma(S) \setminus \sigma_{	ext{Bor}}(S) = E(S); \)
- a – generalized Weyl’s theorem, \( S \in a - gWt, \) if \( \sigma(a) \setminus \sigma_{	ext{Bor}}(S) = E^a(S) \)

(see [1, Definitions 6.59, 6.81]). Let \( S \in Wt, S \in a - Wt, S \in gBt, S \in a - gBt, S \in Bt \) and \( S \in a - Bt \) denote, respectively, that

- \( S \) satisfies Weyl’s theorem : \( \sigma(S) \setminus \sigma_{aw}(S) = E_0(S), \)
- \( S \) satisfies a – Weyl’s theorem : \( \sigma(a) \setminus \sigma_{aw}(S) = E^a_0(S), \)
- \( S \) satisfies generalized Browder’s theorem : \( \sigma(S) \setminus \sigma_{	ext{Bor}}(S) = \Pi(S), \)
- \( S \) satisfies generalized a – Browder’s theorem : \( \sigma(a) \setminus \sigma_{	ext{Bor}}(S) = \Pi^a(S), \)
- \( S \) satisfies Browder’s theorem : \( \sigma(S) \setminus \sigma_{aw}(S) = \Pi_0(S), \)
- \( S \) satisfies a – Browder’s theorem : \( \sigma(a) \setminus \sigma_{aw}(S) = \Pi_0^a(S), \)

(see [1, Chapter 6]). The following implications are well known ([1, Chapters 5, 6]):

\[
S \in a - gWt \implies \begin{cases} S \in a - Wt \\ S \in gWt \end{cases} \implies S \in Wt \implies S \in Bt,
\]

\[
S \in a - gBt \implies \begin{cases} S \in a - Wt \\ S \in a - gBt \end{cases} \implies S \in a - Bt \implies S \in Bt,
\]

\[
S \in a - gBt \iff S \in a - Bt, \quad S \in gBt \iff S \in Bt.
\]

A has SVEP (guarantees \( A \in a - gBt ([1, Therem 5.37]) \)) and \( \sigma(A) = \sigma_{aw}(A) \) guarantee the equivalence of a-gBt and gBt (hence also of a-gBt with a-Bt and Bt) for \( A. \) The fact that \( A \) is polaroid and \( \sigma(A) = \sigma_{aw}(A) \) guarantees also that \( E(A) = E^a(A) = \Pi^a(A) = \Pi_0(A) \) (and \( E_0(A) = E^a_0(A) = \Pi_0^a(A) = \Pi_0(\sigma(A)) \)). Hence all Weyl’s theorems (listed above) are equivalent for \( A \) and :

**Theorem 3.8.** \( A \in a - gWt \)

**Normal A.** For the operator \( A = B_1 \oplus B_0 \) to have any chance of being a normal operator, it is necessary that (either \( B_1 \) is missing, or) \( B_0 = 0 \) is, however, in no way sufficient to ensure the normality of \( A. \) Additional hypotheses are required. An operator \( S \in B(H) \) is said to be dominant (resp., class \( A(1, 1) \)) if to every complex \( \lambda \) there corresponds a real number \( M_1 > 0 \) such that \( \|S - \lambda x\| \leq M_1 \|S - \lambda x\| \) for all \( x \in H \) (resp., \( |S^2| \leq |S^2| \)) ([19], [15]). Recall from [10, Lemma 2.1] that if a dominant or class \( A(1, 1) \) operator \( A \in B(H) \) is a square root of a normal operator, then \( A \) is normal. The following theorem, which uses an argument different from that used in [10], proves that this result extends to \( n \)th roots of \( A. \)

**Theorem 3.9.** Dominant or \( A(1, 1) \) \( n \)th roots of a normal operator in \( B(H) \) are normal.

**Proof.** Recall that the eigenvalues of a dominant operator are normal (i.e., they are simple and the corresponding eigenspace is reducing). Hence if our \( n \)th root of \( A = B_1 \oplus B_0 \) is dominant, then \( A = B_1 \oplus 0 \) is a dominant operator which satisfies

\[
A \left( Y \oplus I \mid_{H_1} \right) = \left( Y \oplus I \mid_{H_1} \right) (N \oplus 0).
\]
The operator $N \oplus 0$ being normal and the operator $Y \oplus I \mid_{\mathcal{H}_1}$ being a quasi-affinity it follows from [19], [8] that $A$ is normal (and unitarily equivalent to $N \oplus 0$). We consider next $A \in \mathcal{A}(1, 1)$.

It is well known that $\mathcal{A}(1, 1)$ operators have ascent less than or equal to one. (Indeed, operators $S \in \mathcal{A}(1, 1)$ are paranormal: $\|Sx\| \leq \|S^2x\|\|x\|$ for all $x \in \mathcal{H}$, hence $\text{asc}(S) \leq 1$.) Hence if $A = B_1 \oplus B_0 \in \mathcal{A}(1, 1)$, then $B_0 = 0$ and $A \in B \left( A^{-1}(0) \oplus A^{-1}(0) \right)^*$ has an upper triangular matrix representation

$$A = \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

Let $N_1 = N \oplus 0 \mid_{\mathcal{H}_1}$ have the representation

$$N_1 = 0 \oplus N_{22} \in B \left( N_1^{-1}(0) \oplus N_1^{-1}(0) \right)^*,$$

and let $Y_1 = Y \oplus I \mid_{\mathcal{H}_1} \in B \left( N_1^{-1}(0) \oplus N_1^{-1}(0) \right)^*$. Applying Proposition 2.5 and Lemma 2.2 of [10], it follows that $A_{22}Y_2 = Y_{22}N_{22}$ implies that $A_{22}$ is quasi-affinity. Hence, since $A^{-n}Y_1 = Y_1N_1^{-n}$, it follows that $A^{-n}N_{22} = A^{-n}N_1^{-n}$, and hence $Y_{12}N_{22} = 0$. Since the normal operator $N_{22}$ has a dense range, $Y_{12} = 0$ (which than implies that $Y_{11}$ and $Y_{22}$ are quasi-affinities). But then $A_{22}Y_2 = Y_{22}N_{22}$ and $A_{22}Y_2 = Y_{22}N_{22}$ imply that $A_{22}$ is quasi-affinity. Hence, since $A^{-n}Y_1 = Y_1N_1^{-n}$ implies also that $A_{12}A_{22}^{-1}Y_{12} = 0$, $A_{12} = 0$. Thus $A = 0 \oplus A_{22}$, where $A_{22} \in \mathcal{A}(1, 1)$. Hence, if $A = 0 \oplus A_{22}$, applying Proposition 2.5 and Lemma 2.2 of [10], it follows that $A_{22}$ and $N_{22}$ are (unitarily equivalent) normal operators. Conclusion: $A = 0 \oplus A_{22}$ is a normal $n$th root.

References