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Coefficient Problems for Certain Subclass of m–Fold Symmetric BI-Univalent Functions by Using Faber Polynomial

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Abstract. In this paper, we apply the Faber polynomial expansions to find upper bounds for the general coefficients $|a_{mk+1}|(k \geqq 3)$ of functions in the subclass $\mathcal{N}_{\Sigma_m}(\tau, \gamma; \beta)$. The results presented in this paper would generalize and improve some recent works.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ denote the class of functions in $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. It is well known (see [6]) that every function $f \in \mathcal{S}$ has an inverse map f^{-1}, defined by $f^{-1}(f(z)) = z, (z \in \mathbb{D})$ and $f(f^{-1}(w)) = w$, $\left(|w| < r_0(f),\ r_0(f) \geqq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$, if both f and f^{-1} are univalent in $\mathbb{D}$. Let $\sigma_{\mathcal{B}}$ denote the class of bi-univalent functions in $\mathbb{D}$.

In 1967, Lewin [11] introduced this class $\sigma_{\mathcal{B}}$ and proved that the bound for the second coefficients of every $f \in \sigma_{\mathcal{B}}$ satisfies the inequality $|a_2| \leqq 1.52$. At same time, Netenyahu [12] showed that max $|a_2| = 4/3$ for $f \in \sigma_{\mathcal{B}}$.

Also, for bi-univalent polynomial $f(z) = z + a_2 z^2 + a_3 z^3$ with real coefficients, Smith [16] showed that $|a_2| \leqq 2/\sqrt{27}$ and $|a_3| \leqq 4/\sqrt{27}$. He also demonstrated fthat if $z + a_n z^n$ is bi-univalent, then $|a_n| \leqq (n-1)^{n-1}/n^n$ with equality best possible for $n = 2, 3$. Kedzierawski and Waniurski [9] proved the conjecture of Smith [16] for $n = 3, 4$ in the case of bi-univalent polynomial of degree n.

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In fact, this widely-cited work by Srivastava et al. [25] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al. [17–20, 22–25, 29, 32, 33] and others [5, 14, 35] .
However, finding upper bounds of the Taylor-Maclaurin coefficients $|a_n|$ $(n \geqq 3)$ for each $f \in \sigma_{\mathcal{B}}$ is coefficient estimate problem and still an open problem. To the best of our knowledge, there is no direct way to obtain bound for coefficients greater than three. For some special cases there are some papers in which the Faber polynomial methods were used for determining upper bounds for higher order coefficients (for example see [28, 34]).

Also the coefficients of $g = f^{-1}$ are given by the Faber polynomial [7] (see also [1, 2]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^n, \tag{3}$$

where

$$\begin{aligned} K_{n-1}^{-n} = {} & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ & + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \times [a_5 + (-n+2) a_3^2] + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] \\ & + \sum_{j \geqq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j with $7 \leqq j \leqq n$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$.
For each function $f \in \mathcal{S}$ function, the function

$$h(z) = \sqrt[m]{f(z^m)} \tag{4}$$

is univalent and maps the unit disk $\mathbb{D}$ into a region with m-fold symmetry. We recal, a function is said to be m-fold symmetric (see [10, 15]) if it has the following normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in \mathbb{D}, m \in \mathbb{N}). \tag{5}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{D}$.
The functions in the class $\mathcal{S}$ are said to be one-fold symmetric. The normalized form of f is given as in (7) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [27], is given as follows:

$$\begin{aligned} f^{-1}(w) &= w + \sum_{k=1}^{\infty} A_{mk+1} w^{mk+1} && (6) \\ &= w - a_{m+1} w^{m+1} + [(m+1) a_{m+1}^2 - a_{2m+1}] w^{2m+1} - [\frac{1}{2}(m+1)(3m+2) a_{m+1}^3 - (3m+2) a_{m+1} a_{2m+1} \\ &\quad + a_{3m+1}] w^{3m+1} + \cdots . && (7) \end{aligned}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{D}$. Thus, when $m = 1$, the formula (7) coincides with the Eq. (2).
Recently, researchers are interented to study the m-fold symmetric bi-univalent functions class Σ_m (see [13, 20, 21, 30, 31]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$.
In this respect, Atshan and Kazim [4] introduced subclasses $\mathcal{N}_{\Sigma_m}(\tau, \gamma; \beta)$ and $\mathcal{N}_{\Sigma_m}^{\beta}$ of m-fold symmetric bi-univalent function class Σ_m and obtained non sharp estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in the subclass $\mathcal{N}_{\Sigma_m}(\tau, \gamma; \beta)$.

Definition 1.1. *[4] A function $f \in \Sigma_m$ given by* (5) *is said to be in the subclass $\mathcal{N}_{\Sigma_m}(\tau,\gamma;\beta)$ $(\tau \in \mathbb{C}-\{0\}, 0 \leqq \gamma < 1, 0 \leqq \beta < 1)$, if the following conditions are satisfied:*

$$\Re\left(1+\frac{1}{\tau}\left[\frac{(1+\gamma)z^2f''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}-1\right]\right) > \beta \quad (z \in \mathbb{D})$$

and

$$\Re\left(1+\frac{1}{\tau}\left[\frac{(1+\gamma)w^2g''(w)+wg'(w)}{(1+\gamma)wg'(w)-\gamma g(w)}-1\right]\right) > \beta \quad (w \in \mathbb{D}),$$

where the function g is the inverse of f given by (6).

It is stated that in Theorem 3.1, the calculations done by Atshan and Kazim (see [4]) for the bound $|a_{2m+1}|$ are inaccurate. To remove this remarkable mistake, we've revised the calculations appropriately (see Theeorem 1.2).

Theorem 1.2. *[4] Let $f \in \mathcal{N}_{\Sigma_m}(\tau,\gamma;\beta)$ $(\tau \in \mathbb{C}-\{0\}, 0 \leqq \gamma < 1, 0 \leqq \beta < 1)$ be of the form* (5). *Then*

$$|a_{m+1}| \leqq \sqrt{\frac{2|\tau|(1-\beta)}{m\Big((m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)}}$$

and

$$|a_{2m+1}| \leqq \frac{2|\tau|^2(1-\beta)^2(m+1)}{m^2[m(1+\gamma)+1]^2}+\frac{|\tau|(1-\beta)}{m[2m(1+\gamma)+1]}.$$

Definition 1.3. *[4] A function $f \in \Sigma_m$ given by* (5) *is said to be in the subclass $\mathcal{N}^{\beta}_{\Sigma_m}$ $(0 \leqq \beta < 1)$, if the following conditions are satisfied:*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta \quad \text{and} \quad \Re\left(1+\frac{wg''(w)}{g'(w)}\right) > \beta \quad (z, w \in \mathbb{D}),$$

where the function g is the inverse of f given by (6).

In 2014, Hamidi and Jahangiri [8] introduced the following subclass of the m-fold symmetric bi-univalent functions Σ_m.

Definition 1.4. *[8] A function $f \in \Sigma_m$ given by* (5) *is said to be m-fold symmetric bi-starlike of order β $(0 \leqq \beta < 1)$, if the following conditions are satisfied:*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad \text{and} \quad \Re\left(\frac{wg'(w)}{g(w)}\right) > \beta \quad (z, w \in \mathbb{D}),$$

where the function g is the inverse of f given by (6).

In this paper, we generalize parameter γ of the subclass $\mathcal{N}_{\Sigma_m}(\tau,\gamma;\beta)$ to $-1 \leqq \gamma < 1$. We use the Faber polynomial expansion [7] to obtain not only improvement of estimates of coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ which obtained by Atshan and Kazim, but also we find estimates of coefficients $|a_{mk+1}| (k \geqq 3)$ of functions in the subclass $\mathcal{N}_{\Sigma_m}(\tau,\gamma;\beta)$ (see section 3).

2. Preliminaries

For finding the coefficients for functions belonging to the subclass $\mathcal{N}_{\Sigma_m}(\tau,\gamma;\beta)$ $(\tau\in\mathbb{C}-\{0\},-1\leqq\gamma<1,0\leqq\beta<1)$, we need the following lemmas.

Lemma 2.1. *[1, 2] Let $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n\in\mathcal{S}$. Then for any $\mu\in\mathbb{R}$, there are the polynomials K_n^{μ}, such that*

$$\left(\frac{f(z)}{z}\right)^{\mu}=1+\sum_{n=1}^{\infty}K_n^{\mu}(a_2,a_3\cdots,a_{n+1})z^n,$$

where

$$K_n^{\mu}(a_2,a_3\cdots,a_{n+1})=\mu a_{n+1}+\frac{\mu(\mu-1)}{2}D_n^2+\frac{\mu!}{(\mu-3)!3!}D_n^3+\cdots+\frac{\mu!}{(\mu-n)!n!}D_n^n,$$

where

$$D_n^k=D_n^k(a_2,a_3,\cdots,a_{n+1})=\sum\frac{k!(a_2)^{\nu_1}\cdots(a_{n+1})^{\nu_n}}{\nu_1!\cdots\nu_n!},$$

where the sum is taken over all non negative integers $\nu_1,\cdots,\nu_n$ satisfying

$$\begin{cases}\nu_1+\nu_2+\cdots+\nu_n=k,\\ \nu_1+2\nu_2+\cdots+n\nu_n=n.\end{cases}$$

Lemma 2.2. *[3] Let $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n\in\mathcal{S}$. Then*

$$\frac{zf'(z)}{f(z)}=1-\sum_{k=1}^{\infty}F_kz^k,$$

where $F_k=F_k(a_2,a_3,\cdots,a_{k+1})$ is a Faber polynomial of degree k,

$$F_k(a_2,a_3,\cdots,a_{k+1})=\sum_{i_1+2i_2+\cdots+ki_k=k}A_{(i_1,i_2,\cdots,i_k)}a_2^{i_1}a_3^{i_2}\cdots a_{k+1}^{i_k} \tag{8}$$

and

$$A_{(i_1,i_2,\cdots,i_k)}:=(-1)^{k+2i_1+3i_2+\cdots+(k+1)i_k}\,\frac{(i_1+i_2+\cdots+i_k-1)!k}{i_1!i_2!\cdots i_k!}. \tag{9}$$

The first Faber polynomials $F_k(a_2,a_3,\cdots,a_{k+1})$ are given by:

$$F_1(a_2)=-a_2,\quad F_2(a_2,a_3)=a_2^2-2a_3\quad\text{and}\quad F_3(a_2,a_3,a_4)=-a_2^3+3a_2a_3-3a_4.$$

Lemma 2.3. *Let $f(z)=z+\sum\limits_{k=1}^{\infty}a_{mk+1}z^{mk+1}\in\mathcal{S}_m$. Then we can write,*

$$\frac{zf'(z)}{f(z)}=1-\sum_{k=1}^{\infty}L_k(a_{m+1},\cdots,a_{mk+1})z^{mk},$$

where

$$\begin{aligned}L_k(a_{m+1},\cdots,a_{mk+1}) &= F_{mk}(\underbrace{0,\cdots,0,a_{m+1},0,\cdots,0,a_{mk+1}}_{mk}) \qquad (10)\\ &= \sum_{mi_m+2mi_{2m}+\cdots+mki_{mk}=mk}A_{(i_1,i_2,\ldots,i_{mk})}a_{m+1}^{i_m}a_{2m+1}^{i_{2m}}\cdots a_{mk+1}^{i_{mk}}.\end{aligned}$$

Proof. By using Lemma 2.2 for function $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$, we have

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k\geqq 1} F_k z^k.$$

Assume that $t, j \in \mathbb{N}$ and $1 \leqq j \leqq m-1$, and consider three cases for k.

(i) If $1 \leqq k \leqq m-1$, then $F_k(\underbrace{0, \cdots, 0}_{k}) = 0$.

(ii) If $k = tm$, then, we have

$$F_{tm}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}}_{tm}) = \sum_{mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm} A_{(i_1, i_2, \ldots, i_{tm})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}}.$$

(iii) If $k = tm + j$, then

$$F_{tm+j}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}, \underbrace{0, \cdots, 0}_{j}}_{tm+j}) =$$

$$\sum_{mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm+j} A_{(i_1, i_2, \ldots, i_{tm+j})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}}.$$

Since the equation

$$mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm + j,$$

does not have positive ineger solution, so

$$F_{tm+j}(0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}, 0, \cdots, 0) = 0.$$

□

Remark 2.4. *In the special case, if* $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, *then*

$$L_i(a_{m+1}, \cdots, a_{mi+1}) = 0\ ; \quad 1 \leqq i \leqq k-1$$

and

$$L_k(a_{m+1}, \cdots, a_{mk+1}) = -mka_{mk+1}.$$

Lemma 2.5. *Let* $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$. *Then*

$$\frac{(1+\gamma)z^2 f''(z) + zf'(z)}{(1+\gamma)zf'(z) - \gamma f(z)} = 1 - \sum_{k=1}^{\infty} L_k\Big([m(1+\gamma)+1]a_{m+1}, \cdots, [mk(1+\gamma)+1]a_{mk+1}\Big) z^{mk},$$

where $L_k\Big([m(1+\gamma)+1]a_{m+1}, \cdots, [mk(1+\gamma)+1]a_{mk+1}\Big)$ *given by* (10).

Proof. For function $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$, we have

$$\frac{(1+\gamma)z^2 f''(z) + zf'(z)}{(1+\gamma)zf'(z) - \gamma f(z)} = \frac{z\Big((1+\gamma)zf'(z) - \gamma f(z)\Big)'}{(1+\gamma)zf'(z) - \gamma f(z)}.$$

Making use of Lemma 2.2, we get

$$\begin{aligned}\frac{z\Big((1+\gamma)zf'(z) - \gamma f(z)\Big)'}{(1+\gamma)zf'(z) - \gamma f(z)} &= \frac{z\left(z + \sum_{k\geqq 1}[mk(1+\gamma)+1]a_{mk+1}z^{mk+1}\right)'}{z + \sum_{k\geqq 1}[mk(1+\gamma)+1]a_{mk+1}z^{mk+1}} \\ &= 1 - \sum_{k=1}^{\infty} L_k\Big([m(1+\gamma)+1]a_{m+1}, \cdots, [mk(1+\gamma)+1]a_{mk+1}\Big)z^{mk}.\end{aligned}$$

□

Remark 2.6. *If* $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, *then*

$$L_k\Big([m(1+\gamma)+1]a_{m+1}, \cdots, [mk(1+\gamma)+1]a_{mk+1}\Big) = 0\,; \quad 1 \leqq i \leqq k-1$$

and

$$L_k\Big([m(1+\gamma)+1]a_{m+1}, \cdots, [mk(1+\gamma)+1]a_{mk+1}\Big) = -mk[mk(1+\gamma)+1]a_{mk+1}.$$

Lemma 2.7. *[15] If $p \in \mathcal{P}$, then $|c_k| \leqq 2$ for each k, where $\mathcal{P}$ is the family of all analytic functions p in $\mathbb{D}$ for which $\Re\,(p(z)) > 0$ where $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$.*

3. Main results

In this section, we give the main results of our contribution.

Theorem 3.1. *Let $f \in \mathcal{N}_{\Sigma_m}(\tau, \gamma; \beta)$ $(\tau \in \mathbb{C} - \{0\}, -1 \leqq \gamma < 1,\ 0 \leqq \beta < 1)$ be of the form* (5). *If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2|\tau|(1-\beta)}{mk[mk(1+\gamma)+1]}\,; \quad (k \geqq 3).$$

Proof. Applying Lemma 2.5, for m-fold symmetric bi-univalent functions f of the form (5), we have:

$$1 + \frac{1}{\tau}\left[\frac{(1+\gamma)z^2 f''(z) + zf'(z)}{(1+\gamma)zf'(z) - \gamma f(z)} - 1\right] = 1 - \sum_{k=1}^{\infty} \frac{L_k(a_{m+1}, \cdots, a_{mk+1})}{\tau} z^{mk}. \tag{11}$$

Similarly, for its inverse map, $g(w) = f^{-1}(w) = w + \sum_{k=1}^{\infty} A_{mk+1} w^{mk+1}$, it yields:

$$1 + \frac{1}{\tau}\left[\frac{(1+\gamma)^2 g''(w) + zg'(w)}{(1+\gamma)wg'(w) - \gamma g(w)} - 1\right] = 1 - \sum_{k=1}^{\infty} \frac{L_k(A_{m+1}, \cdots, A_{mk+1})}{\tau} w^{mk}. \tag{12}$$

Since $f \in \mathcal{N}_{\Sigma_m}(\tau, \gamma; \beta)$, by definition, there exist two positive real-part functions $p(z) = 1 + \sum_{k=1}^{\infty} p_{mk} z^{mk}$ and $q(w) = 1 + \sum_{k=1}^{\infty} q_{mk} w^{mk}$, where $\Re\, p(z) > 0$ and $\Re\, q(w) > 0$ in $\mathbb{D}$ so that:

$$1 + \frac{1}{\tau}\left[\frac{(1+\gamma)z^2 f''(z) + zf'(z)}{(1+\gamma)zf'(z) - \gamma f(z)} - 1\right] = 1 + (1-\beta)\sum_{k=1}^{\infty} K_k^1(p_m, \cdots, p_{mk}) z^{mk} \tag{13}$$

and

$$1+\frac{1}{\tau}\left[\frac{(1+\gamma)w^2g''(w)+wg'(w)}{(1+\gamma)wg'(w)-\gamma g(w)}-1\right]=1+(1-\beta)\sum_{k=1}^{\infty}K_k^1(q_m,\cdots,q_{mk})w^{mk}. \tag{14}$$

Equating the corresponding coefficients of (11) and (13), we have:

$$-\frac{L_k(a_{m+1},\cdots,a_{mk+1})}{\tau}=(1-\beta)K_k^1(p_m,\cdots,p_{mk}). \tag{15}$$

Similarly, from (12) and (14), we obtain

$$-\frac{L_k(A_{m+1},\cdots,A_{mk+1})}{\tau}=(1-\beta)K_k^1(q_m,\cdots,q_{mk}). \tag{16}$$

Note that for $a_{mi+1}=0$ $(1\leqq i\leqq k-1)$, we have $A_{mi+1}=0$ $(1\leqq i\leqq k-1)$ and $A_{mk+1}=-a_{mk+1}$.
By using Remark 2.6 the equalities (15), (16) can be rewritten as follows:

$$\frac{mk[mk(1+\gamma)+1]}{\tau}a_{mk+1}=(1-\beta)p_{mk},$$

$$\frac{mk[mk(1+\gamma)+1]}{\tau}A_{mk+1}=-\frac{mk[mk(1+\gamma)+1]}{\tau}a_{mk+1}=(1-\beta)q_{mk}.$$

Using the absolute values of either of the above two equations and Lemma 2.7, we get:

$$|a_{mk+1}|=\frac{|\tau|(1-\beta)|p_{mk}|}{mk[mk(1+\gamma)+1]}=\frac{|\tau|(1-\beta)|q_{mk}|}{mk[mk(1+\gamma)+1]}\leqq\frac{2|\tau|(1-\beta)}{mk[mk(1+\gamma)+1]}.$$

□

Now, we give estimates for the initial coefficients of functions $f\in\mathcal{N}_{\Sigma_m}(\tau,\gamma;\beta)$.

Theorem 3.2. *Let* $f\in\mathcal{N}_{\Sigma_m}(\tau,\gamma;\beta)$ $(\tau\in\mathbb{C}-\{0\},-1\leqq\gamma<1,\ 0\leqq\beta<1)$ *be of the form* (5). *Then*

$$|a_{m+1}|\leqq\min\left\{\frac{2|\tau|(1-\beta)}{m[m(1+\gamma)+1]},\sqrt{\frac{2|\tau|(1-\beta)}{m\Big((m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)}}\right\}$$

and

$$|a_{2m+1}|\leqq\min\left\{\frac{2|\tau|^2(m+1)(1-\beta)^2}{m^2[m(1+\gamma)+1]^2}+\frac{|\tau|(1-\beta)}{m[2m(1+\gamma)+1]},\right.$$
$$\left.\frac{|\tau|(m+1)(1-\beta)}{m\Big((m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)}\right\}.$$

Proof. Comparing equations (15) and (16) for $k=1,2$, we have

$$\frac{m[m(1+\gamma)+1]}{\tau}a_{m+1}=(1-\beta)p_m, \tag{17}$$

$$\frac{2m[2m(1+\gamma)+1]}{\tau}a_{2m+1}-\frac{m[m(1+\gamma)+1]^2}{\tau}a_{m+1}^2=(1-\beta)p_{2m}, \tag{18}$$

$$-\frac{m[m(1+\gamma)+1]}{\tau}a_{m+1}=(1-\beta)q_m \tag{19}$$

and

$$\frac{2m[2m(1+\gamma)+1][(m+1)a_{m+1}^2-a_{2m+1}]}{\tau}-\frac{m[m(1+\gamma)+1]^2}{\tau}a_{m+1}^2=(1-\beta)q_{2m}. \tag{20}$$

From (17) and (19), we get

$$p_m=-q_m \tag{21}$$

and

$$a_{m+1}^2=\frac{\tau^2(1-\beta)^2(p_m^2+q_m^2)}{2m^2[m(1+\gamma)+1]^2}. \tag{22}$$

Adding (18) and (20), it yields

$$a_{m+1}^2=\frac{\tau(1-\beta)(p_{2m}+q_{2m})}{2m\Big((m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)}. \tag{23}$$

Therefore, from equations (22), (23) and Lemma 2.7, we have

$$|a_{m+1}|\leqq\frac{2|\tau|(1-\beta)}{m[m(1+\gamma)+1]}$$

and

$$|a_{m+1}|\leqq\sqrt{\frac{2|\tau|(1-\beta)}{m\Big((m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)}},$$

respectively. So we get the desired estimate on the coefficient $|a_{m+1}|$.

Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (20) from (18). We thus get

$$a_{2m+1}=\frac{(m+1)}{2}a_{m+1}^2+\frac{\tau(1-\beta)(p_{2m}-q_{2m})}{4m[2m(1+\gamma)+1]}. \tag{24}$$

Substituting the value of a_{m+1}^2 from (22) into (24), it follows that

$$a_{2m+1}=\frac{\tau^2(1-\beta)^2(m+1)(p_m^2+q_m^2)}{4m^2[m(1+\gamma)+1]^2}+\frac{\tau(1-\beta)(p_{2m}-q_{2m})}{4m[2m(1+\gamma)+1]}. \tag{25}$$

On the other hand, substituting the value of a_{m+1}^2 from (23) into (24), it follows that

$$a_{2m+1}=\frac{\tau\Big[\Big(2(m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)p_{2m}+[m(1+\gamma)+1]^2q_{2m}\Big](1-\beta)}{4m\Big((m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)[2m(1+\gamma)+1]}. \tag{26}$$

By applying Lemma 2.7 for equations (25) and (26), we immediately have

$$|a_{2m+1}|\leqq\frac{2|\tau|^2(1-\beta)^2(m+1)}{m^2[m(1+\gamma)+1]^2}+\frac{|\tau|(1-\beta)}{m[2m(1+\gamma)+1]}$$

and

$$|a_{2m+1}|\leqq\frac{|\tau|(m+1)(1-\beta)}{m\Big((m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)}.$$

This evidently completes the proof of Theorem 3.2. □

4. Corollaries and Consequences

By setting $\tau = 1$ and $\gamma = -1$ in Theorem 3.1, we conclude the following result.

Corollary 4.1. *Let f given by* (5) *be m-fold symmetric bi-starlike of order β. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{mk}\ ; \quad (k \geqq 3).$$

By taking $m = 1$ in Corollary 4.1, we have the following result.

Corollary 4.2. *Let f given by* (1) *be bi-starlike of order β. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{k}\ ; \quad (k \geqq 3).$$

Setting $\tau = 1$ and $\gamma = 0$ in Theorem 3.1, we conclude the following result.

Corollary 4.3. *Let f given by* (5) *be in the subclass $\mathcal{N}^{\beta}_{\Sigma_m}$. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{mk(mk+1)}\ ; \quad (k \geqq 3).$$

Taking $m = 1$ in Corollary 4.3, gives the following result.

Corollary 4.4. *Let f given by* (1) *be bi-convex of order β. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{k(k+1)}\ ; \quad (k \geqq 3).$$

Remark 4.5. *The bound on $|a_{m+1}|$ given in Theorem 3.2 is better than that of Theorem 1.2. Because*

$$\frac{2|\tau|(1-\beta)}{m[m(1+\gamma)+1]} \leqq \sqrt{\frac{2|\tau|(1-\beta)}{m\Big((m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)}};$$

$$\beta \geqq 1 - \frac{m[m(1+\gamma)+1]^2}{2|\tau|\Big((m+1)[2m(1+\gamma)+1]-[m(1+\gamma)+1]^2\Big)}.$$

Setting $\tau = 1$ and $\gamma = -1$ in Theorem 3.2, we conclude the following result.

Corollary 4.6. *Let f given by* (5) *be m-fold symmetric bi-starlike of order β. Then*

$$|a_{m+1}| \leqq \begin{cases} \frac{\sqrt{2(1-\beta)}}{m}; & 0 \leqq \beta \leqq \frac{1}{2} \\ \frac{2(1-\beta)}{m}\ ; & \frac{1}{2} \leqq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leqq \begin{cases} \frac{(m+1)(1-\beta)}{m^2}\ ; & 0 \leqq \beta \leqq \frac{1+2m}{2(1+m)} \\ \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}; & \frac{1+2m}{2(1+m)} \leqq \beta < 1. \end{cases}$$

By setting $m = 1$ in Corollary 4.6, we conclude the following result.

Corollary 4.7. *Let f given by* (1) *be bi-starlike of order β. Then*

$$|a_2| \leqq \begin{cases} \sqrt{2(1-\beta)}; & 0 \leqq \beta \leqq \frac{1}{2} \\ 2(1-\beta) \ ; & \frac{1}{2} \leqq \beta < 1 \end{cases}$$

and

$$|a_3| \leqq \begin{cases} 2(1-\beta) & ; \quad 0 \leqq \beta \leqq \frac{3}{4} \\ (1-\beta)(5-4\beta) \ ; & \frac{3}{4} \leqq \beta < 1. \end{cases}$$

By setting $\tau = 1$ and $\gamma = 0$ in Theorem 3.2, we conclude the following result.

Corollary 4.8. *Let f given by* (5) *be in the subclass $\mathcal{N}_{\Sigma_m}^{\beta}$. Then*

$$|a_{m+1}| \leqq \begin{cases} \frac{2(1-\beta)}{m(m+1)} & ; \quad 0 \leqq \beta \leqq 1 - \frac{m^2}{2(m+1)} \\ \sqrt{\frac{2(1-\beta)}{m^2(m+1)}} & ; \quad 1 - \frac{m^2}{2(m+1)} \leqq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leqq \begin{cases} \frac{1-\beta}{m^2} & ; \quad 0 \leqq \beta \leqq \frac{2m-m^2+1}{2(2m+1)} \\ \frac{2(1-\beta)^2}{m^2(m+1)} + \frac{1-\beta}{m(2m+1)} \ ; & \frac{2m-m^2+1}{2(2m+1)} \leqq \beta < 1. \end{cases}$$

By setting $m = 1$ in Corollary 4.8, we conclude the following result.

Corollary 4.9. *Let f given by* (1) *be bi-convex of order β. Then*

$$|a_2| \leqq \begin{cases} 1-\beta & ; \quad 0 \leqq \beta \leqq \frac{3}{4} \\ \sqrt{1-\beta}\,; & \frac{3}{4} \leqq \beta < 1 \end{cases}$$

and

$$|a_3| \leqq \begin{cases} 1-\beta & ; \quad 0 \leqq \beta \leqq \frac{1}{3} \\ \frac{(1-\beta)(4-3\beta)}{3} & ; \quad \frac{1}{3} \leqq \beta < 1. \end{cases}$$

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