Automorphisms and Isomorphisms of Enhanced Hypercubes

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Abstract. Let $\mathbb{Z}_n^k$ be the elementary abelian 2-group, which can be viewed as the vector space of dimension $n$ over $\mathbb{F}_2$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}_n^k$ and $e_k = e_1 + \cdots + e_i$, for some $1 \leq k \leq n - 1$. Denote by $\Gamma_{n,k}$ the Cayley graph over $\mathbb{Z}_n^k$ with generating set $S_i = \{e_1, \ldots, e_k, e_i\}$, that is, $\Gamma_{n,k} = \text{Cay}(\mathbb{Z}_n^k, S_i)$. In this paper, we characterize the automorphism group of $\Gamma_{n,k}$ for $1 \leq k \leq n - 1$ and determine all Cayley graphs over $\mathbb{Z}_n^k$ isomorphic to $\Gamma_{n,k}$. Furthermore, we prove that for any Cayley graph $\Gamma = \text{Cay}(\mathbb{Z}_n^k, T)$, if $\Gamma$ and $\Gamma_{n,k}$ share the same spectrum, then $\Gamma \cong \Gamma_{n,k}$. Note that $\Gamma_{n,k}$ is known as the so called $n$-dimensional folded hypercube $FQ_n$, and $\Gamma_{n,k}$ is known as the $n$-dimensional enhanced hypercube $Q_{n,k}$.

1. Introduction

It is known that an interconnection network is conveniently represented by an undirected graph. For instance, the vertices of the graph represent the nodes of the network and the edges represent the links of the network. The $n$-dimensional hypercube $Q_n$ is the graph whose vertices are the $n$-bit binary strings $(x_1, \ldots, x_n)$ with $x_i \in \{0, 1\}$ for $1 \leq i \leq n$, and whose edges are pairs of vertices differing in exactly one position. As a topology for an interconnection network of a multiprocessor system, the hypercube is a widely used and well-known model, since it possesses many attractive properties such as regularity, symmetry, logarithmic diameter, high connectivity, recursive construction, ease of bisection, and relatively low link complexity [10, 13, 16]. There are many invariants of $Q_n$, for instance, generalized hypercube, folded hypercube, twisted hypercube, argument hypercube and enhanced hypercube. In this paper, we focus on the folded hypercube and the enhance hypercube. As a variant of the hypercube, the $n$-dimensional folded hypercube $FQ_n$, proposed first by El-Amawy and Latifi [2], is obtained from the hypercube $Q_n$ by making each vertex $u$ adjacent to its complementary vertex, denoted by $\bar{u}$ and obtained from $u$ by subtracting each bit from 1. Some properties of the folded hypercube $FQ_n$ are discussed in [11, 16–18]. Another variant of hypercube, the so called enhanced hypercube $Q_{n,k}$, proposed first by Tzeng and Wei [15], is obtained from the hypercube $Q_n$ by adding the edges $\{x, y\}$ if $x = (x_1, \ldots, x_n)$ and $y = (x_1, \ldots, x_{k-1}, \bar{x}_k, \ldots, \bar{x}_n)$ where $\bar{x}_j = x_j + 1$. It is clear that the folded hypercube $FQ_n$ is the special case of the enhanced hypercube $Q_{n,k}$ for $k = 1$, that is, $FQ_n = Q_{n,1}$.

Let $\mathbb{Z}_n^k$ be the elementary abelian 2-group. It can be viewed as the $n$-dimensional vector space over the filed $\mathbb{F}_2$ and $\{e_1, \ldots, e_n\}$ forms the orthonormal basis of $\mathbb{Z}_n^k$, where $e_i$ is the vector whose $i$-th entry is 1 and other entries are 0s. From the definition of the enhanced hypercube $Q_{n,k}$, each vertex of $Q_{n,k}$ is a vector in
Lemma 2.4. Now we present a sufficiently for a Cayley graph over an abelian group to be normal. Many mathematicians have noticed this result, see, for example, [19]. We still give a proof here for the convenience of the reader.

Lemma 2.3 (6). The automorphism group \( \text{Aut}(\Gamma) \) of a normal Cayley graph \( \Gamma \) is given by \( \text{Aut}(\Gamma) = R(\Gamma) \rtimes \text{Aut}_G(\Gamma) \).

Recall that the Cayley graph over a group \( G \) with the generating set \( S \subset G \), denoted by \( \text{Cay}(G, S) \), is the graph with vertex set \( G \) and two vertices \( x \) and \( y \) are adjacent if \( yx^{-1} \in S \). Therefore, the \( n \)-dimensional enhanced hypercube \( Q_{n, k} \) is just the Cayley graph \( \Gamma_{n,k} = \text{Cay}(\mathbb{Z}_2^n, S_k) \), where \( S_k = \{e_1, \ldots, e_n, e_k\} \) and \( e_k = e_k + \cdots + e_n \). Particularly, the \( n \)-dimensional folded hypercube \( \mathcal{F}Q_n \) is the Cayley graph \( \Gamma_{n,1} = \text{Cay}(\mathbb{Z}_2^n, S_1) \), where \( S_1 = \{e_1, \ldots, e_n, e_1\} \) and \( e_1 = e_1 + \cdots + e_n \).

An isomorphism \( \alpha \) from a graph \( \Gamma \) to \( \Gamma' \) is a bijection from \( V(\Gamma) \) to \( V(\Gamma') \) such that \( u \sim v \) in \( \Gamma \) if and only if \( \alpha(u) \sim \alpha(v) \) in \( \Gamma' \). If \( \Gamma \) and \( \Gamma' \) are the same, then \( \alpha \) is called an automorphism of \( \Gamma \). The set of all automorphisms of \( \Gamma \) forms the automorphism group of \( \Gamma \), denoted by \( \text{Aut}(\Gamma) \) [4]. There is a strong connection between the automorphism of a graph and the structure of the graph. For example, a graph with high symmetry always has a large automorphism group and a graph with little symmetry always has a small automorphism group. Therefore, we would like to investigate the automorphism group of the hypercube \( Q_{n,k} \). For a Cayley graph \( \Gamma = \text{Cay}(G, S) \), let \( \text{Aut}_G(\Gamma) = \{\alpha \in \text{Aut}(\Gamma) \mid \alpha(e) = e\} \) and \( \text{Aut}_S(\Gamma) = \{\beta \in \text{Aut}(G) \mid \beta(S) = S\} \), where the subscript \( e \) denotes the identity element of \( G \). If \( G \) is abelian, we write \( \text{Aut}_0(\Gamma) \) for \( \text{Aut}_G(\Gamma) \) since the identity element of abelian groups is always denoted by \( 0 \). It is clear that \( \text{Aut}_0(\Gamma) \geq \text{Aut}_G(\Gamma) \). If \( \text{Aut}_0(\Gamma) = \text{Aut}_G(\Gamma) \) then \( \Gamma \) is called normal. In this paper, we characterize the automorphism group of \( \Gamma_{n,k} \). Moreover, we determine all Cayley graphs over \( \mathbb{Z}_2^n \) isomorphic to \( \Gamma_{n,k} \). Furthermore, we give the spectrum of \( \Gamma_{n,k} \) and show that \( \Gamma_{n,k} \) is determined by its spectrum among the Cayley graphs over \( \mathbb{Z}_2^n \), that is, for any Cayley graph \( \Gamma = \text{Cay}(\mathbb{Z}_2^n, T) \), if \( \Gamma \) and \( \Gamma_{n,k} \) share the same spectrum, then \( \Gamma \cong \Gamma_{n,k} \).

2. Basic properties

In this part, we present some properties of Cayley graphs which will be used latter. At first, we introduce the automorphism group of \( \mathbb{Z}_2^n \).

Lemma 2.1 ([1]). The automorphism group of \( \mathbb{Z}_2^n \) is isomorphic to \( \text{GL}(n, F_2) \) where \( \text{GL}(n, F_2) \) is the set of invertible matrices of order \( n \) over \( F_2 \). Furthermore, we have

\[
|\text{Aut}(\mathbb{Z}_2^n)| = |\text{GL}(n, F_2)| = \prod_{k=1}^{n} (2^{n} - 2^{k-1}).
\]

In fact, each element \( v \) of \( \mathbb{Z}_2^n \) is a \((0,1)\)-vector. For any \( \sigma \in \text{Aut}(\mathbb{Z}_2^n) \), there exists \( M_{\sigma} \in \text{GL}(n, F_2) \) corresponding to \( \sigma \) such that \( \sigma(v) = M_{\sigma}v \). By simple observations, one can obtain the following well-known result.

Lemma 2.2. For any two non-zero vectors \( x, y \in \mathbb{Z}_2^n \), there exists \( A \in \text{GL}(n, F_2) \) such that \( Ax = y \) and thus \( \text{GL}(n, F_2) \) acting on \( \mathbb{Z}_2^n \) has two orbits: \( \{0\} \) and \( \mathbb{Z}_2^n \setminus \{0\} \).

Let \( \Gamma = \text{Cay}(G, S) \) be a Cayley graph over \( G \) with generating set \( S \) satisfying \( S = S^{-1} \) and \( \langle S \rangle = G \). The right regular representation of the group \( G \) is defined as \( R(G) = \{r_g: x \mapsto gx (\forall x \in G) \mid g \in G\} \). Clearly, \( R(G) \leq \text{Aut}(\Gamma) \) acts transitively on \( \Gamma \). Recall that \( \text{Aut}_G(\Gamma) = \{\alpha \in \text{Aut}(\Gamma) \mid \alpha(e) = e\} \). The following result follows.

Lemma 2.3 ([6]). The automorphism group \( \text{Aut}(\Gamma) \) of a normal Cayley graph \( \Gamma \) is given by \( \text{Aut}(\Gamma) = R(G) \rtimes \text{Aut}_G(\Gamma) \).

Now we present a sufficiently for a Cayley graph over an abelian group to be normal. Many mathematicians have noticed this result, see, for example, [19]. We still give a proof here for the convenience of the reader.

Lemma 2.4. Let \( \Gamma = \text{Cay}(G, S) \) be a connected Cayley graph on the abelian group \( G \). If \( N(s) \cap N(t) = \{0, s + t\} \) for any distinct \( s, t \in S \) then \( \Gamma \) is normal, that is, \( \text{Aut}_G(\Gamma) = \text{Aut}_G(S) \).
Proof. It is clear that $Aut_S(G) \leq Aut_0(\Gamma)$ and it remains to show that $Aut_0(\Gamma) \leq Aut_S(G)$. For any $\sigma \in Aut_0(\Gamma)$, we have $\sigma(S) = S$ and thus it only needs to show that $\sigma \in Aut(G)$. Clearly, $\sigma$ is a bijection from $G$ to itself, and thus it needs to show that $\sigma(x + y) = \sigma(x) + \sigma(y)$ for any $x, y \in G$.

For any distinct $s, t \in S$, since $N(s) \cap N(t) = \{0, s + t\}$, we have $N(\sigma(s)) \cap N(\sigma(t)) = \{0, \sigma(s + t)\}$. Note that $\sigma(s), \sigma(t) \in S$. We also have $N(\sigma(s)) \cap N(\sigma(t)) = \{0, \sigma(s + t)\}$. It leads to that $\sigma(s + t) = \sigma(\sigma(s) + \sigma(t))$. In addition, we show that $\sigma(2s) = \sigma(s) + \sigma(s)$. Since $s \sim 2s$, we have $\sigma(s) \sim \sigma(2s)$. It means that $\sigma(2s) = \sigma(s) + \sigma(s')$. If $s' \neq s$ then it is proved that $\sigma(s) + \sigma(s') = \sigma(s + s')$ and thus $\sigma(2s) = \sigma(s) + \sigma(s') = \sigma(s + s')$. It leads to $2s = s + s'$ and thus $s = s'$, a contradiction. Hence, we have $\sigma(s + t) = \sigma(s) + \sigma(t)$ for any $s, t \in S$, where $s$ and $t$ may be equal.

In general, for $s_1, \ldots, s_k \in S$, we will show that $\sigma(s_1 + \ldots + s_k) = \sigma(s_1) + \cdots + \sigma(s_k)$ for any $\sigma \in Aut_0(\Gamma)$. Note that $r_{\sigma} \in Aut(\Gamma)$ for any $u \in G$. Thus $r_{\sigma} r_{\sigma} r_{\sigma} \in Aut(\Gamma)$. Note that $r_{\sigma} r_{\sigma} \sigma r_{\sigma} = 0$. We have $r_{\sigma} r_{\sigma} \sigma r_{\sigma} \in Aut_0(\Gamma)$, by inductive assumption, we have

$$r_{\sigma(s_1)} r_{\sigma(s_2)} \cdots r_{\sigma(s_k)} (s_1 + \ldots + s_k) = r_{\sigma(s_1)} r_{\sigma(s_2)} \cdots r_{\sigma(s_k)} (s_1 + \ldots + s_k)$$

$$r_{\sigma(s_1)} r_{\sigma(s_2)} \cdots r_{\sigma(s_k)} (s_1 + \ldots + s_k)$$

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$$r_{\sigma(s_1)} r_{\sigma(s_2)} \cdots r_{\sigma(s_k)} (s_1 + \ldots + s_k)$$

Thus, we have

$$\sigma(s_1 + \ldots + s_k) = \sigma(s_1) + \cdots + \sigma(s_k).$$

Since $(S) = G$, for any $x, y \in G$, we have $x = s_1 + \ldots + s_a$ and $y = s'_1 + \ldots + s'_b$, where $s_i, s'_i \in S$. Therefore, we have

$$\sigma(x + y) = \sigma(s_1 + \ldots + s_a + s'_1 + \ldots + s'_b)$$

$$\sigma(s_1 + \ldots + s_a + s'_1 + \ldots + s'_b)$$

$$\sigma(s_1 + \ldots + s_a + s'_1 + \ldots + s'_b)$$

$$\sigma(s_1 + \ldots + s_a + s'_1 + \ldots + s'_b)$$

It completes the proof. \square

Let $X_1$ and $X_2$ be two graphs with $V(X_1) = \{u_1, \ldots, u_m\}$ and $V(X_2) = \{v_1, \ldots, v_n\}$. The Cartesian product $X_1 \square X_2$ is the graph with vertex set $V(X_1) \times V(X_2)$ and two vertices $(u_i, v_j)$ and $(u'_i, v'_j)$ are connected if $u_i \sim u'_i$ in $X_1$ and $v_j = v'_j$ or $u_i = u'_i$ and $v_j \sim v'_j$ in $X_2$.

Lemma 2.5. Let $\Gamma = Cay(G, S)$ be a Cayley graph over the group $G$. If $S = T_1 \cup T_2$ such that $G = \langle T_1 \rangle \cdot \langle T_2 \rangle$ the internal direct product of $\langle T_1 \rangle$ and $\langle T_2 \rangle$, then $\Gamma \cong \Gamma_1 \square \Gamma_2$ where $\Gamma_1 = Cay(\langle T_1 \rangle, T_1)$ and $\Gamma_2 = Cay(\langle T_2 \rangle, T_2)$.

Proof. Since $G = \langle T_1 \rangle \cdot \langle T_2 \rangle$ is the internal direct product of $\langle T_1 \rangle$ and $\langle T_2 \rangle$, each element $v \in G$ can be uniquely written as $v = v_1 v_2$ for $v_1 \in \langle T_1 \rangle$ and $v_2 \in \langle T_2 \rangle$. Moreover, it is well known that $G \cong \langle T_1 \rangle \times \langle T_2 \rangle$ the external direct product of $\langle T_1 \rangle$ and $\langle T_2 \rangle$ [7]. In fact, let $\phi: G \to \langle T_1 \rangle \times \langle T_2 \rangle$ be the map defined by $\phi(v) = \phi(v_1 v_2) = (v_1, v_2)$, then $\phi$ is the isomorphism from $G$ to $\langle T_1 \rangle \times \langle T_2 \rangle$. Therefore, the map $\phi$ gives a bijection from $V(G)$ to $V(\Gamma_1 \square \Gamma_2)$. Next, we show that $\phi$ is also a graph isomorphism from $\Gamma$ to $\Gamma_1 \square \Gamma_2$.

Since $S = T_1 \cup T_2$ for each $s \in S$, we have $s \in T_1$ or $s \in T_2$. If $s \in T_1$ then $\phi(s) = (s, e_2)$, where $e_2$ is the identity element of $\langle T_2 \rangle$. If $s \in T_2$ then $\phi(s) = (e_1, s)$, where $e_1$ is the identity element of $\langle T_1 \rangle$. Let $v = v_1 v_2$ and $v' = v'_1 v'_2$ be two vertices of $V(G)$. It is seen that $v \sim v'$ in $\Gamma$ if and only if $\phi(v' v^{-1}) = (v'_1 v_1^{-1}, v'_2 v_2^{-1}) = \phi(s) = (s, e_2)$ when $s \in T_1$, or $\phi(v' v^{-1}) = \phi(s) = (v'_1 v_1^{-1}, v'_2 v_2^{-1}) = (e_1, s)$ when $s \in T_2$ if and only if $v_1 \sim v'_1$ in $\Gamma_1$ and $v_2 = v'_2$, or $v_1 = v'_1$ and $v_2 \sim v'_2$ in $\Gamma_2$ if and only if $\phi(v) = (v_1, v_2) \sim (v'_1, v'_2) = \phi(v')$ in $\Gamma_1 \square \Gamma_2$. It leads to that $\Gamma \cong \Gamma_1 \square \Gamma_2$.

This completes the proof. \square
3. Automorphism group of the enhanced hypercube \(Q_{n,k}\)

In this part, we first give the automorphism group of the folded hypercube \(Q_{n,1}\) and determine the Cayley graphs over \(Z^n_2\) isomorphic to \(Q_{n,1}\). Next, we extend such results to the enhanced hypercube \(Q_{n,k}\) for any \(2 \leq k \leq n - 1\). Keep in mind that \(Q_{n,k}\) is just the Cayley graph \(\Gamma_{n,k} = \text{Cay}(Z^n_2, S_k)\) with \(S_k = \{e_1, \ldots, e_n, e_k\}\) and \(e_k = e_k + \cdots + e_n\).

**Lemma 3.1.** Let \(\Gamma_{n,1} = \text{Cay}(Z^n_2, S_1)\) be the \(n\)-dimensional folded hypercube with \(n \geq 4\). Then \(\Gamma_{n,1}\) is normal and \(\text{Aut}_0(\Gamma_{n,1}) \cong S_{n+1}\), where \(S_{n+1}\) is the symmetric group of degree \(n + 1\).

**Proof.** Denote by \(v_i = e_i\) for \(1 \leq i \leq n\) and \(v_{n+1} = e_1\). Clearly, \([0, v_i + v_j] \subseteq N(v_i) \cap N(v_j)\). Now assume that \(x\) is a vertex of \(N(v_i) \cap N(v_j)\). We have \(x = v_i + v_s = v_j + v_t\) for some \(v_s, v_t \in S\) and thus \(v_i + v_s - v_j - v_t = 0\), that is \(v_i, v_j, v_s, v_t\) are linear dependent. If \(v_i, v_j, v_s, v_t\) are distinct then \([v_i, v_j, v_s, v_t]\) must be linear independent since any \(n\) elements of \(S_1\) form a basis of \(Z^n_2\) and \(n \geq 4\). Therefore, we have \(v_i = v_j\) or \(v_s = v_t\) and \(v_i = v_t\). If the former occurs then \(x = 0\); if the latter occurs then \(x = v_i + v_j\). Thus, we have \(N(v_i) \cap N(v_j) = \{0, v_i + v_j\}\). Since \(Z^n_2\) is abelian, Lemma 2.4 implies that \(\text{Aut}_0(\Gamma_{n,1}) = \text{Aut}_0(S_{n+1})\).

In what follows, we show that \(\text{Aut}_0(S_{n+1}) \cong S_{n+1}\). From Lemma 2.1, we have \(\text{Aut}(Z^n_2) = \text{GL}(n, F_2)\) and thus \(\text{Aut}_0(S_{n+1}) = \{A \in \text{GL}(n, F_2) \mid S_1 = S_1\}\). Since \(S_1 = S_1\) for any \(A \in \text{Aut}_0(S_{n+1})\), we have \(A(v_i, v_1, \ldots, v_{n+1}) = (v_i, v_1, \ldots, v_{n+1})\), which implies that \(A = (v_i, \ldots, v_{n+1})\). Define the map \(\phi: \text{Aut}_0(S_{n+1}) \rightarrow S_{n+1}\) by setting \(\phi(A) = (1,1)(1,2)\ldots(n+1)\). In fact, \(\phi\) is well defined since \(1,1,\ldots,1\) is a reset of \(1,2,\ldots,n+1\). It is clear that \(\phi_A = \phi_B\) if and only if \((v_i, v_1, \ldots, v_{n+1}) = (v_i, v_1, \ldots, v_{n+1})\) if and only if \(A = B\). Besides, for each \(\theta \in S_{n+1}\), we construct \(A = (v_1(\theta), \ldots, v_{n+1}(\theta))\) and thus \(A(v_1(\theta), \ldots, v_{n+1}(\theta)) = (v_1(\theta), \ldots, v_{n+1}(\theta))\) for any \(1 \leq i \leq n+1\), we have \(\phi_A = \phi_B\). It leads to that \(\phi\) is an isomorphism and thus \(\text{Aut}_0(S_{n+1}) \cong S_{n+1}\).

Combining Lemmas 2.3 and 3.1, we get the automorphism group of the folded hypercube \(\Gamma_{n,1} = \text{Cay}(Z^n_2, S_1)\), where \(S_1 = \{e_1, \ldots, e_n, e_1\}\) and \(e_1 = e_1 + \cdots + e_n\).

**Theorem 3.2.** The automorphism group of the folded hypercube \(\Gamma_{n,1}\) is given by

\[
\text{Aut}(\Gamma_{n,1}) = \begin{cases} 
S_4 & \text{if } n = 2 \\
(S_4 \times S_4) \rtimes S_2 & \text{if } n = 3 \\
Z^n_2 \rtimes S_{n+1} & \text{if } n \geq 4
\end{cases}
\]

**Proof.** It is easy to see that \(\Gamma_{2,1} \cong K_4\) and \(\Gamma_{3,1} \cong K_4 \rtimes K_2\). Therefore, we have \(\text{Aut}(\Gamma_{2,1}) = \text{Aut}(K_4) = S_4\) and \(\text{Aut}(\Gamma_{3,1}) = \text{Aut}(K_4 \rtimes K_2) = (S_4 \times S_4) \rtimes S_2\). For \(n \geq 4\), by Lemmas 2.3 and 3.1, we have \(\text{Aut}(\Gamma_{n,1}) = Z^n_2 \rtimes S_{n+1}\).

A Cayley graph \(\text{Cay}(Z^n_2, T)\) is called perfect if \(T = [t_1, t_2, \ldots, t_n, t]\) such that \([t_1, \ldots, t_n]\) is a basis of \(Z^n_2\) and \(t = t_1 + \cdots + t_n\). Clearly, \(\Gamma_{n,1}\) is perfect.

**Theorem 3.3.** Let \(\Gamma_{n,1} = \text{Cay}(Z^n_2, S_1)\) be the Cayley graph with generating set \(S_1 = \{e_1, \ldots, e_n, e_1\}\) where \(e_1 = e_1 + \cdots + e_n\). Then the Cayley graph \(\text{Cay}(Z^n_2, T)\) is isomorphic to \(\Gamma_{n,1}\), if and only if \(\text{Cay}(Z^n_2, T)\) is perfect.

**Proof.** If \(\text{Cay}(Z^n_2, T)\) is perfect, then \(T = \{t_1, \ldots, t_n, t\}\) such that \([t_1, \ldots, t_n]\) is a basis of \(Z^n_2\) and \(t = t_1 + \cdots + t_n\). Now we define \(\varphi(x) = Ax\) for any \(x \in V(\Gamma_{n,1})\) where \(A = \{t_1, \ldots, t_n, t\}\). It is clear that \(AS_1 = T\). Since \([t_1, \ldots, t_n]\) is a basis of \(Z^n_2\), we have \(A \in \text{GL}(n, F_2)\). Therefore, \(\varphi\) is a bijection between \(V(\Gamma_{n,1})\) and \(V(\text{Cay}(Z^n_2, T))\). Moreover, \(\varphi(x) \sim \varphi(y)\) in \(\text{Cay}(Z^n_2, T)\) if and only if \(\varphi(y) - \varphi(x) = A(y - x)\) in \(T\) if and only if \((y - x) \in A^{-1}T = S_1\) if and only if \(x \sim y\) in \(\Gamma_{n,1}\). It leads to that \(\text{Cay}(Z^n_2, T) \cong \Gamma_{n,1}\).

Conversely, assume that \(\text{Cay}(Z^n_2, T)\) is a Cayley graph isomorphic to \(\Gamma_{n,1}\). Let \(\varphi\) be the isomorphism from \(\Gamma_{n,1}\) to \(\text{Cay}(Z^n_2, T)\) with \(\varphi(0) = 0\). We have \(T = \varphi(S_1) = \{\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_n), \varphi(e_1)\}\). Since \(\text{Cay}(Z^n_2, T)\) is also connected, we have \(Z^n_2 = \langle T \rangle\). Therefore, there is a basis \(B^n_2\) contained in \(T\). Without loss of generality, we may assume that \(\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_n)\) is a basis. Thus, we have

\[
\varphi(e_1) = \varphi(e_1) + \varphi(e_2) + \cdots + \varphi(e_n),
\]
where \(i_1, i_2, \ldots, i_k\) are distinct. Now, it only needs to show that \(\{i_1, i_2, \ldots, i_k\} = \{1, 2, \ldots, n\}\), i.e., \(s = n\). Note that \(\{0, \varphi(e_{i_1}), \varphi(e_{i_2}), \ldots, \varphi(e_{i_s})\}\) forms a cycle of length \(s+1\) in \(\text{Cay}(\mathbb{Z}_2^n, T)\). It follows that \(\{0, e_{i_1}, \varphi^{-1}(\varphi(e_{i_1}) + \varphi(e_{i_2}))) + \varphi(e_{i_s})\}\) forms a cycle in \(\Gamma_{n,1}\). Therefore, for \(2 \leq k \leq s - 1\), we have:

\[
\varphi^{-1}(\varphi(e_{i_1}) + \varphi(e_{i_2})) = \varphi^{-1}(\varphi(e_{i_3}) + \varphi(e_{i_4})) + v_k,
\]

and \(e_1 = \varphi^{-1}(\varphi(e_{i_1}) + \varphi(e_{i_2})) + v_s\), where \(v_{2s}, \ldots, v_s \in \{e_1, e_2, \ldots, e_n, e_1\}\). Thus, we have:

\[
e_1 = v_1 + v_2 + \cdots + v_s,
\]

where \(v_1 = e_i\) and \(v_2, \ldots, v_s \in \{e_i, e_{i+1}, \ldots, e_n, e_1\}\). Note that \(v_1, v_2, \ldots, v_s\) may be not distinct, and \(2v_j = 0\) for any \(1 \leq j \leq s\). Each term appears even times vanishes and we have:

\[
e_1 = v_1' + v_2' + \cdots + v_s',
\]

where \(\{v_1', v_2', \ldots, v_s'\} \subseteq \{e_1, e_2, e_3, \ldots, e_n, e_1\}\) and \(l \leq s\). If there is one of \(v_1', v_2', \ldots, v_s'\), then \(v_1' + v_2' + \cdots + v_s' = 0\). It is a contradiction because \(v_1', v_2', \ldots, v_s' \in \{e_1, e_2, \ldots, e_n\}\), which is linear independent. Therefore, we have \(\{v_1', v_2', \ldots, v_s'\} \subseteq \{e_1, e_2, \ldots, e_n\}\). It follows that there are \(l\) positions of \(e_1\) is 1, and thus \(l = s = n\).

The proof is completed. □

Recall that, if \(\Gamma = \Gamma_1 \sqcup \Gamma_2\), then \(\Gamma_1\) and \(\Gamma_2\) are called factors of \(\Gamma\). Two graphs \(\Gamma_1\) and \(\Gamma_2\) are called relatively prime if there is no non-trivial graph that is a factor of both of them.

**Lemma 3.4 (12, Corollary 6.12).** If \(\Gamma = \Gamma_1 \sqcup \Gamma_2\) where \(\Gamma_1, \Gamma_2\) are two connected relative prime graphs, then \(\text{Aut}(\Gamma) = \text{Aut}(\Gamma_1) \times \text{Aut}(\Gamma_2)\).

**Lemma 3.5.** For any \(n \geq 2\), the graph \(\Gamma_{n,1} = \text{Cay}(\mathbb{Z}_2^n, S_1)\) has no factor \(K_2\).

**Proof.** Suppose to the contrary that \(\Gamma_{n,1} = K_2 \sqcup \Delta\). Therefore, the vertex set of \(\Gamma_{n,1}\) can be partitioned as \(V(\Gamma_{n,1}) = V \cup V'\), where each of \(V\) and \(V'\) induces a graph isomorphic to \(\Gamma\). Moreover, each vertex in \(V\) (resp. \(V'\)) has exactly one neighbor in \(V\) (resp. \(V'\)). We will use this fact frequently. Without loss of generality, assume that \(0 \in V\). Therefore, all but one neighbors of 0 are in \(V\). Without loss of generality, assume that \(e_1 \in V\) and \(e_2, \ldots, e_n, e_1 \in V\).

In what follows, we show that, for any \(2 \leq i \leq n-1\), \(e_i \in V\) and \(e_1 + e_i + e_{i+1} + \cdots + e_n \in V\). Clearly, \(e_1 + e_2 + e_3 + \cdots + e_{n-1} = e_1 \in V\). Note that \(e_1 \in V'\) and 0 is the only neighbor of \(e_1\) in \(V\). We have \(e_2 + e_3 + \cdots + e_n \in V'\) because \(e_1 \sim e_2 + e_3 + \cdots + e_n\). Therefore, the statement is true for \(i = 2\). Now we assume that the statement is true for \(i\). It suffices to show that the statement is also true for the case of \(i + 1\). Note that \(e_1 + e_{i+1} + \cdots + e_n \in V'\) and \(e_1 + e_i + \cdots + e_n\) is the only neighbor of \(e_1 + e_{i+1} + \cdots + e_n \in V'\). We have \(e_{i+1} + \cdots + e_n \in V'\) because \(e_1 + e_{i+1} + \cdots + e_n \sim e_{i+1} + \cdots + e_n\).

By the arguments above, we have \(e_{n-1} + e_n \in V'\). However, if \(n = 2\), then \(e_{n-1} + e_n = e_1 \in V\), a contradiction; if \(n \geq 3\), then \(e_{n-1} + e_n \sim e_{n-1} \in V\) and \(e_{n-1} + e_n \sim e_n \in V\), which contradicts the fact that \(e_{n-1} + e_n\) has exactly one neighbor in \(V\).

This completes the proof. □

Now we are in the position to present one of our main results.

**Theorem 3.6.** The automorphism group of \(\Gamma_{n,k}\) is given by

\[
\text{Aut}(\Gamma_{n,k}) = \begin{cases}
S_1 & \text{if } n = 2 \text{ and } k = 1 \\
S_2 \times S_4 & \text{if } n = 3 \text{ and } k = 2 \\
(S_1 \times S_2) \times S_2 & \text{if } n = 3 \text{ and } k = 1 \\
(\mathbb{Z}_2^{-1} \times S_{n-1}) \times S_4 & \text{if } n \geq 4 \text{ and } k = n - 1 \\
(\mathbb{Z}_2^{-1} \times S_{n-1}) \times ((S_1 \times S_4) \times S_2) & \text{if } n \geq 4 \text{ and } k = n - 2 \\
(\mathbb{Z}_2^{-1} \times S_{n-1}) \times (\mathbb{Z}_2^{k+1} \times S_{n-k+2}) & \text{if } n \geq 4 \text{ and } k \leq n - 3
\end{cases}
\]
Proof. The cases of \( n = 2, k = 1 \) and \( n = 3, k = 1 \) were considered in Theorem 3.2. If \( n = 3 \) and \( k = 2 \), then \( \Gamma_{3,2} \cong K_2 \circ K_3 \). It is clear that \( K_2 \) is not a factor of \( K_3 \). Lemma 3.4 implies that \( \text{Aut}(\Gamma_{3,2}) \cong \text{Aut}(K_2) \times \text{Aut}(K_3) = S_2 \times S_3 \)

Now we consider the case of \( n \geq 4 \). Let \( S_k^{(1)} = \{e_1, e_2, \ldots, e_{k-1} \} \) and \( S_k^{(2)} = \{e_{k}, e_{k+1}, \ldots, e_n, e_1, \ldots, e_{k-1} \} \). It is clear that \( S_k = S_k^{(1)} \cup S_k^{(2)} \) and \( S_k^2 = (S_k^{(1)})^2 \circ (S_k^{(2)})^2 \). Therefore, Lemma 2.5 indicates that \( \Gamma_{n,k} = \text{Cay}(S_k^{(1)}, S_k^{(2)}) \square \text{Cay}(S_k^{(2)}, S_k^{(2)}) \). Note that \( \text{Cay}(S_k^{(1)}, S_k^{(1)}) \cong S_k \) and \( \text{Cay}(S_k^{(1)}, S_k^{(2)}) \cong \Gamma_{n-k+1,1} \). We have \( \Gamma_{n,k} \cong \Gamma_{n-k+1,1} \). Since \( \Gamma_{k-1} \) is the Cartesian product of \( k-1 \) and \( \Gamma_2 \) and Lemma 3.5 indicates that \( K_2 \) is not a factor of \( \Gamma_{n-k+1,1} \), Lemma 3.4 implies that \( \text{Aut}(\Gamma_{n,k}) \cong \text{Aut}(\Gamma_{k-1}) \times \text{Aut}(\Gamma_{n-k+1,1}) \). Note that \( \text{Aut}(\Gamma_{k-1}) = Z_2^{k-1} \rtimes S_{k-1} \) and \( \text{Aut}(\Gamma_{n-k+1,1}) = \text{Aut}(\Gamma) \), \( \text{Aut}(\Gamma) \)

A Cayley graph \( \text{Cay}(Z_n^a, T) \) is called \( k \)-perfect if \( T = \{t_1, t_2, \ldots, t_n, t\} \) such that \( t_1, \ldots, t_n \) is a basis of \( Z_2^n \) and \( t \) is a sum of \( n-k+1 \) elements of \( \{t_1, t_2, \ldots, t_n\} \), i.e., \( t = t_1 + \cdots + t_{k-1} \). Particularly, when \( k = 1 \), the concept of \( 1 \)-perfect is coincident with the concept of \( \text{DS} \). It is clear that \( \Gamma_{n,k} = \text{Cay}(Z_n^a, S_k) \) is \( k \)-perfect, where \( S_k = \{e_1, \ldots, e_n, e_1\} \) and \( e_k = e_k + e_{k+1} + \cdots + e_n \).

**Theorem 3.7.** The Cayley graph \( \Gamma = \text{Cay}(Z_n^a, T) \) is isomorphic to \( \Gamma_{n,k} \) if and only if \( k \) is \( k \)-perfect.

**Proof.** From Lemma 2.5, we have \( \Gamma_{n,k} = Q_{k} - \bigcup_{\Gamma_{n-k+1,1}} \). Note that \( Q_{k-1} \) is a \( k-1 \) Cartesian products of \( K_2 \) and Lemma 3.5 indicate that \( \Gamma_{n-k+1,1} \) has no factor \( K_2 \). It implies that \( \Gamma_{n,k} \cong \Gamma_{n,k} \) if \( k \neq 1 \).

Assume \( \text{Cay}(Z_n^a, T) \cong \Gamma_{n,k} \). By the connectivity of \( \text{Cay}(Z_n^a, T) \), \( T \) contains a basis, and the other element in \( T \) is a sum of \( l \) distinct elements in the basis. Then, \( \text{Cay}(Z_n^a, T) \cong \Gamma_{n,l} \), and hence \( \Gamma_{n,l} \cong \Gamma_{n,k} \). It follows that \( l = k \), that is, \( \text{Cay}(Z_n^a, T) \) is \( k \)-perfect.

We know that up to isomorphism there is only one \( n \)-regular connected Cayley graph over \( Z_n^a \). Theorem 3.7 implies the following result.

**Corollary 3.8.** Up to isomorphism, \( \Gamma_{n,1}, \Gamma_{n,2}, \ldots, \Gamma_{n,n-1} \) are the only \( (n+1) \)-regular connected Cayley graphs over \( Z_n^a \).

4. The G-DS property of \( Q_n \)

For a graph \( \Gamma \) with vertex set \( V(\Gamma) = \{v_1, \ldots, v_n\} \), its adjacency matrix \( A = (a_{ij})_{n \times n} \) is the \( n \times n \) matrix with \( a_{ij} = 1 \) if \( v_i \sim v_j \) in \( \Gamma \) and \( a_{ij} = 0 \) otherwise. The eigenvalues of \( A \) are called the eigenvalues of \( \Gamma \) and the set of such eigenvalues together with their multiplicities forms the spectrum of \( \Gamma \), denoted by \( \text{Spec}(\Gamma) \). A graph \( \Gamma \) is called determined by its spectrum (DS for short) if, for any graph \( \Gamma' \), \( \text{Spec}(\Gamma') = \text{Spec}(\Gamma) \) implies that \( \Gamma' \cong \Gamma \). The question ‘which graphs are DS?’ goes back for about half a century, and originates from chemistry. In 1956, Günther and Primas [8] raised the question in a paper that relates the theory of graph spectra to Hückel’s theory from chemistry. For more details about this problem, we would like to refer the reader to [5].

Let \( \Gamma = \text{Cay}(G, S) \) be a Cayley graph over \( G \). To investigate whether the Cayley graph \( \Gamma \) is DS, it should be first discussed whether it is determined by its spectrum among the Cayley graphs over \( G \). This thought leads to the concept of ‘G-DS’. A Cayley graph \( \Gamma = \text{Cay}(G, S) \) is called \( G \)-spectrum determined (G-DS for short) if, for any \( \Gamma' = \text{Cay}(G, T) \), \( \text{Spec}(\Gamma') = \text{Spec}(\Gamma) \) implies that \( \Gamma' \cong \Gamma \). Clearly, if a Cayley graph is DS then it must be G-DS, but the converse is not true. In this part, we discuss the G-DS property of \( Q_n \).

**Lemma 4.1 (3).** Let \( G \) be an abelian group of order \( n \) and \( S \) a subset of \( G \) such that \( 1 \not\in S \) and \( S^{-1} = S \). If \( \chi_1, \ldots, \chi_n \) are all irreducible characters of \( G \), then the eigenvalues of the Cayley graph \( \text{Cay}(G, S) \) are \( \lambda_i = \sum_{s \in S} \chi_i(s) \) for \( 1 \leq i \leq n \).

According to Lemma 4.1, to get the eigenvalues of \( Q_n = \Gamma_{n,k} \), we should know the irreducible characters of \( Z_n^a \).

**Lemma 4.2 (14).** The irreducible characters of \( Z_n^a \) are \( \chi_{i_1, \ldots, i_n} \) for \( i_j \in \{0, 1\} \) and \( 1 \leq j \leq n \), where \( \chi_{i_1, \ldots, i_n}(v) = (-1)^{\sum_{j=1}^{n} i_j} \) for \( v = (a_1, \ldots, a_n) \in Z_n^a \).

Now we are ready to give the eigenvalues of \( Q_n \).

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Lemma 4.3. The eigenvalues of $\Gamma_{n,k} = \text{Cay}(\mathbb{Z}_2^n, S_k)$ consist of $\lambda_{t,0}$ and $\lambda_{t,e}$ for $0 \leq t \leq n$ with multiplicities $\eta_{t,0}$ and $\eta_{t,e}$, respectively, where

$$
\begin{align*}
\lambda_{t,0} &= n - 2t - 1 \\
\lambda_{t,e} &= n - 2t + 1
\end{align*}
$$

and

$$
\eta_{t,0} = \sum_{t=0}^{n} \left( \frac{k-1}{2^t-1} (n-2^{t+1}) \right) \\
\eta_{t,e} = \sum_{t=0}^{n} \left( \frac{k-1}{2^t-1} (n-2^{t+1}) \right)
$$

Proof. From Lemmas 4.1 and 4.2, the eigenvalues of $\Gamma_{n,k}$ are given by

$$
\lambda_{i_1,\ldots,i_n} = \sum_{v \in S_i} \chi_{i_1,\ldots,i_n}(v) = \sum_{j=1}^{n} \chi_{i_1,\ldots,i_n}(e_j) + \chi_{i_1,\ldots,i_n}(e_k)
$$

where $\chi_{i_1,\ldots,i_n}$ is the irreducible characters of $\mathbb{Z}_2^n$ given in Lemma 4.2. Denote by $\Lambda_t = \{ \chi_{i_1,\ldots,i_n} | i_1 + \cdots + i_n = t \}$ for $0 \leq t \leq n$. It is clear that $\eta_t = |\Lambda_t| = \binom{n}{t}$. Furthermore, denote by $\Lambda_{t,0} = \{ \chi_{i_1,\ldots,i_n} \in \Lambda_t | i_1 + \cdots + i_n \equiv 1 \pmod{2} \}$ and $\Lambda_{t,e} = \{ \chi_{i_1,\ldots,i_n} \in \Lambda_t | i_1 + \cdots + i_n \equiv 0 \pmod{2} \}$. It is clear that $\eta_{t,0} = |\Lambda_{t,0}| = \sum_{t=0}^{n} \left( \frac{k-1}{2^t-1} (n-2^{t+1}) \right)$ and $\eta_{t,e} = |\Lambda_{t,e}| = \sum_{t=0}^{n} \left( \frac{k-1}{2^t-1} (n-2^{t+1}) \right)$. Note that $\chi_{i_1,\ldots,i_n}(e_j) = (-1)^j$ and $\chi_{i_1,\ldots,i_n}(e_k) = (-1)^{i_1+\cdots+i_n}$. It is seen that, for any $\chi_{i_1,\ldots,i_n} \in \Lambda_t$,

$$
\begin{align*}
\sum_{j=1}^{n} \chi_{i_1,\ldots,i_n}(e_j) + \chi_{i_1,\ldots,i_n}(e_k) &= \sum_{j=1}^{n} (-1)^j + (-1)^{i_1+\cdots+i_n} \\
&= \begin{cases} 
-n - 2t - 1, & \text{if } \chi_{i_1,\ldots,i_n} \in \Lambda_{t,0} \\
-n - 2t + 1, & \text{if } \chi_{i_1,\ldots,i_n} \in \Lambda_{t,e}
\end{cases}
\end{align*}
$$

It means that all characters in $\Lambda_{t,0}$ lead to the same eigenvalue $\lambda_{t,0} = n - 2t - 1$ and all characters in $\Lambda_{t,e}$ lead to the same eigenvalue $\lambda_{t,e} = n - 2t + 1$. This completes the proof. \(\Box\)

From Lemma 4.3, it is easy to see that $\lambda_{1,e} = n - 1$ is an eigenvalue of $\Gamma_{n,k}$ with multiplicity $k - 1$. Therefore, the following result follows immediately.

Corollary 4.4. The Cayley graphs $\Gamma_{n,k}$ and $\Gamma_{n,k'}$ cannot share the same spectrum if $k \neq k'$.\(\Box\)

Now we are ready to present the main result of this part.

Theorem 4.5. The enhanced hypercube $\Gamma_{n,k}$ is $\mathbb{Z}_2^n$-DS.

Proof. Let $\Gamma = \text{Cay}(\mathbb{Z}_2^n, T)$ be the Cayley graph such that $\text{Spec}(\Gamma) = \text{Spec}(\Gamma_{n,k})$. It leads to that $\Gamma$ is also $n + 1$-regular because the two graphs share the same largest eigenvalue which is the valency of them. By Corollary 3.8, it is seen that $\Gamma' = \Gamma_{n,k'}$. However, Corollary 4.4 implies that $\text{Spec}(\Gamma_{n,k}) = \text{Spec}(\Gamma_{n,k'})$ if and only if $k = k'$. It follows the result. \(\Box\)

5. Conclusion

The enhanced hypercube $Q_{n,k}$ for $1 \leq k \leq n - 1$ is an important network topology for parallel processing computer systems. It is proved that a message routed algorithm can always follow a shortest path in any enhanced hypercube. Besides, though the hardware cost to construct enhanced hypercubes is greater than that of the normal hypercubes, the overhead is negligible when the order is large, and thus is more cost-effective when compared to a normal hypercube [15]. Therefore, the structural properties, such as the connectivity, the diameter and so on, of enhanced hypercubes play a very important role in the interconnection network. It is effective to obtain the structural properties of a graph from its algebraic properties. As an important algebraic property, the automorphism group of a graph not only reveals the symmetry of the graph but also reflects the complexity of the construction of the graph. In this paper, we completely determine the automorphism group of the enhanced hypercube $Q_{n,k}$ by regarding $Q_{n,k}$ as a Cayley graph over $\mathbb{Z}_2^n$. Moreover, we prove that all Cayley graphs over $\mathbb{Z}_2^n$ isomorphic to $Q_{n,k}$ must be the so called $k$-perfect Cayley graphs. Furthermore, we show that no two distinct enhanced hypercube can share the same spectrum, and $Q_{n,k}$ is determined by its spectrum among all Cayley graphs over $\mathbb{Z}_2^n$.\(\Box\)
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