



A Discrete Probability Model Suitable for Both Symmetric and Asymmetric Count Data

Deepesh Bhati^a, Subrata Chakraborty^b, Snober Gowhar Lateef^a

^aDepartment of Statistics, Central University of Rajasthan, Kishangarh, Rajasthan, India.

^bDepartment of Statistics, Dibrugarh University, Assam, India.

Abstract. In this paper, an alternative discrete probability model, namely the discrete skew logistic distribution, suitable for both asymmetric and symmetric count data is proposed. Some important properties of the distribution along with the estimation of the parameters are discussed. A detailed Monte Carlo simulation study is carried out to assess the performance of the maximum likelihood method and the method of proportion for parameter estimation. Finally, the application of the proposed model is discussed by considering two real-life datasets.

1. Introduction

Quite often, in the field of medical, sports, health, ecology, life testing experiments etc. researchers come across situations where certain phenomenon or processes over continuous domain are measured over a discrete support. For example, in some life testing experiments, there are instances where it is inconvenient to measure characteristics, such as life length of a device, on a continuous scale and hence measured on a discrete scale. Therefore, it is pertinent to construct discrete probability distributions to model such characteristics.

Further, it is often of interest to the practitioner to analyse count data distributed over the integer support \mathbb{Z} . Discrete distributions over \mathbb{Z} are rare in literature with only limited choices available including the famous Skellam distribution (Skellam [14]) and a few others, namely, the discrete normal (Roy [11]), Discrete Laplace (Kozubowski and Inusah [9]), discrete logistic (Chakraborty and Chakravarty [4]) etc. Such distributions have a natural application when one is interested to model the difference of counts rather than the count itself. For example, the difference in goals scored by two opponent teams (Karlis and Ntzoufras [8]), the difference in assessment score in count scales by two experts, the difference in the count and estimated count in a survey and other similar situations which can easily be conceived off.

Discrete distributions such as the discrete normal in Roy [11], discrete Laplace distribution in Kozubowski and Inusah [9], discrete logistic models in Chakraborty and Chakravarty [4] are suitable for symmetric count data, whereas only few distributions such as Skellam distribution, Skew Laplace discrete distribution are available for both symmetric or asymmetric count data sets. Because of meagreness in distributions relevant

2010 *Mathematics Subject Classification.* 60E05, 62E15, 62F10

Keywords. Discretization, Asymmetric count data, Skew Logistics Distribution, Generalized Geometric Distribution.

Received: 05 August 2019; Revised: 26 October 2019; Accepted: 11 February 2020

Communicated by Biljana Popović

Email addresses: deepesh.bhati@curaj.ac.in (Deepesh Bhati), subrata_arya@yahoo.co.in (Subrata Chakraborty), snobergauher786@gmail.com (Snober Gowhar Lateef)

to both asymmetric or symmetric count data, in this article, we explore and develop an alternative discrete distribution suitable for count data. For a comprehensive review on techniques to generate new class of discrete distributions see Chakraborty [3].

In this article, we discretize the recently introduced two-parameter skew logistic distribution, $\mathcal{SL}(\kappa, \beta)$, introduced in Sastry and Bhati [13]. As the skew logistic distribution of Sastry and Bhati possesses many properties in closed form, this motivates us to explore the properties and applications of its discretized version. The rest of the article is structured as follows: we start with a brief introduction of the skew logistic distribution of Sastry and Bhati [13] and introduce the new discrete counterpart of the skew logistic distribution in section 2; distributional properties like quantile function, mean, median, variance, mode of the distribution and its relation with other distributions are presented in section 3. Section 3 also discusses an algorithm for generating the skew logistic random variables which will be later used in simulation studies in section 4. In Section 4, methods of estimation are discussed. Finally, suitability and superiority of the proposed model compared to the other existing models is presented in section 6.

2. Proposed two parameter discrete skew logistic distribution

Here we begin our discussion by briefly introducing the $\mathcal{SL}(\kappa, \beta)$ of Sastry and Bhati [13] before proposing its discrete version.

2.1. Continuous skew logistic distribution

The probability density function (pdf) of $\mathcal{SL}(\kappa, \beta)$ with skew parameter $\kappa > 0$ and scale parameter $\beta > 0$ is given by

$$f(x; \kappa, \beta) = \frac{2\kappa}{1 + \kappa^2} \begin{cases} \frac{e^{-\frac{x}{\kappa\beta}}}{\beta\left(1+e^{-\frac{x}{\kappa\beta}}\right)^2} & \text{if } x < 0 \\ \frac{e^{-\frac{x\kappa}{\beta}}}{\beta\left(1+e^{-\frac{x\kappa}{\beta}}\right)^2} & \text{if } x \geq 0 \end{cases} \quad (1)$$

Letting $\kappa = 1$, the model reduces to the standard logistic distribution (see Johnson et al. [7]), values for $\kappa < 1$ lead to left-skewed logistic distribution whereas $\kappa > 1$ leads to right-skewed logistic distribution. Corresponding cumulative distribution function (cdf) and survival function (sf) of $\mathcal{SL}(\kappa, \beta)$ are given respectively as

$$F(x; \kappa, \beta) = \begin{cases} \frac{2\kappa^2}{(1+\kappa^2)} \frac{1}{1+e^{-\frac{x}{\kappa\beta}}} & \text{if } x < 0 \\ \frac{\kappa^2}{1+\kappa^2} + \frac{2}{1+\kappa^2} \left(\frac{1}{1+e^{-\frac{x\kappa}{\beta}}} - \frac{1}{2} \right) & \text{if } x \geq 0 \end{cases} \quad (2)$$

and

$$S(x; \kappa, \beta) = \begin{cases} 1 - \frac{2\kappa^2}{(1+\kappa^2)} \frac{1}{1+e^{-\frac{x}{\kappa\beta}}} & \text{if } x < 0 \\ \frac{2}{1+\kappa^2} \left(\frac{e^{-\frac{x\kappa}{\beta}}}{1+e^{-\frac{x\kappa}{\beta}}} \right) & \text{if } x \geq 0 \end{cases}, \quad (3)$$

with this prior knowledge of $\mathcal{SL}(\kappa, \beta)$, we now discuss the construction of the proposed distribution.

2.2. New discrete skew logistic distribution

Given a continuous random variable (rv) X with survival function (sf) $S_X(\cdot)$, discrete rv Y defined as $Y = \lfloor X \rfloor$, where $\lfloor X \rfloor$ is largest integer less or equal to X and its probability mass function (pmf) $P(Y = y)$ of Y is obtained as

$$P(Y = y) = S_X(y) - S_X(y + 1).$$

The pmf of rv Y as defined may be viewed as a discrete concentration of pdf of X and retains the same functional form of the sf as that of the continuous one. As a result, many characteristics remain unchanged. Discretization of many well-known distributions is studied using this approach (for detail see Chakraborty [3]). Hence, we re-parametrized the $\mathcal{SL}(\kappa, \beta)$ distribution given in (1), by assuming $p = e^{-\frac{\kappa}{\beta}}$ and $q = e^{-\frac{1}{\kappa\beta}}$, obviously $0 < p < 1, 0 < q < 1$ and p and q are related to κ and β as

$$\kappa = \sqrt{\frac{\ln p}{\ln q}} \quad \text{and} \quad \beta = \sqrt{\frac{1}{\ln p \ln q}}.$$

This leads us to the following definition of the proposed discrete skew logistic distribution.

Definition 1: A continuous rv X with sf $S_X(\cdot)$ follows \mathcal{SL} distribution then the rv $Y = \lfloor X \rfloor$ follows the Discrete skew logistic distribution with parameters p and q denoted as $\mathcal{DSL}(p, q)$ and its pmf $P(Y = y)$ is given by

$$P(Y = y) = \begin{cases} \frac{2 \ln p}{\ln(pq)} \left(\frac{q^{-(y+1)}}{1+q^{-(y+1)}} - \frac{q^{-y}}{1+q^{-y}} \right) & \text{if } y = \dots, -2, -1 \\ \frac{2 \ln q}{\ln(pq)} \left(\frac{p^y}{1+p^y} - \frac{p^{y+1}}{1+p^{y+1}} \right) & \text{if } y = 0, 1, 2, \dots \end{cases} \quad (4)$$

where $0 < p < 1, 0 < q < 1$. The cdf, sf and hf respectively given as

$$F_Y(y) = \begin{cases} \frac{2 \ln p}{\ln(pq)} \frac{q^{-(y+1)}}{(1+q^{-(y+1)})} & \text{if } y = \dots, -2, -1 \\ 1 - \frac{2 \ln q}{\ln(pq)} \frac{p^{(y+1)}}{(1+p^{(y+1)})} & \text{if } y = 0, 1, 2, \dots \end{cases} \quad (5)$$

$$S_Y(y) = \begin{cases} 1 - \frac{2 \ln p}{\ln(pq)} \frac{q^{-y}}{1+q^{-y}} & \text{if } y = \dots, -2, -1 \\ \frac{2 \ln q}{\ln(pq)} \frac{p^y}{1+p^y} & \text{if } y = 0, 1, 2, \dots \end{cases} \quad (6)$$

and

$$h_Y(y) = \frac{P(Y = y)}{S_Y(y)} = \begin{cases} \frac{2 \ln p(1-q)}{(1+q^{y+1})(\ln(pq)+q^{-y} \ln(q/p))} & \text{if } y = \dots, -2, -1 \\ \frac{1-p}{1+p^{y+1}} & \text{if } y = 0, 1, 2, \dots \end{cases} \quad (7)$$

Remark: For $p = q$ in (4), the model reduces to a symmetric distribution with integer support on $(-\infty, \infty)$ which was discussed in Chakraborty and Chakravarty [4], for $p > q$ the pmf will be right-skewed whereas for $p < q$ it will be left-skewed discrete skew logistic distribution, which can also be observed in Figure 1. Beside skew parameters p and q , the location version of $\mathcal{DSL}(p, q)$ can be obtained by introducing location parameter μ and the resulting locational $\mathcal{DSL}(p, q)$ distribution will be written as

$$P(Y = y) = P(Y = y; p, q, \mu) = \begin{cases} \frac{2 \ln p}{\ln(pq)} \left(\frac{q^{-(y-\mu+1)}}{1+q^{-(y-\mu+1)}} - \frac{q^{-(y-\mu)}}{1+q^{-(y-\mu)}} \right) & \text{if } y = \dots, \mu - 2, \mu - 1 \\ \frac{2 \ln q}{\ln(pq)} \left(\frac{p^{y-\mu}}{1+p^{y-\mu}} - \frac{p^{y-\mu+1}}{1+p^{y-\mu+1}} \right) & \text{if } y = \mu, \mu + 1, \mu + 2, \dots \end{cases} \quad (8)$$

Further, it can be noted that the mode of the location family of $\mathcal{DSL}(\mu, p, q)$ is either μ or $\mu - 1$ depending on $p > q$ or $p < q$, the proof of this result will be discussed in the coming section.

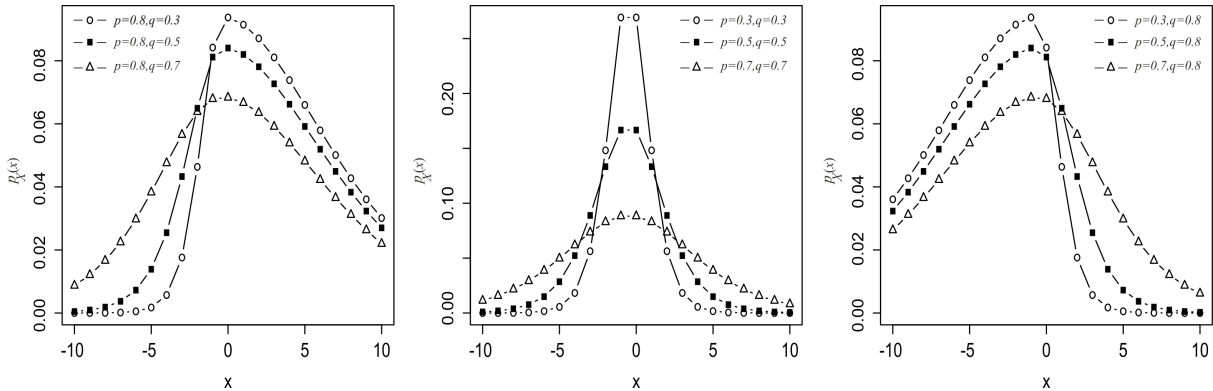


Figure 1: Probability mass function of $DS\mathcal{L}(p, q)$ distribution for several combinations of the parameter values.

3. Distributional Properties

In this section, we show the connection between $DS\mathcal{L}(p, q)$ and some other existing distributions and study the distributional properties like quantile function, moments, and mode.

3.1. Relation with DLoG(p)

Chakraborty and Chakravarty [4] obtained a one parameter symmetric discrete logistic distribution (denoted as DLoG(p)) with support on \mathbb{Z} by discretizing logistic distribution and its pmf given by

$$f_Y(y; p) = P(Y = y) = \frac{(1 - p)p^y}{(1 + p^y)(1 + p^{y+1})}, \quad y = 0, \pm 1, \pm 2, \dots \tag{9}$$

This distribution is symmetric in the sense that $f_{-(Y+1)}(y; p) = f_Y(y; p), y = 0, \pm 1, \pm 2, \dots$ with $P(Y \geq 0) = P(Y \leq -1) = 1/2$. Clearly the pmf of truncated DLoG(p) on either $y = 0, 1, \dots$ or on $y = -1, -2, \dots$ is then given by

$$2f_Y(y; p) = \frac{2(1 - p)p^y}{(1 + p^y)(1 + p^{y+1})}. \tag{10}$$

Thus $DS\mathcal{L}(p, q)$ reduces to DLoG(p) when $p = q$. In the following result, we show that the $DS\mathcal{L}(p, q)$ can be obtained as a mixture of two truncated DLoG distributions with proper mixing proportions.

Result 1: The $DS\mathcal{L}(p, q)$, is the mixture of truncated DLoG(p) over support $y = 0, 1, \dots$ and DLoG(q) distributions over support $y = -1, -2, \dots$ with respective mixing proportions $\ln(q) / \ln(pq)$ and $\ln(p) / \ln(pq)$ respectively. The $DS\mathcal{L}(p, q)$ can therefore be thought of as being generated by

$$\frac{\ln q}{\ln pq} 2f_Y(y; q)\mathbb{I}_{y \geq 0}(y) + \frac{\ln p}{\ln pq} 2f_Y(y; q)\mathbb{I}_{y \leq -1}(y).$$

Denoting the mean of truncated DLoG(p) over support $y = 0, 1, \dots$ by $\mu^+(p)$ and that of truncated DLoG(q) over support $y = -1, -2, \dots$ by $\mu^-(q)$ we can write

$$\begin{aligned} \mathbb{E}(Y) &= \frac{\ln q}{\ln(pq)} \mu^+(p) + \frac{\ln p}{\ln(pq)} \mu^-(q) \\ &= \frac{\ln q}{\ln(pq)} \mu^+(p) - \frac{\ln p}{\ln(pq)} (1 + \mu^+(q)) \end{aligned} \tag{11}$$

It may be noted here that truncated DLoG(p) over support $y = 0, 1, \dots$ is actually the generalized geometric distribution $(\mathcal{GGD})(2, q)$ of Gómez [6] discussed in the next section. But \mathcal{GGD} is defined over the

support $y = 0, 1, \dots$ while we have a truncated DLoG(p) is also available over support $y = -1, -2, \dots$. The above formula for the mean can, therefore, be written in terms of the mean of \mathcal{GGD} as

$$\mathbb{E}(X) = \frac{\ln q}{\ln(pq)} \mu^+(2, p) - \frac{\ln p}{\ln(pq)} (1 + \mu^+(2, q)), \tag{12}$$

where $\mu^+(2, \theta)$ is the mean of the $\mathcal{GGD}(2, \theta)$ that is of truncated DLoG(θ) over the support $y = 0, 1, \dots$.

3.2. Relation with generalized Geometric(α, θ)

Goméz [6] obtained a two-parameter generalization of geometric distribution (denoted as $(\mathcal{GGD})(2, q)$) by utilizing the discretization technique to Marshall-Olkin exponential distribution. The pmf of $(\mathcal{GGD})(2, q)$ is given as

$$P(Y = y) = \frac{\alpha q^y (1 - q)}{(1 - \bar{\alpha} q^{y+1})(1 - \bar{\alpha} q^y)} \quad y = 0, 1, \dots \tag{13}$$

where $\alpha > 0$, $\bar{\alpha} = 1 - \alpha$ and $0 < q < 1$, and it reduces to geometric for $\alpha = 1$.

Proposition 1: If $Y \sim \mathcal{DSL}(p, q)$, then r.v. $Y|Y \geq 0$ follows $\mathcal{GGD}(2, p)$.

Proof: The proof is straight forward by using the relation $P(X = x|X \geq 0) = \frac{P(X=x)}{P(X \geq 0)}$. \square

Proposition 2: If $Y \sim \mathcal{DSL}(p, q)$ then r.v. $Y|Y < 0$ follows $\mathcal{GGD}(2, q)$ with support $\{\dots, -2, -1\}$.

Proposition 3: If $Y \sim \mathcal{DSL}(p, q)$, then r.v. $Z = |Y| \stackrel{d}{=} wZ_1 + (1 - w)Z_2$, where mixing proportion $w = \frac{\ln p}{\ln pq}$, $Z_1 - 1 \sim \mathcal{GGD}(2, q)$ and $Z_2 \sim \mathcal{GGD}(2, p)$.

Proof: We know that, $Z = |Y| = \begin{cases} -Y & Y < 0 \\ +Y & Y \geq 0 \end{cases}$, hence the df of rv Z is given as

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ P(|Y| \leq z) &= P(-z \leq Y \leq z) \\ &= P(-z \leq Y \leq -1) + P(0 \leq Y \leq z) \\ F_Z(z) &= \frac{2 \ln p}{\ln pq} \left(\frac{1}{2} - \frac{q^z}{1 + q^z} \right) + \frac{2 \ln q}{\ln pq} \left(\frac{1}{2} - \frac{p^{z+1}}{1 + p^{z+1}} \right). \end{aligned}$$

Therefore, $P(Z = z) = F_Z(z) - F_Z(z - 1)$, hence

$$\begin{aligned} P(Z = z) &= \frac{2 \ln p}{\ln pq} \left(\frac{q^z}{1 + q^z} - \frac{q^{z-1}}{1 + q^{z-1}} \right) + \frac{2 \ln q}{\ln pq} \left(\frac{p^{z+1}}{1 + p^{z+1}} - \frac{p^z}{1 + p^z} \right) \\ &= \frac{\ln p}{\ln pq} \left(\frac{2q^{z-1}(1 - q)}{(1 + q^z)(1 + q^{z-1})} \right) + \frac{\ln q}{\ln pq} \left(\frac{2p^z(1 - p)}{(1 + p^{z+1})(1 + p^z)} \right). \end{aligned}$$

hence substituting $w = \frac{\ln p}{\ln pq}$, we obtained the desired results. \square

In view of the proposition 3, we can write the following

$$\begin{aligned} \mathbb{E}|Y| &= \frac{\ln p}{\ln(pq)} + \frac{\ln p}{\ln(pq)} \mathbb{E}(Z_1 - 1) + \frac{\ln q}{\ln(pq)} \mathbb{E}(Z_2) \\ &= \frac{\ln p}{\ln(pq)} + \frac{\ln p}{\ln(pq)} \mu^+(2, p) + \frac{\ln q}{\ln(pq)} \mu^+(2, q) \\ &= \frac{\ln p}{\ln(pq)} (1 + \mu^+(2, p)) + \frac{\ln q}{\ln(pq)} \mu^+(2, q). \end{aligned}$$

It is known from Gómez [6] that the mean $\mu^+(2, \theta)$ of $\mathcal{GGD}(2, \theta)$ tends to 0 as θ tends to 0. Therefore under the assumption that $\frac{\ln p}{\ln(pq)}$ and hence $\frac{\ln q}{\ln(pq)}$ is finite for p, q or both very small we can state that

$$E|Y| \rightarrow \begin{cases} \frac{\ln p}{\ln(pq)} + \frac{\ln q}{\ln(pq)} \mu^+(2, p) & \text{as } q \rightarrow 0 \\ \frac{\ln p}{\ln(pq)} + \frac{\ln p}{\ln(pq)} \mu^+(2, q) & \text{as } p \rightarrow 0 \\ \frac{\ln p}{\ln(pq)} & \text{as } p, q \rightarrow 0 \\ 0.5 + \mu^+(2, p) & \text{as } p \rightarrow q \end{cases} \tag{14}$$

3.3. Quantile function

The quantile of order $0 < \gamma < 1$, y_γ , can be obtained by inverting the cdf (5). Then for $\gamma \geq 1 - \frac{2 \ln q}{\ln(pq)} \frac{p}{1+p}$, the corresponding quantile is

$$y_\gamma = \left\lceil \ln_p \left(\frac{(1 - \gamma) \ln(pq)}{2 \ln q - (1 - \gamma) \ln(pq)} \right) \right\rceil - 1, \tag{15}$$

where $\lceil z \rceil$ represents the smallest integer greater than or equal to z ; otherwise for $\gamma < 1 - \frac{2 \ln q}{\ln(pq)} \frac{p}{1+p}$, γ -th quantile is

$$y_\gamma = \left\lceil \ln_q \left(\frac{2 \ln p}{\gamma \ln pq} - 1 \right) \right\rceil - 1. \tag{16}$$

Usually, quantiles are used to simulate observations from a distribution. For which, we draw a random number, say γ , from a standard uniform distribution, and the corresponding quantile y_γ obtained by using (15) or (16) is a random value from the \mathcal{DSL} with assigned parameters p and q .

The median (*Med*) of $\mathcal{DSL}(p, q)$ obtained by substituting $\gamma = \frac{1}{2}$ in the above two equations

$$Med = \begin{cases} \left\lceil \ln_p \left(\frac{\ln(pq)}{\ln\left(\frac{q^3}{p}\right)} \right) \right\rceil - 1 & \text{if } q > p^{\frac{1+p}{3p-1}} \\ \left\lceil \ln_q \left(\frac{\ln\left(\frac{p^3}{q}\right)}{\ln(pq)} \right) \right\rceil - 1 & \text{if } q \leq p^{\frac{1+p}{3p-1}} \end{cases}$$

3.4. Moments

Theorem 1: If $Y \sim \mathcal{DSL}(p, q)$, then

- (i) $\frac{2 \ln(\frac{p}{q})}{\ln p \ln q} \ln(2) - 1 \leq E(Y) \leq \frac{2 \ln(\frac{p}{q})}{\ln p \ln q} \ln(2)$.
- (ii) $\frac{(\ln p)^3 + (\ln q)^3}{(\ln pq)(\ln p \ln q)^2} \frac{\pi^2}{3} - \left(\frac{2 \ln(\frac{p}{q})}{\ln p \ln q} \ln(2) \right)^2 \leq V(Y) \leq \frac{(\ln p)^3 + (\ln q)^3}{(\ln pq)(\ln p \ln q)^2} \frac{\pi^2}{3} - \left(\frac{2 \ln(\frac{p}{q})}{\ln p \ln q} \ln(2) \right)^2 + \frac{1}{4}$.

Proof: For skew logistic distribution with pdf in (1), it is known that

$$E(X) = \frac{2 \ln(\frac{p}{q})}{\ln p \ln q} \ln 2 \quad \text{and} \quad V(X) = \frac{(\ln p)^3 + (\ln q)^3}{(\ln pq)(\ln p \ln q)^2} \frac{\pi^2}{3} - \left(\frac{2 \ln(\frac{p}{q})}{\ln p \ln q} \ln(2) \right)^2$$

Now the discretized version Y of X that is $\mathcal{DSL}(p, q)$ is defined as $Y = \lfloor X \rfloor$, where $p = e^{-\frac{\kappa}{\beta}}$ and $q = e^{-\frac{1}{\kappa\beta}}$. Further, it can be assumed that $X = Y + U$, where U is the fractional part of X which is chopped off from X

to obtain Y . then

$$\mathbb{E}(Y) = \mathbb{E}(X) - \mathbb{E}(U) = \frac{2 \ln(\frac{p}{q})}{\ln p \ln q} \ln(2) - \mathbb{E}(U)$$

and

$$\mathbb{V}(Y) = \mathbb{V}(X) + \mathbb{V}(U)$$

$$\mathbb{V}(Y) = \frac{(\ln p)^3 + (\ln q)^3}{(\ln pq)(\ln p \ln q)^2} \frac{\pi^2}{3} - \left(\frac{2 \ln(\frac{q}{p})}{\ln p \ln q} \ln(2) \right)^2 + \mathbb{V}(U).$$

But for a continuous r.v. U with support $(0, 1)$, $0 < \mathbb{E}(U) < 1$ and $0 \leq \mathbb{V}(U) \leq \frac{1}{4}$, (Popoviciu [10]). Hence the result follows.

3.5. Mode

Theorem 2: $\mathcal{DSL}(p, q)$ has an unique mode at 0 if $p > q$, and at -1, if $p < q$ and is bimodal with modes at -1, 0 if $p = q$.

Proof: Let us define $\Delta f_Y(y)$ as

$$\Delta f_Y(y) = f(y + 1) - f(y)$$

$$= \begin{cases} \left(\frac{2 \ln p}{\ln(pq)} \left(\frac{q^{-y}}{1+q^{-y}} - \frac{2q^{-(y+1)}}{1+q^{-(y+1)}} + \frac{q^{-(y+2)}}{1+q^{-(y+2)}} \right) \right) & \text{if } y = \dots, -2, -1 \\ \left(\frac{2 \ln q}{\ln(pq)} \left(\frac{2p^{y+1}}{1+p^{y+1}} - \frac{p^{y+2}}{1+p^{y+2}} - \frac{p^y}{1+p^y} \right) \right) & \text{if } y = 0, 1, 2, \dots \end{cases}$$

It can be further observed that

$$\Delta f_Y(y) \begin{cases} > 0 & \text{for } y < 0 \\ < 0 & \text{for } y \geq 0 \end{cases}$$

implies $f(y)$ is monotonically increasing for $y < 0$ and decreasing for $y \geq 0$. Hence the only modal value for the pmf are -1 and/or 0. 0 is the unique mode iff

$$f(0) > f(-1)$$

$$\Rightarrow \frac{\log q}{\log pq} \frac{1-p}{1+p} > \frac{\log p}{\log pq} \frac{1-q}{1+q}, \tag{17}$$

since, $0 < p < 1$ and $0 < q < 1$, (17) holds iff

$$(\ln q) \left(\frac{1+q}{1-q} \right) < (\ln p) \left(\frac{1+p}{1-p} \right).$$

Similarly, -1 will be the unique mode iff

$$f(0) < f(-1)$$

$$\Rightarrow \frac{\log q}{\log pq} \frac{1-p}{1+p} < \frac{\log p}{\log pq} \frac{1-q}{1+q}, \tag{18}$$

$$(\ln q) \left(\frac{1+q}{1-q} \right) > (\ln p) \left(\frac{1+p}{1-p} \right).$$

Consider, the function $g(\xi) = \ln \xi \left(\frac{1+\xi}{1-\xi} \right)$ for $\xi \in (0, 1)$, with derivative $g'(\xi) = \frac{1-\xi^2+2\xi \ln \xi}{\xi(1-\xi)^2} = \frac{h(\xi)}{\xi(1-\xi)^2}$, in order to prove $g(\xi)$ is increasing it is sufficient to prove $g'(\xi)$ or $h(\xi)$ is positive in $(0, 1)$. Further, we have $h(0) = 1$, $h(1) = 0$ and the $h'(\xi) = 2(1 - \xi + \log \xi) < 0$ for $\xi \in (0, 1)$. Hence the function $h(\xi)$ is decreasing function decreasing to zero from 1 implies $g'(\xi) > 0$ means increasing function. Hence, it follows that 0(-1) is the unique mode if $p > (<)q$, and if $p = q$ the pmf is bimodal with modes at 0 and at -1.

4. Methods of Estimation

In this section we consider two methods of estimation of parameters p and q namely (i) Method of proportion and zero's (ii) Maximum likelihood method.

4.1. Method of Proportion

From (4), (5) and (6), we can obtain following:

$$p_0 = P(Y = 0) = \frac{(1 - p) \ln q}{(1 + p) \ln pq}, \quad p^+ = P(Y \geq 0) = \frac{\ln p}{\ln pq} \quad \text{and} \quad p^- = P(Y \leq -1) = \frac{\ln q}{\ln pq} \tag{19}$$

and solving these equations, we obtain

$$p = \frac{p^+ - p_0}{p^+ + p_0} \quad \text{and} \quad q = \left(\frac{p^+ - p_0}{p^+ + p_0} \right)^{p^+ / p^-} .$$

Since a straightforward estimate of p_0 is the proportion of sample values equal to zero to the total sample size, denote it with $r_0 = \sum_{i=1}^n \mathbb{I}_{\{y_i=0\}}/n$, analogously an estimate for p^- and p^+ are the proportion of sample values less and greater than or equal to zero, i.e. $r^- = \sum_{i=1}^n \mathbb{I}_{\{y_i < 0\}}/n, r^+ = \sum_{i=1}^n \mathbb{I}_{\{y_i \geq 0\}}/n$ respectively. Hence the estimates of p and q are respectively given as

$$\tilde{p} = \frac{r^+ - r_0}{r^+ + r_0} \quad \text{and} \quad \tilde{q} = \left(\frac{r^+ - r_0}{r^+ + r_0} \right)^{r^+ / r^-} . \tag{20}$$

As it is well know that the r_0, r^- and r^+ are unbiased and consistent estimators of p_0, p^- and p^+ respectively, thus these can be used as an initial guess for searching global maxima of log-likelihood surface. It should be noted that this method fails to provide estimate of parameters if the sample contains no zero's or have all negative observations. Hence, in such situation, we recommend to use method of maximum likelihood estimation.

4.2. Maximum Likelihood estimation

The log-likelihood function of the discrete skew logistic model based on a iid sample Y_1, Y_2, \dots, Y_n is

$$l = n \ln 2 - n \ln(\ln pq) + s^- \ln(\ln p) + s^+ \ln(\ln q) + \sum_{i=1}^n \left(\ln \left(\frac{q^{-(y_i+1)}}{1 + q^{-(y_i+1)}} - \frac{q^{-y_i}}{1 + q^{-y_i}} \right) \cdot \mathbb{I}_{\{y_i < 0\}} \right) + \sum_{i=1}^n \left(\ln \left(\frac{p^{y_i}}{1 + p^{y_i}} - \frac{p^{y_i+1}}{1 + p^{y_i+1}} \right) \cdot \mathbb{I}_{\{y_i \geq 0\}} \right)$$

where $s^- = \sum_{i=1}^n \mathbb{I}_{\{y_i < 0\}}$ and $s^+ = \sum_{i=1}^n \mathbb{I}_{\{y_i \geq 0\}}$ express the number of negative and non-negative values in the sample, respectively.

Further differentiating the log-likelihood function partially w.r.t p and q , we get

$$\frac{\partial l}{\partial q} = -\frac{n}{q \ln(pq)} + \frac{s^+}{q \ln q} + \sum_{i=1}^n \left(\frac{y_i + y_i q^{2y_i+2} - 2q^{y_i+1} - (y_i + 1)q^{2y_i+1} - q(y_i + 1)}{(1 - q)q (q^{y_i} + 1) (q^{y_i+1} + 1)} \cdot \mathbb{I}_{\{y_i < 0\}} \right) \tag{21}$$

$$\frac{\partial l}{\partial p} = -\frac{n}{p \ln(pq)} + \frac{s^-}{p \ln p} + \sum_{i=1}^n \left(\frac{y_i + y_i p^{2y_i+2} - 2p^{y_i+1} - (y_i + 1)p^{2y_i+1} - p(y_i + 1)}{(1 - p)p (p^{y_i} + 1) (p^{y_i+1} + 1)} \cdot \mathbb{I}_{\{y_i \geq 0\}} \right). \tag{22}$$

The solution to above equations provides the maximum likelihood estimates(MLEs) of p and q . It is quite clear that no close analytical expression can be derived for the MLEs, and they have to be computed with some numerical procedure. We use the maxlik() function available in the maxLik package in R environment

to carry out this task.

Further differentiating (12) and (13), we have

$$\begin{aligned} \frac{\partial^2 l}{\partial q^2} &= n \left(\frac{1}{q^2 \ln^2(pq)} + \frac{1}{q^2 \ln(pq)} \right) - s^+ \left(\frac{1}{q^2 \ln(q)^2} + \frac{1}{(q^2 \ln(q))} \right) \\ &\quad + \sum_{i=1}^n \left(\left(\frac{y_i}{q^2} + \frac{1-2q}{q^2(1-q)^2} + \frac{(y_i+1)^2}{q^2(q^{y_i+1}+1)^2} - \frac{(y_i+1)(y_i+2)}{q^2(q^{y_i+1}+1)} + \frac{y_i^2}{q^2(q^{y_i}+1)^2} - \frac{(y_i+1)y_i}{q y_i^2 (q^{y_i}+1)} \right) \mathbb{I}_{\{y_i < 0\}} \right), \\ \frac{\partial^2 l}{\partial p^2} &= n \left(\frac{1}{p^2 \ln^2(pq)} + \frac{1}{p^2 \ln(pq)} \right) - s^- \left(\frac{1}{p^2 \ln(p)^2} + \frac{1}{p^2 \ln(p)} \right) \\ &\quad + \sum_{i=1}^n \left(\left(\frac{y_i}{p^2} + \frac{1-2p}{p^2(p-1)^2} + \frac{(y_i+1)^2}{p^2(p^{y_i+1}+1)^2} - \frac{(y_i+1)(y_i+2)}{p^2(p^{y_i+1}+1)} + \frac{y_i^2}{p^2(p^{y_i}+1)^2} - \frac{(y_i+1)y_i}{p^2(p^{y_i}+1)} \right) \mathbb{I}_{\{y_i \geq 0\}} \right), \\ \frac{\partial^2 l}{\partial p \partial q} &= \frac{n}{pq(\log(pq))^2}. \end{aligned}$$

Remembering that $\mathbb{E}(\mathbb{I}_{y \geq 0}) = P[Y \geq 0] = \frac{\ln q}{\ln(pq)}$ and $\mathbb{E}(\mathbb{I}_{y \leq 0}) = P[Y < 0] = \frac{\ln p}{\ln(pq)}$, we can compute the elements of information matrix $I(p, q)$. If the true values of p and q are not available, one can plug in \hat{p}_{ML} and \hat{q}_{ML} for p and q respectively and the naïve large-sample confidence intervals at the nominal level $(1 - \alpha)$ can be separately provided for q and p respectively as $\hat{q}_{ML} \pm z_{(1-\frac{\alpha}{2})} \sqrt{\hat{I}_{(n)11}^{-1}}$, and $\hat{p}_{ML} \pm z_{(1-\frac{\alpha}{2})} \sqrt{\hat{I}_{(n)22}^{-1}}$, where z is quantile of standard normal variate.

5. Simulation study

In this section we employ Monte Carlo simulations with 1000 repetition to assess the performance of the estimation methods. Simulation study consisting of the following steps is carried out for different (p, q) , namely for each combination of $p = 0.25, 0.50, 0.75$ and $q = 0.25, 0.50, 0.75$ and the values of sample size are $n = 25, 50, 75$ and 100 . We obtained the point estimate using method of Maximum Likelihood (ML) and Method of Proportion(MP) as discussed previously. The sample for each simulation run is generated using the procedure discussed in section 3.3. The estimation methods have been compared through bias, mean square error (MSE) and the average width (aw) of the obtained estimates over all 1000 samples. For these, the bias and mean square error (MSE) and the proportion of 95% confidence interval covering the true value of the parameter known as Coverage probability (CP), defined respectively as

$$bias(\hat{\Theta}) = \frac{1}{n} \sum_{i=1}^n (\hat{\Theta}_i - \Theta_0), \quad mse(\hat{\Theta}) = \frac{1}{n} \sum_{i=1}^n (\hat{\Theta}_i - \Theta_0)^2$$

and

$$CP(\hat{\Theta}) = \frac{1}{n} \sum_{i=1}^n I(\hat{\Theta}_i - 1.96SE < \Theta_0 < \hat{\Theta}_i + 1.96SE),$$

where SE represent the standard error of Θ . The simulation results are shown in Table 1 and 2. We can observe that the value of average bias obtained from ML estimate of (p, q) is negative for each n , whereas the bias of MP estimators for p and q are positive. But for both parameters, for all cases, average bias goes to zero as sample size n increases. Further, we can observe that the mean square error of ML and MP estimators decreases as sample size n increases. Moreover, the average width of ML estimator decreases with increase in the sample size n . The simulation study also gives an evidence that the CP of ML estimator of p increases up to 0.95 more slowly than the CP of ML estimator of q .

Table 1: Simulation study for bias, mean square error(MSE),average width(aw), and Converge Probability (CP).

Parameters	n	Method of Maximum Likelihood						Method of Proportion					
		bias(p)	bias(q)	mse(p)	mse(q)	aw(p)	aw(q)	CP(p)	CP(q)	bias(p)	bias(q)	mse(p)	mse(q)
$p=0.25, q=0.25$	25	-0.0077	-0.0140	0.0057	0.0056	0.2836	0.2835	0.9029	0.9089	0.0189	0.0202	0.0139	0.0296
	50	-0.0063	-0.0085	0.0029	0.0027	0.2050	0.2052	0.9320	0.9430	0.0046	0.0082	0.0061	0.0147
	75	-0.0032	-0.0046	0.0018	0.0018	0.1690	0.1691	0.9500	0.9440	0.0041	0.0059	0.0039	0.0099
	100	-0.0035	-0.0031	0.0014	0.0014	0.1467	0.1468	0.9440	0.9470	0.0040	0.0085	0.0032	0.0082
$p=0.25, q=0.50$	25	-0.0169	-0.0148	0.0081	0.0051	0.3222	0.2612	0.8980	0.9370	0.0136	0.0182	0.0229	0.0409
	50	-0.0106	-0.0074	0.0040	0.0022	0.2370	0.1849	0.9200	0.9500	0.0083	0.0022	0.0092	0.0181
	75	-0.0062	-0.0049	0.0026	0.0015	0.1959	0.1510	0.9330	0.9550	0.0017	0.0072	0.0063	0.0135
	100	-0.0057	-0.0022	0.0019	0.0012	0.1707	0.1305	0.9470	0.9400	0.0020	0.0011	0.0044	0.0094
$p=0.25, q=0.75$	25	-0.0266	-0.0072	0.0135	0.0017	0.4028	0.1526	0.8610	0.9496	0.0412	0.1232	0.0615	0.1015
	50	-0.0197	-0.0034	0.0076	0.0008	0.3055	0.1070	0.9010	0.9480	0.0224	0.0321	0.0250	0.0287
	75	-0.0115	-0.0031	0.0048	0.0005	0.2556	0.0874	0.9160	0.9450	0.0088	0.0211	0.0134	0.0117
	100	-0.0088	-0.0027	0.0038	0.0004	0.2227	0.0757	0.9280	0.9480	0.0091	0.0052	0.0100	0.0072
$p=0.50, q=0.25$	25	-0.0125	-0.0198	0.0050	0.0079	0.2593	0.3239	0.9300	0.9000	0.0543	0.0543	0.0723	0.0225
	50	-0.0051	-0.0106	0.0022	0.0039	0.1839	0.2375	0.9470	0.9340	0.0417	0.0417	0.0484	0.0105
	75	-0.0011	-0.0037	0.0015	0.0027	0.1500	0.1965	0.9470	0.9300	0.0426	0.0426	0.0480	0.0075
	100	-0.0007	-0.0052	0.0010	0.0018	0.1299	0.1709	0.9570	0.9460	0.0443	0.0443	0.0506	0.0064
$p=0.50, q=0.50$	25	-0.0066	-0.0095	0.0061	0.0061	0.2934	0.2910	0.9310	0.9280	0.0132	0.0131	0.0091	0.0263
	50	-0.0031	-0.0027	0.0027	0.0029	0.2053	0.2046	0.9470	0.9420	0.0091	0.0091	0.0061	0.0122
	75	-0.0008	-0.0024	0.0019	0.0019	0.1675	0.1669	0.9470	0.9470	0.0022	0.0022	0.0024	0.0071
	100	-0.0013	-0.0011	0.0013	0.0013	0.1447	0.1448	0.9550	0.9580	0.0010	0.0010	0.0014	0.0056
$p=0.50, q=0.75$	25	-0.0303	-0.0112	0.0120	0.0020	0.3582	0.1631	0.9240	0.9520	0.03289	0.0329	0.0072	0.0508
	50	-0.0178	-0.0050	0.0054	0.0009	0.2552	0.1132	0.9340	0.9520	0.0075	0.0075	0.0034	0.0206
	75	-0.0071	-0.0042	0.0028	0.0006	0.2062	0.0925	0.9560	0.9520	0.0049	0.0049	0.0054	0.0127
	100	-0.0068	-0.0018	0.0021	0.0004	0.1791	0.0794	0.9610	0.9370	0.0029	0.0030	0.0040	0.0104

Table 2: Simulation study for bias, mean square error(MSE),average width(aw), and Converge Probability (CP).

Parameters	n	Method of Maximum Likelihood						Method of Proportion					
		bias(p)	bias(q)	mse(p)	mse(q)	aw(p)	aw(q)	CP(p)	CP(q)	bias(p)	bias(q)	mse(p)	mse(q)
$p=0.75, q=0.25$	25	-0.0103	-0.0295	0.0018	0.0135	0.1544	0.4008	0.9500	0.8500	0.0311	0.1050	0.0135	0.0815
	50	-0.0057	-0.0174	0.0008	0.0073	0.1079	0.3050	0.9460	0.9010	0.0286	0.0689	0.0071	0.0395
	75	-0.0036	-0.0087	0.0005	0.0051	0.0876	0.2545	0.9560	0.9170	0.0284	0.0636	0.0050	0.0291
	100	-0.0016	-0.0090	0.0004	0.0036	0.0753	0.2237	0.9350	0.9240	0.0332	0.0704	0.0043	0.0251
$p=0.75, q=0.50$	25	-0.0118	-0.0273	0.0020	0.0104	0.1633	0.3588	0.9550	0.9320	0.0149	0.0336	0.0162	0.0614
	50	-0.0038	-0.0158	0.0009	0.0050	0.1128	0.2547	0.9510	0.9370	0.0145	0.0246	0.0087	0.0330
	75	-0.0032	-0.0105	0.0005	0.0031	0.0921	0.2071	0.9460	0.9410	0.0062	0.0129	0.0053	0.0212
	100	-0.0018	-0.0070	0.0004	0.0022	0.0794	0.1790	0.9510	0.9470	0.0124	0.0226	0.0043	0.0167
$p=0.75, q=0.75$	25	-0.0068	-0.0040	0.0025	0.0025	0.1845	0.1815	0.9360	0.9300	0.0141	0.0043	0.0229	0.0295
	50	-0.0022	-0.0029	0.0012	0.0011	0.1276	0.1282	0.9510	0.9520	0.0102	0.0039	0.0111	0.0146
	75	-0.0018	-0.0017	0.0007	0.0007	0.1042	0.1037	0.9370	0.9480	0.0014	0.0013	0.0074	0.0097
	100	-0.0031	-0.0011	0.0006	0.0005	0.0904	0.0898	0.9460	0.9520	0.0049	0.0009	0.0059	0.0078

6. Applications

Here the applicability of the proposed DSL distribution is discussed for two real datasets. The description and the source of both the dataset are as follows:

- i. The frequency data in Table 3 is taken from Chesneau and Kachour [5] is the difference (y) between post perception of the teaching reputation measured (based on a seven-point Likert-scale) one year after the entrance test) and the prior expectation measured at the entrance test among the candidates who have passed the entrance test to IDRAC Business School.

Table 3: Dataset 1

y	-2	-1	0	1	2	3
frequency	10	20	115	60	24	11

- ii. The data set in Table 4 represents the differences in assessment of number of palpable lymph nodes among sexual contacts of AIDS or an AIDS-related condition (ARC) patients by two physicians on 32 randomly selected participants, see Rosner [12].

Table 4: Dataset 2

y	-4	-3	-2	-1	0	1	2	3	4	5	6	7
frequency	0	1	2	1	3	3	1	10	4	3	2	2

Table 5: Descriptive Statistics of both the data sets.

	Min.	Q_1	Median	Mean	Q_3	Max.	st. dev.	Skewness	Kurtosis
Data set 1	-2	0	0	0.421	1	3	1.067	0.299	3.4626
Data set 2	-3	1	3	2.625	4	7	2.562	-0.381	2.5921

Table 5 gives the statistical description of both the data sets and it is clear that coefficient of skewness for data set 1 is positive and for data set 2 is negative. Thus, for comparative performance appraisal, we consider four other discrete distributions over the integer support \mathbb{Z} suitable for symmetric and asymmetric count data, namely, a new discrete Logistic (DLoG(p, μ)), discrete skew Laplace (DSLap(p, q)), discrete Normal (DNr(μ, σ)) and Skellam(p, q) distribution.

- (i) DLoG(p, μ)(Chakraborty and Chakravarty [4])

$$P(Y = y) = \frac{(1 - p)p^{y-\mu}}{(1 + p^{y-\mu})(1 + p^{y-\mu+1})}$$

where $y \in \mathbb{Z}, -\infty < \mu < \infty, 0 < p < 1$.

- (ii) DSLap(p, q)(Barbiero [2])

$$P(Y = y) = \frac{1}{\ln(pq)} \begin{cases} \ln(p)(q^{-(y+1)}(1 - q)) & \text{if } y < 0 \\ \ln(q)(p^y(1 - p)) & \text{if } y \geq 0 \end{cases}$$

where $0 < p < 1, 0 < q < 1$.

- (iii) DNr(μ, σ) (Roy [11])

$$P(Y = y) = \Phi\left(\frac{y + 1 - \mu}{\sigma}\right) - \Phi\left(\frac{y - \mu}{\sigma}\right)$$

$y = 0, \pm 1, \pm 2, \dots, \infty; -\infty < \mu < \infty, \sigma > 0$, and $\Phi(y)$ is the cdf of Standard Normal distribution.

(iv) Skellam(p, q) (Skellam [14])

$$P(Y = y) = e^{-(p+q)} \left(\frac{p}{q}\right)^{y/2} I_{|y|}(2\sqrt{pq}).$$

where $y = 0, \pm 1, \pm 2, \dots, p, q > 0$, and $I_r(x)$ is the modified Bessel function of order r (see Abramowitz and Stegun [1], p. 375).

We use maximum likelihood estimates of the parameters, that were obtained numerically by searching for global maxima of log-likelihood surface using method of proportion estimates as initial values. The findings of distribution fit for both datasets are presented in Table 6-7. For model assessment and comparison, we consider three criteria namely: the log-likelihood (LL), Akaike Information criteria(AIC) defined as $-2 \cdot LL + 2k$, and the Hannan–Quinn information-criterion (HQIC) defined as $-2 \cdot LL + 2(k + 1) \log(\log(n))$, where k is the number of estimated parameters and n refers to the sample size. It can be further observed that the \mathcal{DSL} distribution has maximum LL and minimum AIC and HQIC values for data set 1 and while for data set 2 the proposed distribution gives equally good fit when compared with Skellam distribution. With these illustrations, we conclude that \mathcal{DSL} distribution can be considered as a suitable alternative for asymmetric count model having integer \mathbb{Z} as support.

Table 6: Data fitting for dataset 1

parameter(mse)	$\mathcal{DSL}(\mu, p, q)$	DLoG(μ, p)	DSLap(p, q)	DNr(μ, σ)	Skellam(p, q)
$\hat{\mu}$	0.731 (0.086)	0.882(0.066)		0.9212(0.068)	-
\hat{p}	0.215 (0.023)	0.168(0.017)	0.446(0.026)	-	0.742(0.068)
\hat{q}	0.088 (0.019)	-	0.059(0.019)	-	0.321(0.054)
$\hat{\sigma}$	-	-		1.026(0.050)	-
LL	-343.281	-353.156	-365.861	-355.818	-347.341
AIC	692.562	710.312	735.722	715.636	698.681
HQIC	700.172	716.519	741.929	721.843	704.887

Table 7: Data fitting for dataset 2

parameter(mse)	$\mathcal{DSL}(\mu, p, q)$	DLoG(μ, p)	DSLap(p, q)	DNr(μ, σ)	Skellam(p, q)
$\hat{\mu}$	3.045(0.549)	3.258(0.446)		3.125(0.444)	-
\hat{p}	0.531(0.062)	0.498(0.052)	0.782(0.036)	-	4.757(0.893)
\hat{q}	0.440(0.079)	-	0.345(0.118)	-	2.132(0.842)
$\hat{\sigma}$	-	-		2.505(0.317)	-
LL	-74.963	-75.458	-83.735	-75.004	-75.432
AIC	155.925	154.915	171.471	154.009	154.864
HQIC	150.457	151.270	167.826	150.365	151.220

7. Conclusions

A new discrete distribution defined on \mathbb{Z} suitable for symmetric/asymmetric count data is proposed here. Some of its important probabilistic properties and its relation with other distribution(s) were also discussed. Monte Carlo simulations were carried out to illustrate the behaviour of the estimation methods. From the results of the data fitting examples considered here, the proposed discrete distribution is suitable and preferred than other count distributions over \mathbb{Z} .

Acknowledgment

The authors wish to thank the Associate Editor and anonymous referee(s) for their relevant and useful comments which resulted in this improved version.

References

- [1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1974.
- [2] A. Barbiero, An alternative discrete skew Laplace distribution, *Statistical Methodology* 16 (2014) 47–67.
- [3] S. Chakraborty, Generation of discrete analogues of continuous distributions- A Survey of Methods and Constructions, *Journal of Statistical Distributions and Applications* 2 (1) (2015) 6.
- [4] S. Chakraborty, D. Chakravarty, A new discrete probability distribution with integer support on $(-\infty, \infty)$, *Communication in Statistics-Theory and Methods* 45 (2) (2016) 492–505.
- [5] C. Chesneau, M. Kachour, A new parametric mixture class of distributions on \mathbb{Z} , *ProbStat Forum* 10 (2017) 1–15.
- [6] E. Gómez-Déniz, Another generalization of the geometric distribution, *Test* 19 (2) (2010) 399–415.
- [7] N.L. Johnson, S. Kotz, N. Balakrishnan, *Continuous univariate distributions*, Vol. 2, Wiley Series in Probability and Mathematical Statistics, Applied Probability and Statistics, 1995.
- [8] D. Karlis, I. Ntzoufras, Bayesian modeling of football outcomes: Using the Skellam distribution for the goal difference, *IMA Journal of Management Mathematics* 20 (2) (2008) 133–145.
- [9] T.J. Kozubowski, S. Inusah, A Skew Laplace Distribution on Integers, *Annals of the Institute of Statistical Mathematics* 58 (3) (2006) 555–571.
- [10] T. Popoviciu, Sur les équations algébriques ayant toutes leurs racines réelles”, *Mathematica (Cluj)* 9 (1935) 129–145.
- [11] D. Roy, The discrete normal distribution. *Communications in Statistics-Theory and Methods* 32 (10) (2003) 1871–1883.
- [12] B. Rosner, *Fundamentals of Biostatistics*, (Fifth edition) Duxbury, California 2000.
- [13] D.V.S. Sastry, D. Bhati, A new skew logistic distribution: Properties and applications, *Brazilian Journal of Probability and Statistics* 30 (2) (2016) 248–271.
- [14] J.G. Skellam, The frequency distribution of the difference between two Poisson variates belonging to different populations, *Journal of the Royal Statistical Society Series B* 109 (1946) 296–296.