Variational Inequalities, Variational Inclusions and Common Fixed Point Problems in Banach Spaces

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Abstract. In this paper, let $X$ be a uniformly convex and $q$-uniformly smooth Banach space with $1 < q \leq 2$. We introduce and study modified implicit extragradient iterations for treating a common solution of a common fixed-point problem of a countable family of nonexpansive mappings, a general system of variational inequalities, and a variational inclusion in $X$.

1. Introduction

Multivalued monotone inclusion is an important optimization problem, which can be viewed as a real mathematical modelling for many engineering design, such as, transportation, single processing and image reconstruction; see, e.g., [1–3, 5, 14, 24, 27, 35]. There are a huge number of approximation methods for solving multivalued monotone inclusion problems; see, e.g., [6, 8, 16, 21, 25, 26]. Most efficient one is the resolvent method, which transfers the inclusion problem into a fixed point problem via an inverse problem; see, e.g., [12, 13, 26, 30]. Let $X$ be a real Banach space with the dual space $X^\ast$. Both the norms of $X$ and dual space $X^\ast$ are presented by $\| \cdot \|$. Let $C$ be a convex and closed set in $X$. Let $T : C \to C$ be a nonlinear single-valued mapping. From now on, one employs $\text{Fix}(T)$ to represent the set of fixed points of $T$. Recall that $T$ is said to be strictly contractive iff $\|Tx - Ty\| \leq \delta \|x - y\|, \forall x, y \in C$, where constant $\delta \in (0, 1)$. $T$ is said to be nonexpansive iff $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$. The theory of nonexpansive mappings, whose complementary mappings are monotone, is interesting and important in operator theory. Some efficient approximation methods were studied for fixed points of nonexpansive mappings and their extensions; see [15, 19, 22, 23, 32] and the references therein. Recall that $J(x) := \{ \phi \in X^\ast : \langle x, \phi \rangle = \|x\|^2 = \|\phi\|^2 \} \forall x \in X$, where $\langle \cdot, \cdot \rangle$ represents the generalized duality pairing between $X$ and $X^\ast$. Recall a Banach space is said to be smooth (it has Gâteaux differentiable norm) if $\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$ exists for all $\|x\| = \|y\| = 1$. $J$ is norm-to-weak$^\ast$ continuous in such a space. Moreover, the norm of $X$ is said to be Fréchet differentiable, if for each $\|x\| = 1$, the limit is attained uniformly for $\|y\| = 1$. The norm of $X$ is said to be uniformly Fréchet differentiable, if for each $\|x\| = 1$ and $\|y\| = 1$, the limit is attained uniformly for $\|t\| = 1$. The dual norm of $X$ is said to be uniformly Fréchet differentiable, if for each $\|x\| = 1$ and $\|y\| = 1$, the limit is attained uniformly for $\|t\| = 1$.

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differentiable, if the limit is attained uniformly. \( J \) is norm-to-norm uniformly continuous on bounded sets in such a space. A space \( X \) is said to be uniformly convex if for each \( \varepsilon \in (0, 2] \), there exists \( \delta > 0 \) such that for any \( \|x - y\| = \|y - z\| = 1, \|\frac{x + y}{2}\| > 1 - \delta \Rightarrow \|x - y\| < \varepsilon. \) It is known that a uniformly convex Banach space is reflexive. Let \( A_1, A_2 : C 
rightarrow X \) be two nonlinear mappings. In the framework that \( J \) is single-valued, consider the following problem of finding \((x', y') \in C \times C \) such that

\[
\begin{align*}
\langle x' + \mu_1 A_1 y' - y', J(x' - x) \rangle & \geq 0, \quad \forall x \in C, \\
\langle y' + \mu_2 A_2 x' - x, J(y' - y) \rangle & \geq 0, \quad \forall x \in C,
\end{align*}
\]

(1)

with constants \( \mu_1, \mu_2 > 0 \), which is called a general system of variational inequalities (GSVI). In particular, if \( X = H \) a Hilbert space, then GSVI (1) reduces to the following GSVI of finding \((x', y') \in C \times C \) such that

\[
\begin{align*}
\langle x' + \mu_1 A_1 y' - y', x - x' \rangle & \geq 0, \quad \forall x \in C, \\
\langle y' + \mu_2 A_2 x' - x, x - y' \rangle & \geq 0, \quad \forall x \in C,
\end{align*}
\]

(2)

with constants \( \mu_1, \mu_2 > 0 \). The literature on the system of variational inequalities is vast and gradient-like methods have received great attention; see, e.g., [9–11, 17, 20, 21, 28, 31] and references therein. In addition, if \( A_1 = A_2 = A \) and \( x' = y' \), then GSVI (1.1) reduces to the variational inequality of finding \( x' \in C \) such that \( \langle x', J(x' - x) \rangle \geq 0 \ \forall x \in C \). In 2006, Aoyama, Iiduka and Takahashi [4] proposed an iterative scheme of finding its approximate solutions and proved the weak convergence of the sequences generated by the proposed scheme.

Recently Ceng et al. [11] suggested and analyzed an implicit iterative algorithm by the two-step relaxed extragradient method in the setting of uniformly convex and 2-uniformly smooth Banach space \( X \) with 2-uniform smoothness coefficient \( \kappa_2 \). Let \( \Pi_C \) be a sunny nonexpansive retraction from \( X \) onto \( C \). Let the mapping \( A_i : C \rightarrow X \) be \( \alpha_i \)-inverse-strongly accretive for \( i = 1, 2 \). Let \( f : C \rightarrow X \) be a contraction with constant \( \delta \in (0, 1) \). Let \( \{S_n\}_{n=1}^\infty \) be a countable family of nonexpansive self-mappings on \( C \) such that \( \Omega = \cap_{n=0}^\infty \text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2) \neq \emptyset \), where \( \text{GSVI}(C, A_1, A_2) \) is the fixed point set of the mapping \( G := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2) \). For arbitrarily given \( x_0 \in C \), let \( \{x_n\} \) be the sequence generated by

\[
\begin{align*}
y_n &= (1 - \alpha_n)\Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)x_n + \alpha_n f(y_n), \\
x_{n+1} &= (1 - \beta_n)S_n y_n + \beta_n x_n, \quad \forall n \geq 0,
\end{align*}
\]

(3)

with \( 0 < \mu_i < \frac{2\kappa_2}{\kappa_2} \) for \( i = 1, 2 \), where \( \alpha_n \) and \( \beta_n \) are sequences in \((0, 1)\) satisfying the conditions: \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=0}^\infty \alpha_n = \infty \) and \( \lim inf_{n \to \infty} \beta_n > 0 \) and \( \lim sup_{n \to \infty} \beta_n < 1 \). They proved the strong convergence of \( \{x_n\} \) to \( x' \in \Omega \), which solved the variational inequality: \( \langle (I - f)x', J(x' - p) \rangle \leq 0 \ \forall p \in \Omega \). Furthermore, let \( X \) be a uniformly convex and \( q \)-uniformly smooth Banach space with \( q \)-uniform smoothness coefficient \( \kappa_q \), where \( 1 < q \leq 2 \). Let \( \Pi_{C_1}, A_1, A_2, G, \{S_n\}_{n=1}^\infty \) be the same mappings as above. Assume that \( \Omega = \cap_{n=0}^\infty \text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2) \neq \emptyset \). Suppose that \( F : C \rightarrow X \) is a \( k \)-Lipschitzian and \( \eta \)-strongly accretive operator with constants \( k, \eta > 0 \). Let \( C \rightarrow X \) be \( L \)-Lipschitzian mapping with constant \( L \geq 0 \). Assume \( 0 < \rho < (\frac{\eta}{L})^{\frac{1}{\tau}} \), \( 0 < \mu_i < (\frac{\alpha}{\kappa_2})^{\frac{1}{\tau}}, i = 1, 2 \), and \( \alpha \leq \gamma \). Song and Ceng [28] proposed and considered a general iterative scheme by the modified relaxed extragradient method, that is, for arbitrarily given \( x_0 \in C \), let \( \{x_n\} \) be the sequence generated by

\[
\begin{align*}
y_n &= (1 - \beta_n)x_n + \beta_n\Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)x_n, \\
x_{n+1} &= \Pi_C[y_n + ((1 - \gamma_n)I - \alpha_n \rho F)Sy_n + \alpha_n \gamma f(x_n)] \quad \forall n \geq 0,
\end{align*}
\]

(4)

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1) \) satisfying the conditions: (i) \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=0}^\infty \alpha_n = \infty \), \( \sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty \); (ii) \( 0 < \lim inf_{n \to \infty} \gamma_n \leq \lim sup_{n \to \infty} \gamma_n < 1 \), \( \sum_{n=0}^\infty |\gamma_{n+1} - \gamma_n| < \infty \); and (iii) \( \sum_{n=0}^\infty |\beta_{n+1} - \beta_n| < \infty \), \( \lim inf_{n \to \infty} \beta_n > 0 \). They proved the strong convergence of \( \{x_n\} \) to \( x' \in \Omega \), which solves the variational inequality: \( \langle (\rho F - \gamma f)x', J(x' - p) \rangle \leq 0 \ \forall p \in \Omega \).

The purpose of this paper is to find a common solution of GSVI (1), a variational inclusion (VI) and a common fixed point problem (CFPP) of a countable family of nonexpansive mappings in a uniformly
convex and \( q \)-uniformly smooth Banach space where \( 1 < q \leq 2 \). We introduce the modified implicit
extragradient iterations, which are based on Korpelevich’s extragradient method, viscosity approximation
method and Mann’s iteration method. We then prove the strong convergence of the sequences generated
by modified implicit extragradient iterations to a common solution of the GSVI, VI and CFPP, which solves
a hierarchical variational inequality (HVI).

2. Preliminaries

Let \( X \) be a real Banach space with the dual \( X^* \). For simplicity, the norms of \( X \) and \( X^* \) are denoted by
the same symbol \( \| \cdot \| \). Let \( C \) be a convex and closed set in \( X \). We write \( x_n \to x \) (respectively, \( x_n \rightharpoonup x \)) to indicate
the weak (respectively, strong) convergence of the sequence \( \{x_n\} \) to \( x \). It is known that the normalized
duality mapping \( J \) from \( X \) into the family of nonempty (by Hahn-Banach’s theorem) weak∗
compact subsets of \( X^* \) satisfies \( f(tx) = tf(x) \) and \( f(-x) = -f(x) \) for all \( t > 0 \) and \( x \in X \).

Let \( A : C \to 2^X \) be a set-valued operator with \( Ax \neq \emptyset \ \forall x \in C \). Let \( q > 1 \). An operator \( A \) is said to be
accrative if for each \( x, y \in C \), there exists \( j_q(x - y) \in j_q(x - y) \) such that \( (u - v, j_q(x - y)) \geq 0 \ \forall u \in Ax, v \in Ay \).
An accretive operator \( A \) is said to be \( \alpha \)-inverse-strongly accretive of order \( q \) if for each \( x, y \in C \), there exist
\( \alpha > 0 \) and \( j_q(x - y) \in j_q(x - y) \) such that \( (u - v, j_q(x - y)) \geq \alpha \|Ax - Ay\|^q \ \forall u \in Ax, v \in Ay \).
An accretive operator \( A \) is said to be \( m \)-accretive if and only if \( A \) is accretive and \((I + \lambda AA)C = X \) for all \( \lambda > 0 \). For an accretive operator \( A \), we define the mapping \( f_0^A : (I + \lambda AA)C \to C \) by \( f_0^A = (I + \lambda AA)^{-1} \) for each \( \lambda > 0 \). Such \( f_0^A \) is called the resolvent of \( A \) for each \( \lambda > 0 \).

Lemma 1. \([18]\) Let \( X \) be smooth and uniformly convex, and \( r > 0 \). Then there exists a strictly increasing,
continuous and convex function \( g : [0, 2r] \to \mathbb{R} \) such that \( g(0) = 0 \) and \( g(|x - y|) \leq \|x\|^2 - 2\langle x, j(y) \rangle + \|y\|^2 \)
for all \( x, y \in B_r = \{ y \in X : \|y\| \leq r \} \).

Let \( \rho_X : [0, \infty) \to [0, \infty) \) be the modulus of smoothness of \( X \) defined by \( \rho_X(t) = \sup\{|\|x + y\| + \|x - y\|\}/2 - 1 : x \in U, \|y\| \leq t \} \). A Banach space \( X \) is said to be uniformly smooth if \( \lim_{t \to 0^+} \rho_X(t)/t = 0 \). Let \( q \in (1, 2] \) be a fixed real number. A Banach space \( X \) is said to be \( q \)-uniformly smooth if there exists a constant \( c > 0 \) such that \( \rho_X(t) \leq ct^q \) \( \forall t > 0 \). It is well known that each Hilbert, \( L^p \) and \( \ell^p \) spaces are uniformly smooth where \( p > 1 \).

Let \( q > 1 \). The generalized duality mapping \( J_q : X \to 2^{X^*} \) is defined by
\[
J_q(x) := \{ \phi \in X^* : \langle x, \phi \rangle = \|x\|^q \ \text{and} \ \|\phi\| = \|x\|^{q - 1} \} \quad \forall x \in X,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the generalized duality pairing between \( X \) and \( X^* \). It is easy to see that \( f_q(x) = J(x)\|x\|^{q/2} \),
and if \( H = H \), then \( J_2 = f = 1 \) the identity mapping of \( H \).

Lemma 2. \([33]\) Let \( q \in (1, 2] \) be a given real number and let \( X \) be \( q \)-uniformly smooth. Then, for any
given \( x, y \in X \) the inequality holds: \( \|x + y\|^q \leq \|x\|^q + \langle q(y, j_q(x + y)) \rangle \quad \forall x, y \in j_q(x + y) \). Moreover,
\( \|x + y\|^q \leq \|x\|^q + q(y, \|j_q(x + y)\|) \quad \forall x, y \in X, \) where \( k_q \) is the \( q \)-uniformly smooth constant of \( X \). In particular, if \( X \) is \( 2 \)-uniformly smooth, then \( \|x + y\|^2 \leq 2\langle y, j(x) \rangle + v_2\|y\|^2 \quad \forall x, y \in X, \) where \( v_2 \) is the \( 2 \)-uniformly smooth constant of \( X \).

Let \( D \) be a set in \( C \) and let \( \Pi : C \to D \) be a mapping. Then \( \Pi \) is said to be sunny if \( \Pi\Pi(x) + t(\Pi(x) - \Pi(x)) = \Pi(x) \), whenever \( \Pi(x) + t(\Pi(x) - \Pi(x)) \in C \) for \( x \in C \) and \( t \geq 0 \). A mapping \( \Pi \) of \( C \) into itself is called a retraction if \( \Pi^2 = \Pi \). If a self-mapping \( \Pi \) of \( C \) is a retraction, then \( \Pi(z) = z \) for each \( z \in R(\Pi) \). A subset \( D \) of \( C \) is called a sunny nonexpansive retract of \( C \) if there exists a sunny nonexpansive retraction from \( C \) onto \( D \). Then the following are equivalent: (i) \( \Pi \) is sunny and nonexpansive; (ii) \( \|\Pi(x) - \Pi(y)\|^q \leq \langle x - y, j(\Pi(x) - \Pi(y)) \rangle \quad \forall x, y \in \Pi(x) \); (ii) \( (x - \Pi(x), j(y - \Pi(x))) \leq 0 \quad \forall x \in C, y \in D \). It is known that the following statements hold: (i) the resolvent identity: \( J_qx = j_q(\lambda x + (1 - \lambda)j_qx) \quad \forall \lambda, \mu > 0, x \in X; \) (ii) \( f_{\Pi}^{\lambda} \) is a resolvent of \( A \) for \( \lambda > 0 \), then \( f_{\Pi}^{\lambda} \) is a single-valued nonexpansive mapping with \( \text{Fix}(f_{\Pi}^{\lambda}) = \lambda - \Pi(\lambda - \Pi(y)) \) \( \forall y \in 0 \). From \([4]\), one has \( \text{Fix}(T_{\lambda}) = (A + B)^{-1}0 \ \forall \lambda > 0 \) and \( \|T_{\lambda}x - T_{\lambda}y\| \leq \|x - y\| \) if \( 0 < \lambda \leq (\frac{\|x\|}{\|y\|})^{1/2} \).
Lemma 3. [4] Let $X$ be $q$-uniformly smooth. Let the mapping $A : C \to X$ be $\alpha$-inverse-strongly accretive of order $q$. Then the following inequality holds: for $\lambda > 0$,

$$
\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(q\alpha - \kappa_q\lambda^{q-1})\|Ax - Ay\|^q \quad \forall x, y \in C.
$$

In particular, if $0 < \lambda \leq (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Lemma 4. [28] Let $X$ be $q$-uniformly smooth. Suppose that $\Pi_C$ is a sunny nonexpansive retraction from $X$ onto $C$. Let the mapping $A_i : C \to X$ be $\alpha_i$-inverse-strongly accretive of order $q$ for $i = 1, 2$. Let the mapping $G : C \to C$ be defined as $Gx := \Pi_C(l - \mu_1A_1)\Pi_C(l - \mu_2A_2) \forall x \in C$. If $0 < \mu_i \leq (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ for $i = 1, 2$, then $G : C \to C$ is nonexpansive. For given $(x', y') \in C \times C$, $(x', y')$ is a solution of GSVI (1.1) if and only if $x' = \Pi_C(y' - \mu_1A_1y')$ where $y' = \Pi_C(x' - \mu_2A_2x')$, that is, $x' = Gx'$.

Lemma 5. [7] Let $X$ be strictly convex, and $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on $C$. Suppose that $\cap_{n=0}^{\infty}{\text{Fix}}(T_n) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty}{\lambda_n} = 1$. Then a mapping $S$ on $C$ defined by $Sx = \sum_{n=0}^{\infty}{\lambda_n}T_nx$ for $x \in C$ is well defined, nonexpansive and $\text{Fix}(S) = \cap_{n=0}^{\infty}{\text{Fix}}(T_n)$ holds.

Lemma 6. [28] Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of self-mappings on $C$. Suppose that $\sum_{n=0}^{\infty}{\text{sup}}{\|S_nx - S_{n-1}x\|} : x \in C < \infty$. Then for each $x \in C, \{S_nx\}$ converges strongly to some point of $C$. Moreover, let $S$ be a self-mapping on $C$ defined by $Sx = \lim_{n \to \infty}{S_nx}, \forall x \in C$. Then $\lim_{n \to \infty}{\text{sup}}{\|S_nx - Sx\|} : x \in C = 0$.

Lemma 7. [34] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the conditions: $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \gamma_n \forall n \geq 1,$ where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty}{\lambda_n} = \infty$, and (ii) $\text{lim sup}_{n \to \infty}{\gamma_n} \leq 0$ or $\sum_{n=0}^{\infty}{\lambda_n} \gamma_n < \infty$. Then $\lim_{n \to \infty}{a_n} = 0$.

3. Iterative Algorithms and Convergence Criteria

Theorem 1. Let $X$ be uniformly convex and $q$-uniformly smooth with $1 < q \leq 2$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Assume that for $i = 1, 2$, the mappings $A, A_i : C \to X$ are $\alpha_i$-inverse-strongly accretive of order $q$ and $\alpha_i$-inverse-strongly accretive of order $q$, respectively. Let $B : C \to 2^X$ be an $m$-accretive operator, and let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on $C$ such that $\Omega = \cap_{n=0}^{\infty}{\text{Fix}}(S_n) \cap \text{GSVI}(C, A_1, A_2) \cap (A + B)^{-1}0 \neq \emptyset$ where $\text{GSVI}(C, A_1, A_2)$ is the fixed point set of $G := \Pi_C(l - \mu_1A_1)\Pi_C(l - \mu_2A_2) \forall 0 < \mu_i \leq (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ for $i = 1, 2$. Let $f : C \to C$ be a $\delta$-contraction with constant $\delta \in (0, 1)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$
\begin{cases}
\nu_n = \Pi_C(l - \mu_1A_1)\Pi_C(l - \mu_2A_2)x_n, \\
y_n = (1 - \alpha_n)S_\nu \nu_n + \alpha_n f(y_n), \\
x_{n+1} = (1 - \beta_n)T_n y_n + \beta_n x_n, \\
\end{cases}
$$

where $\{\lambda_n\} \subset (0, \frac{q\alpha_n}{\kappa_q}^{-1})$, and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty}{\alpha_n} = \infty$ and $\lim_{n \to \infty}{\alpha_n} = 0$;

(ii) $\limsup_{n \to \infty}{\beta_n} < 1$ and $\liminf_{n \to \infty}{\beta_n} > 0$;

(iii) $0 < \lambda \leq \lambda_n \forall n \geq 0$ and $\lim_{n \to \infty}{\lambda_n} = \lambda < \frac{q\alpha}{\kappa_q}^{-1}$.

Assume that $\sum_{n=0}^{\infty}{\sup_{x \in D}{\|S_n x - S_{n-1}x\|}} < \infty$ for any bounded subset $D$ of $C$ and let $S$ be a self-mapping on $C$ defined by $Sx = \lim_{n \to \infty}{S_n x}, \forall x \in C$ and suppose that $\text{Fix}(S) = \cap_{n=0}^{\infty}{\text{Fix}}(S_n)$. Then $x_n \to x' \in \Omega \Rightarrow x_n - y_n \to 0$, where $x' \in \Omega$ is a unique solution to the variational inequality: $\langle (I - f)x', f(x' - p) \rangle \leq 0 \forall p \in \Omega$.

Proof. Set $u_n = \Pi_C(x_n - \mu_2A_2x_n)$. It is easy to see that scheme (4) can be rewritten as

$$
\begin{cases}
y_n = (1 - \alpha_n)S_\nu \nu_n + \alpha_n f(y_n), \\
x_{n+1} = (1 - \beta_n)T_n y_n + \beta_n x_n, \\
\end{cases}
$$

where $\nu_n \in C$ and $\nu_n \to x' \in \Omega$, that is, $\nu_n \to x' = Gx'$.
where \( T_n := f_n^0 (I - \lambda_n A) \) \( \forall n \geq 0 \). Now, we claim that the necessity of the theorem is valid. Indeed, if \( x_n \to x' \in \Omega = \cap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2) \cap (A + B)^{-1} 0 \), then \( S_n x' = x' \), \( G x' = x' \) and \( T_n x' = x' \). Also, according to Lemma 4, we know that the mapping \( G : C \to C \) is nonexpansive. So it follows from (5) that

\[
\|y_n - x_n\| \leq \|y_n - x'\| + \|x' - x_n\| \\
\leq (1 - \alpha_n)\|S_n G x_n - x'\| + \alpha_n\|f(y_n) - x'\| + \|x_n - x'\| \\
\leq \|f(x_n) - x'\| + (1 - \alpha_n)\|x_n - x'\| + \alpha_n(\|f(y_n) - f(x_n)\| + \|x_n - x'\|) \\
\leq 2\|x_n - x'\| + \alpha_n(\|y_n - x_n\| + \|f(x_n) - x'\|),
\]

which immediately yields

\[
\|y_n - x_n\| \leq 2 \frac{1}{1 - \alpha_n} \|x_n - x'\| + \frac{\alpha_n}{1 - \alpha_n} \|f(x_n) - x'\| \to 0 \quad (n \to \infty)
\]

(due to \( x_n \to x' \), \( \alpha_n \to 0 \) and the boundedness of \( \{f(x_n)\} \)).

Next we show the sufficiency of the theorem. To reach the aim, we assume \( x_n - y_n \to 0 \) and divide the proof of the sufficiency as follows. Pick a fixed \( p \in \Omega = \cap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2) \cap (A + B)^{-1} 0 \) arbitrarily. Then \( S_n p = p \), \( G p = p \) and \( T_n p = p \). Moreover, by Lemma 5 we have

\[
\|y_n - p\| = \|(1 - \alpha_n)(S_n G x_n - p) + \alpha_n f(y_n) - p\| \\
\leq (1 - \alpha_n)\|S_n G x_n - p\| + \alpha_n\|f(y_n) - f(p)\| + \alpha_n\|f(p) - p\| \\
\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\| + \alpha_n\|f(p) - p\|,
\]

which hence implies that

\[
\|y_n - p\| \leq (1 - \frac{1 - \delta}{1 - \alpha_n} \alpha_n)\|x_n - p\| + \frac{1}{1 - \alpha_n} \alpha_n\|f(p) - p\|.
\]

Thus, from (5) and Lemma 5, one has

\[
\|x_{n+1} - p\| \leq (1 - \beta_n)\|T_n y_n - p\| + \beta_n\|x_n - p\| \\
\leq (1 - \beta_n)\|y_n - p\| + \beta_n\|x_n - p\| \\
\leq (1 - \beta_n)(1 - \frac{1 - \beta_n}{1 - \alpha_n} \alpha_n)\|x_n - p\| + \frac{1 - \beta_n}{1 - \alpha_n} \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| \\
= (1 - \frac{1 - \beta_n}{1 - \alpha_n} \alpha_n)\|x_n - p\| + \frac{1 - \beta_n}{1 - \alpha_n} \alpha_n\|f(p) - p\|.
\]

By induction, we get \( \{x_n\} \) is bounded, and so are the sequences \( \{u_n\}, \{v_n\}, \{y_n\}, (G x_n), \{S_n y_n\}, \{T_n y_n\} \) due to (6) and the nonexpansivity of \( I - \mu_1 A_1, I - \mu_2 A_2, G, S_n, T_n \). Using (5), we have

\[
\left\{\begin{array}{l}
y_n = (1 - \alpha_n)S_n G x_n + \alpha_n f(y_n), \\
y_{n+1} = (1 - \alpha_n)S_n G x_{n+1} + \alpha_n f(y_{n+1}) \quad \forall n \geq 1.
\end{array}\right.
\]

Simple calculations show that

\[
\|y_n - y_{n-1}\| \leq \|\alpha_n - \alpha_{n-1}\|\|f(y_{n-1}) - S_{n-1} G x_{n-1}\| + \|\alpha_n\|\|f(y_n) - f(y_{n-1})\| \\
+ (1 - \alpha_n)\|S_n G x_n - S_{n-1} G x_{n-1}\| \\
\leq \|\alpha_n - \alpha_{n-1}\|\|f(y_{n-1}) - S_{n-1} G x_{n-1}\| + \|\alpha_n\|\|y_{n-1} - y_{n-1}\| + (1 - \alpha_n)\|S_n G x_n - S_{n-1} G x_{n-1}\| \\
\leq \|\alpha_n - \alpha_{n-1}\|\|f(y_{n-1}) - S_{n-1} G x_{n-1}\| + \|\alpha_n\|\|y_{n-1} - y_{n-1}\| \\
+ (1 - \alpha_n)\|S_n G x_n - S_{n-1} G x_{n-1}\| + \|S_n G x_n - S_{n-1} G x_{n-1}\| + \|S_n G x_n - S_{n-1} G x_{n-1}\| \\
\leq \|\alpha_n - \alpha_{n-1}\|\|f(y_{n-1}) - S_{n-1} G x_{n-1}\| + \|\alpha_n\|\|y_{n-1} - y_{n-1}\| \\
+ (1 - \alpha_n)\|S_n G x_n - S_{n-1} G x_{n-1}\| + \|S_n G x_n - S_{n-1} G x_{n-1}\|,
\]

which hence yields

\[
\|y_n - y_{n-1}\| \leq (1 - \frac{1 - \beta_n}{1 - \alpha_n} \alpha_n)\|x_n - x_{n-1}\| + \frac{1 - \beta_n}{1 - \alpha_n} \alpha_n\|f(y_{n-1}) - S_{n-1} G x_{n-1}\| \\
+ \frac{1 - \beta_n}{1 - \alpha_n} \alpha_n\|S_n G x_n - S_{n-1} G x_{n-1}\|.
\]
From Lemma 2 and Lemma 5, one deduces that

\[
\|T_ny_n - T_{n-1}y_{n-1}\| \leq \|T_ny_n - T_ny_{n-1}\| + \|T_ny_{n-1} - T_{n-1}y_{n-1}\|
\]

\[
\leq \|y_n - y_{n-1}\| + \|p^\alpha_n (I - \lambda_nA)y_{n-1} - f^\alpha_{n-1} (I - \lambda_nA)y_{n-1}\|
\]

\[
\leq \|y_n - y_{n-1}\| + \|p^\alpha_n (I - \lambda_nA)y_{n-1} - f^\alpha_{n-1} (I - \lambda_nA)y_{n-1}\|
\]

\[
+ \|f^\alpha_{n-1} (I - \lambda_nA)y_{n-1} - f^\beta_{n-1} (I - \lambda_nA)y_{n-1}\|
\]

\[
= \|y_n - y_{n-1}\| + \|p^\beta_n (I - \lambda_nA)y_{n-1} - f^\beta_{n-1} (I - \lambda_nA)y_{n-1}\|
\]

\[
+ \|f^\beta_{n-1} (I - \lambda_nA)y_{n-1} - f^\beta_n (I - \lambda_nA)y_{n-1}\|
\]

\[
\leq \|y_n - y_{n-1}\| + \|p^\beta_n (I - \lambda_nA)y_{n-1} - f^\beta_{n-1} (I - \lambda_nA)y_{n-1}\|
\]

\[
+ \|f^\beta_{n-1} (I - \lambda_nA)y_{n-1} - f^\beta_n (I - \lambda_nA)y_{n-1}\|
\]

\[
\leq \|y_n - y_{n-1}\| + \|\lambda_n - \lambda_{n-1}\| \|A_{n}y_{n-1}\|
\]

where \(\sup_{n \geq 1} \|p^\beta_n (I - \lambda_nA)y_{n-1} - (I - \lambda_nA)y_{n-1}\| + \|A_{n}y_{n-1}\| \leq M_1\) for some \(M_1 > 0\). This together with (7), implies that

\[
\|T_ny_n - T_{n-1}y_{n-1}\| \leq (1 - \frac{1}{\alpha_n})\|x_n - x_{n-1}\| + \|p_n\| + \|f(y_{n-1}) - S_{n-1}Gx_{n-1}\|
\]

\[
+ \|x_n - x_{n-1}\| + \|p_n\| + \|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| + \|\lambda_n - \lambda_{n-1}\| M_1.
\]

So it follows that

\[
\|T_ny_n - T_{n-1}y_{n-1}\| - \|x_n - x_{n-1}\|
\]

\[
\leq \|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| + \|\lambda_n - \lambda_{n-1}\| M_1.
\]

Since \(\sum_{n=1}^{\infty} \|S_nx - S_{n-1}x\| < \infty\) for bounded subset \(D = \{Gx : n \geq 0\}\) of \(C\) (due to the assumption), we know that \(\lim_{n \to \infty} \|S_nx_n - S_{n-1}x_n\| = 0\). Note that \(\alpha_n \to 0\) and \(\|\lambda_n - \lambda_{n-1}\| \to 0\) as \(n \to \infty\). Thus, from the boundedness of \(\{f(y_{n})\}\) and \(\{S_nGx_{n}\}\) we get

\[
\limsup_{n \to \infty} \|T_ny_n - T_{n-1}y_{n-1}\| - \|x_n - x_{n-1}\| \leq 0.
\]

Using Suzuki’s lemma [29], one yields that \(\lim_{n \to \infty} \|T_ny_n - x_n\| = 0\). Hence

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n)\|T_ny_n - x_n\| = 0.
\]

We next denote \(\rho := \Pi_C(I - \mu_2A_2)p\). Note that \(u_n = \Pi_C(I - \mu_2A_2)x_n\) and \(v_n = \Pi_C(I - \mu_1A_1)u_n\). Then \(v_n = Gx_n\). From Lemma 4, we have

\[
\|u_n - p\|^2 \leq \|(I - \mu_2A_2)x_n - (I - \mu_2A_2)p\|^2
\]

\[
\leq \|x_n - p\|^2 + \|\mu_1\|^2\|A_2x_n - A_2p\|^2
\]

and

\[
\|v_n - p\|^2 \leq \|(I - \mu_1A_1)u_n - (I - \mu_1A_1)p\|^2
\]

\[
\leq \|u_n - p\|^2 + \|\mu_1\|^2\|A_1u_n - A_1p\|^2.
\]

Substituting (10) into (11), we obtain

\[
\|v_n - p\|^2 \leq \|x_n - p\|^2 + \|\mu_1\|^2\|A_2x_n - A_2p\|^2
\]

\[
- \mu_1\|A_1u_n - A_1p\|^2.
\]

According to Lemma 1, we have

\[
\|y_n - p\|^2 \leq (1 - \alpha_n)\|S_nv_n - p\|^2 + \|\eta_n\|(f(p) - p, f(y_n) - p) + \alpha_n\|f(y_n) - f(p)\|^2
\]

\[
\leq (1 - \alpha_n)\|y_n - p\|^2 + \|\eta_n\|\|f(p) - p\|\|y_n - p\|^{\alpha + 1} + \alpha_n\|y_n - p\|^2.
\]
which hence yields
\[ \|y_n - p\|^\rho \leq \left( 1 - \frac{1}{1 - \alpha_n} \alpha_n \right) \|v_n - p\|^\rho + \frac{\|q\alpha_n}{1 - \alpha_n} \|f(p) - p\|^\rho \|y_n - p\|^{\rho - 1}. \]

That together with (12) and the nonexpansivity of \( T_n \) leads to
\[\begin{align*}
\|x_{n+1} - p\|^\rho &= \|\beta_n(x_n - p) + (1 - \beta_n)(T_n y_n - p)\|^\rho \\
&\leq \beta_n\|x_n - p\|^\rho + (1 - \beta_n)\|T_n y_n - p\|^\rho \\
&\leq \beta_n\|x_n - p\|^\rho + (1 - \beta_n)(1 - \frac{\mu_1}{1 - \alpha_n} \alpha_n)\|v_n - p\|^\rho + \frac{\|q\alpha_n}{1 - \alpha_n} \|f(p) - p\|^\rho \|y_n - p\|^\rho - 1 \\
&\leq \beta_n\|x_n - p\|^\rho + (1 - \beta_n)(1 - \frac{\mu_1}{1 - \alpha_n} \alpha_n)\|v_n - p\|^\rho \\
&\quad - \mu_2(qa_2 - \kappa_3 \mu_2^{-1})\|T_2 x_n - A_2 p\|^\rho + \mu_1(qa_1 - \kappa_4 \mu_1^{-1})\|A_1 u_n - A_1 p\|^\rho + \alpha_n M_2 \\
&= (1 - \frac{\mu_1}{1 - \alpha_n} \alpha_n)\|x_n - p\|^\rho + (1 - \beta_n)(1 - \frac{\mu_1}{1 - \alpha_n} \alpha_n)\|v_n - p\|^\rho + \frac{\|q\alpha_n}{1 - \alpha_n} \|f(p) - p\|^\rho \|y_n - p\|^\rho - 1 \\
&\quad + \mu_1(qa_1 - \kappa_4 \mu_1^{-1})\|A_1 u_n - A_1 p\|^\rho + \alpha_n M_2 \\
&\leq \|x_n - p\|^\rho - (1 - \beta_n)(1 - \frac{\mu_1}{1 - \alpha_n} \alpha_n)\|v_n - p\|^\rho + \|A_2 x_n - A_2 p\| \|A_1 u_n - A_1 p\| + \alpha_n M_2 \\
&\quad + \mu_1(qa_1 - \kappa_4 \mu_1^{-1})\|A_1 u_n - A_1 p\|^\rho + \alpha_n M_2,
\end{align*}\]

where \( \sup_{\alpha > 0} \left( \frac{\alpha}{1 - \alpha_n} \|f(p) - p\| \|y_n - p\|^{\rho - 1} \right) = M_2 \) for some \( M_2 > 0 \). So it follows from (13) and Lemma 2 that
\begin{align*}
(1 - \beta_n)(1 - \frac{\mu_1}{1 - \alpha_n} \alpha_n)\|v_n - p\|^\rho &+ \mu_1(qa_1 - \kappa_4 \mu_1^{-1})\|A_1 u_n - A_1 p\|^\rho \\
\leq &\\|x_n - p\|^\rho - \|v_n+1 - p\|^\rho + \alpha_n M_2, \\
\leq &\\|x_n - x_{n+1}\| \|x_{n+1} - p\|^{\rho - 1} + \kappa \|x_n - x_{n+1}\|^\rho + \alpha_n M_2.
\end{align*}

Since \( \left( \frac{\mu_1}{1 - \alpha_n} \right) \left( \frac{\mu_1}{1 - \alpha_n} \right) > \mu_1 \) and \( \left( \frac{\mu_1}{1 - \alpha_n} \right) \left( \frac{\mu_1}{1 - \alpha_n} \right) > \mu_2 \), we get from conditions (i), (ii) and (9)
\[\lim_{n \to \infty} \|A_2 x_n - A_2 p\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|A_1 u_n - A_1 p\| = 0.\] (14)

Utilizing Lemma 4, we have
\[\|u_n - p\|^2 = \|\Pi_C(I - \mu_2 A_2)x_n - \Pi_C(I - \mu_2 A_2)p\|^2 \leq \|(I - \mu_2 A_2)x_n - (I - \mu_2 A_2)p, (u_n - p)\| \]
\[= (x_n - p, (u_n - p)) + \mu_2(A_2 p - A_2 x_n, (u_n - p)) \leq \frac{1}{2} \|x_n - p\|^2 + \|u_n - p\|^2 - g_1(\|x_n - u_n - (p - p)\|) + \mu_2|A_2 p - A_2 x_n|\|u_n - p\|,
\]
which implies that
\[\|u_n - p\|^2 \leq \|x_n - p\|^2 - g_1(\|x_n - u_n - (p - p)\|) + 2\mu_2\|A_2 p - A_2 x_n\|\|u_n - p\|.\] (15)

In the same way, we derive
\[2\|v_n - p\|^2 \leq 2\|(I - \mu_1 A_1)u_n - (I - \mu_1 A_1)p, (v_n - p)\| \leq \|u_n - p\|^2 + \|v_n - p\|^2 - g_2(\|u_n - v_n + (p - p)\|) + 2\mu_1|A_1 p - A_1 u_n|\|v_n - p\|,
\]
which implies that
\[\|v_n - p\|^2 \leq \|u_n - p\|^2 - g_2(\|u_n - v_n + (p - p)\|) + 2\mu_1\|A_1 p - A_1 u_n\|\|v_n - p\|.\] (16)

Substituting (15) into (16), we get
\[\|v_n - p\|^2 \leq \|x_n - p\|^2 - g_1(\|x_n - u_n - (p - p)\|) - g_2(\|u_n - v_n + (p - p)\|) + 2\mu_2\|A_2 p - A_2 x_n\|\|u_n - p\| + 2\mu_1\|A_1 p - A_1 u_n\|\|v_n - p\|.\] (17)
Furthermore,\[
\|y_n - p\|^2 = \|\alpha_n(f(y_n) - f(p)) + (1 - \alpha_n)(S_n v_n - p) + \alpha_n(f(p) - p)\|^2
\]
\[
\leq \alpha_n\|f(y_n) - f(p)\|^2 + (1 - \alpha_n)(S_n v_n - p)^2 + 2\alpha_n(f(p) - p, (y_n - p))
\]
\[
\leq \alpha_n\|f(y_n) - f(p)\|^2 + (1 - \alpha_n)||S_n v_n - p||^2 + 2\alpha_n\|f(p) - p, f(y_n - p)\|
\]
\[
\leq \alpha_n\|f(y_n) - p\|^2 + (1 - \alpha_n)||v_n - p||^2 + 2\alpha_n\|f(p) - p, ||y_n - p||,
\]
which together with (17), leads to\[
\|y_n - p\|^2 \leq (1 - \frac{1 - \bar{\alpha}}{1 - \alpha_n}\alpha_n)||v_n - p||^2 + \frac{2\alpha_n}{1 - \alpha_n}\|f(p) - p\||||y_n - p||\]
\[
\leq (1 - \frac{1 - \bar{\alpha}}{1 - \alpha_n}\alpha_n)||x_n - p||^2 - g_1(||x_n - u_n - (p - \bar{\alpha})||) + 2\mu_1\|A_2 p - A_2 x_n\||||u_n - p||
\]
\[
+ 2\mu_1\|A_1 p - A_1 u_n\||||v_n - p|| + \frac{2\alpha_n}{1 - \alpha_n}\|f(p) - p\||||y_n - p||\]
\[
\leq (1 - \frac{1 - \bar{\alpha}}{1 - \alpha_n}\alpha_n)||x_n - p||^2 - (1 - \bar{\alpha})\|g_1(||x_n - u_n - (p - \bar{\alpha})||) + 2\mu_1\|A_2 p - A_2 x_n\||||u_n - p||
\]
\[
+ 2\mu_1\|A_1 p - A_1 u_n\||||v_n - p|| + \frac{2\alpha_n}{1 - \alpha_n}\|f(p) - p\||||y_n - p||\]
\[
\leq (1 - \frac{1 - \bar{\alpha}}{1 - \alpha_n}\alpha_n)||x_n - p||^2 - (1 - \bar{\alpha})\|g_1(||x_n - u_n - (p - \bar{\alpha})||) + 2\mu_1\|A_2 p - A_2 x_n\||||u_n - p||
\]
\[
+ 2\mu_1\|A_1 p - A_1 u_n\||||v_n - p|| + \frac{2\alpha_n}{1 - \alpha_n}\|f(p) - p\||||y_n - p||\]
which yields\[
(1 - \bar{\alpha})(1 - \frac{1 - \bar{\alpha}}{1 - \alpha_n}\alpha_n)\|g_1(||x_n - u_n - (p - \bar{\alpha})||) + 2\mu_1\|A_2 p - A_2 x_n\||||u_n - p||
\]
\[
+ \frac{2\alpha_n}{1 - \alpha_n}\|f(p) - p\||||y_n - p|| - \|x_n - x_{n+1}||^2 - \|x_{n+1} - p||^2
\]
\[
\leq ||x_n - x_{n+1}||(||x_n - p|| + \|x_{n+1} - p||) + 2\mu_1\|A_2 p - A_2 x_n\||||u_n - p||
\]
\[
+ 2\mu_1\|A_1 p - A_1 u_n\||||v_n - p|| + \frac{2\alpha_n}{1 - \alpha_n}\|f(p) - p\||||y_n - p||.
\]
Using conditions (i), (ii), (9) and (14), we have\[
\lim_{n \to \infty} g_1(||x_n - u_n - (p - \bar{\alpha})||) = 0 \quad \text{and} \quad \lim_{n \to \infty} g_2(||u_n - v_n + (p - \bar{\alpha})||) = 0.
\]
Utilizing the properties of $g_1$ and $g_2$, we deduce that\[
\lim_{n \to \infty} ||x_n - u_n - (p - \bar{\alpha})|| = 0 \quad \text{and} \quad \lim_{n \to \infty} ||u_n - v_n + (p - \bar{\alpha})|| = 0.
\]
(19)

From (19) we obtain\[
||x_n - Gx_n|| = ||x_n - v_n|| \leq ||x_n - u_n - (p - \bar{\alpha})|| + ||u_n - v_n + (p - \bar{\alpha})|| \to 0 \quad \text{as} \ n \to \infty.
\]
(20)

Next, we claim $||x_n - Sx_n|| \to 0, ||x_n - T_1 x_n|| \to 0$ and $||x_n - Wx_n|| \to 0$ as $n \to \infty$, where $Sx = \lim_{n \to \infty} S_n x \forall x \in C$, $T_1 = \lim_{n \to \infty} (I - \theta_1 A)$ and $Wx = \theta_1 Sx + \theta_2 Gx + \theta_3 T_1 x \forall x \in C$ for constants $\theta_1, \theta_2, \theta_3 \in (0, 1)$ satisfying $\theta_1 + \theta_2 + \theta_3 = 1$. Indeed, since $y_n = \alpha_n f(y_n) + (1 - \alpha_n) S_n Gx_n \to \|S_n Gx_n - y_n\| = \frac{\alpha_n}{1 - \alpha_n}\|f(y_n) - y_n\|$, we deduce from (20), $\alpha_n \to 0$ and $x_n - y_n \to 0$ (due to the assumption of the sufficiency) that\[
||S_n x_n - x_n|| \leq ||S_n x_n - S_n Gx_n|| + ||S_n Gx_n - y_n|| + ||y_n - x_n||
\]
\[
\leq \alpha_n\|f(y_n) - y_n\|^2 + \frac{\alpha_n}{1 - \alpha_n}\|f(y_n) - y_n\| + ||y_n - x_n|| \to 0 \quad (n \to \infty),
\]
Taking into account condition (iii), i.e., $0 < \lambda \leq \lambda_n \forall n \geq 0$ and $\lim_{n \to \infty} \lambda_n = \lambda = \left(\frac{\beta}{\delta}\right)\frac{1}{\gamma}$, we know that $0 < \lambda \leq \lambda < \left(\frac{\beta}{\delta}\right)\frac{1}{\gamma}$. So it follows that $\text{Fix}(T_\lambda) = (A + B)^{-1}0$ and $T_\lambda : C \to C$ is nonexpansive. Since $x_{n+1} = \beta_n x_n + (1 - \beta_n) T_\lambda y_n$ leads to $\|T_n y_n - x_n\| = \frac{1}{\lambda} \|x_{n+1} - x_n\|$, we deduce from (19), (22), $x_n - y_n \to 0$ and $\lim \inf_{n \to \infty} (1 - \beta_n) > 0$ that
\[
\|T_n x_n - x_n\| \leq \|T_n x_n - T_\lambda y_n\| + \|T_\lambda y_n - T_n y_n\| + \|T_n y_n - x_n\| \\
\leq \|x_n - y_n\| + \|T_\lambda y_n - T_n y_n\| + \frac{1}{\lambda^n} \|x_{n+1} - x_n\| \to 0 \quad (n \to \infty). 
\]
(23)

We now define the mapping $W x = \theta_1 S x + \theta_2 G x + \theta_3 T_\lambda x \forall x \in C$ for constants $\theta_1, \theta_2, \theta_3 \in (0, 1)$ satisfying $\theta_1 + \theta_2 + \theta_3 = 1$. Then by Lemma 5 we know that $\text{Fix}(W) = \text{Fix}(S) \cap \text{Fix}(G) \cap \text{Fix}(T_\lambda) = \Omega$. Observe that
\[
\|x_n - W x_n\| = \|\theta_1 (x_n - S x_n) + \theta_2 (x_n - G x_n) + \theta_3 (x_n - T_\lambda x_n)\| \\
\leq \theta_1 \|x_n - S x_n\| + \theta_2 \|x_n - G x_n\| + \theta_3 \|x_n - T_\lambda x_n\|. 
\]
(24)

From (20), (21), (23) and (24), we get
\[
\lim_{n \to \infty} \|x_n - W x_n\| = 0. 
\]
(25)

Next, we focus on
\[
\lim \sup_{n \to \infty} (f(x^*) - x_n, f(x^*) - x_n) \leq 0, 
\]
(26)

where $x^* = \text{lim}_{n \to \infty} x_n$ with $x_n$ being a fixed point of the contraction $x_t \mapsto t f(x) + (1 - t) W x$ for each $t \in (0, 1)$. Indeed, one guarantees that for each $t \in (0, 1)$, $x_t$ solves the fixed point equation $x_t = t f(x_t) + (1 - t) W x_t$. Hence, $\|x_n - x_t\| = \|(1 - t) (W x_t - x_n) + t (f(x_t) - x_n)\|$. By Lemma 1, we conclude that
\[
\|x_n - x_t\|^2 = \|(1 - t)(W x_t - x_n) + t (f(x_t) - x_n)\|^2 \\
\leq (1 - t)^2 \|W x_t - x_n\|^2 + 2t (f(x_t) - x_n, (x_n - x_n)) \\
\leq (1 - t)^2 \|W x_t - W x_n\|^2 + \|W x_n - x_n\|^2 + 2t (f(x_t) - x_n, (x_n - x_n)) \\
\leq (1 - t)^2 \|(x_t - x_n) + \|W x_n - x_n\|^2 + 2t (f(x_t) - x_n, (x_n - x_n)) \\
= (1 - t)^2 \|x_t - x_n\|^2 + 2t (f(x_t) - x_n, (x_n - x_n)) \\
\leq 2t (f(x_t) - x_n, (x_n - x_n)) + 2t (f(x_t) - x_n, (x_n - x_n)) + 2t (\|x_t - x_n\|^2, \\
(27)

where
\[
f_t(t) = (1 - t)^2 (2 \|x_t - x_n\|^2 + \|x_n - W x_n\|) x_n - W x_n) \to 0 \quad (n \to \infty). 
\]
(28)

It follows from (27) that
\[
(x_t - f(x_t), f(x_t) - x_n) \leq \frac{t}{2} \|x_t - x_n\|^2 + \frac{1}{24} f_t(t). 
\]
(29)
Letting $n \to \infty$ in (29) and noticing (28), we derive
\[
\limsup_{n \to \infty} (x_n - f(x_n), J(x_n - x_n)) \leq \frac{1}{2} M_3,
\]
where $\sup_{t \in (0,1), n \geq 0} \|x_t - x_n\|^2 \leq M_3$ for some $M_3 > 0$. Taking $t \to 0$ in (30), we have
\[
\limsup_{t \to 0} \limsup_{n \to \infty} (x_t - f(x_t), J(x_t - x_n)) \leq 0.
\]

On the other hand, we have
\[
\langle f(x^*) - x^*, J(x_n - x) \rangle = \langle f(x^*) - x^*, J(x_n - x) \rangle - \langle f(x^*) - x^*, J(x_n - x_t) \rangle
\]
\[
+ \langle f(x^*) - x^*, J(x_n - x_t) \rangle (f(x_1) - x, J(x_1 - x_t)) + \langle f(x^*) - x^*, J(x_n - x_t) \rangle
\]
\[
= \langle f(x^*) - x^*, J(x_n - x) \rangle + (x_n - x^*) \langle x_n - x_t, J(x_n - x_t) \rangle
\]
\[
+ \langle f(x^*) - f(x_1), J(x_n - x_1) \rangle < \langle f(x^*) - x^*, J(x_n - x_1) \rangle.
\]
So it follows that
\[
\limsup_{n \to \infty} (f(x^*) - x^*, J(x_n - x_1)) \leq \limsup_{n \to \infty} (f(x^*) - x^*, J(x_n - x_t))
\]
\[
+ (1 + \delta)\|x_t - x^*\| \limsup_{n \to \infty} \|x_n - x_t\| + \limsup_{n \to \infty} (f(x_t) - x_t, J(x_t - x_n)).
\]

Taking into account that $x_t \to x$ as $t \to 0$, we have
\[
\limsup_{n \to \infty} (f(x^*) - x^*, J(x_n - x_1)) = \limsup_{t \to 0} \limsup_{n \to \infty} (f(x^*) - x^*, J(x_n - x_t))
\]
\[
\leq \limsup_{t \to 0} \limsup_{n \to \infty} (f(x^*) - x^*, J(x_n - x_1)).
\]

Since $X$ is uniformly smooth, the normalized duality mapping $J$ is norm-to-norm uniformly continuous on bounded subsets of $X$. Therefore, the two limits are interchangeable and hence (26) holds. According to the assumption $x_n - y_n \to 0$ of the sufficiency, we get $J(y_n - x^*) - J(x_n - x^*) \to 0$. Thus, we conclude from (26) that
\[
\limsup_{n \to \infty} (f(x^*) - x^*, J(y_n - x^*)) = \limsup_{n \to \infty} (f(x^*) - x^*, J(y_n - x^*))
\]
\[
\leq \limsup_{n \to \infty} (f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_n)) \leq 0.
\]

Finally,
\[
\|y_n - x^*\|^2 = \|(1 - \alpha_n)(S_n G x_n - x^*) + \alpha_n (f(x^*) - x^*) + \alpha_n (f(y_n) - f(x^*))\|^2
\]
\[
\leq \|\alpha_n (f(y_n) - f(x^*)) + (1 - \alpha_n)(S_n G x_n - x^*)\|^2 + 2\alpha_n \|f(x^*) - x^*, J(y_n - x^*)\|^2
\]
\[
\leq \alpha_n \|f(y_n) - f(x^*)\|^2 + (1 - \alpha_n)\|S_n G x_n - x^*\|^2 + 2\alpha_n \|f(x^*) - x^*, J(y_n - x^*)\|^2
\]
\[
\leq \alpha_n \|y_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 + 2\alpha_n \|f(x^*) - x^*, J(y_n - x^*)\|^2,
\]
which hence yields
\[
\|y_n - x^*\|^2 \leq (1 - \frac{\alpha_n (1 - \delta)}{1 - \alpha_n \delta})\|y_n - x^*\|^2 + \frac{\alpha_n (1 - \delta)}{1 - \alpha_n \delta} \cdot \frac{2\|f(x^*) - x^*, J(y_n - x^*)\|^2}{1 - \delta}.
\]

By the convexity of $\|\cdot\|^2$, the nonexpansivity of $T_n$ and (5), we get
\[
\|x_{n+1} - x^*\|^2 \leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|T_n y_n - x^*\|^2
\]
\[
\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|y_n - x^*\|^2,
\]
which together with (33) leads to
\[
\|x_{n+1} - x^*\|^2 \
\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \left[ (1 - \alpha_n(1 - \delta)) \|x_n - x^*\|^2 + \frac{\alpha_n(1 - \delta)}{2(1 - \delta) - \alpha_n} \|x_n - x^*\|^2 \right] \
\leq [1 - \frac{\alpha_n(1 - \delta)}{2(1 - \delta) - \alpha_n}] \|x_n - x^*\|^2 + \frac{\alpha_n(1 - \delta)}{2(1 - \delta) - \alpha_n} \|x_n - x^*\|^2.
\]

(34)

Since \(\liminf_{n \to \infty} \frac{\alpha_n(1 - \delta)}{2(1 - \delta) - \alpha_n} > 0\), \(\{\frac{\alpha_n(1 - \delta)}{2(1 - \delta) - \alpha_n}\} \subset (0,1)\) and \(\sum_{n=0}^{\infty} \alpha_n = \infty\), we know that \(\frac{\alpha_n(1 - \delta)}{2(1 - \delta) - \alpha_n} \in (0,1)\) and 
\[\sum_{n=0}^{\infty} \alpha_n = \infty.\]

Utilizing (32) and Lemma 7, we conclude from (34) that \(\|x_n - x^*\| \to 0\) as \(n \to \infty\). This completes the proof.

**Remark 1.** The problem of finding an element of \(\cap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2)\) in [20, Theorem 3.1] is extended to develop our problem of finding an element of \(\cap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2) \cap (A + B)^{-1}0\) where \((A + B)^{-1}0\) is the solution set of the VI: \(0 \in (A + B)x\). The implicit (two-step) relaxed extragradient method in [28], Theorem 1 is extended to develop our modified implicit extragradient method (4). That is, two iterative steps \(y_n = \alpha_n f(y_n) + (1 - \alpha_n)Gx_n\) and \(x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n y_n\) in [20], Theorem 1 is extended to develop our two iterative steps \(y_n = \alpha_n f(y_n) + (1 - \alpha_n)Gx_n\) and \(x_{n+1} = \beta_n x_n + (1 - \beta_n)T_n y_n\), where \(T_n = \frac{\beta_n}{\lambda_n} (I - \lambda_n A)\).

In addition, the uniformly convex and \(q\)-uniformly smooth Banach space \(X\) in [20], Theorem 1 is extended to the uniformly convex and \(q\)-uniformly smooth Banach space \(X\) in our Theorem 1, where \(1 < q \leq 2\). The problem of finding an element of \(\cap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2)\) in [28], Theorem 1 is extended to develop our problem of finding an element of \(\cap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2) \cap (A + B)^{-1}0\) where \((A + B)^{-1}0\) is the solution set of the VI: \(0 \in (A + B)x\). The modified relaxed extragradient method in [28], Theorem 1 is extended to develop our modified implicit extragradient method (4). That is, two iterative steps \(y_n = (1 - \beta_n)x_n + \beta_n Gx_n\) and \(x_{n+1} = \Pi_{C} [\alpha_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \alpha_n \rho F)S_n y_n]\) in [28], Theorem 1 is extended to develop our two iterative steps \(y_n = \alpha_n f(y_n) + (1 - \alpha_n)Gx_n\) and \(x_{n+1} = \beta_n x_n + (1 - \beta_n)T_n y_n\), where \(T_n = \frac{\beta_n}{\lambda_n} (I - \lambda_n A)\).

Let \(g : H \to \mathbb{R}\) be a convex smooth function and \(h : H \to \mathbb{R}\) be a proper convex and lower semicontinuous function. The convex minimization problem is to find \(x^* \in H\) such that
\[
g(x^*) + h(x^*) = \min_{x \in H} [g(x) + h(x)].
\]

By Fermat’s rule, we know that the above problem is equivalent to the problem of finding \(x^* \in H\) such that \(0 \in \nabla g(x^*) + \partial h(x^*)\), where \(\nabla g\) is the gradient of \(g\) and \(\partial h\) is the subdifferential of \(h\). It is also known that if \(\nabla g\) is \(\frac{1}{2}\)-Lipschitz continuous, then it is also \(\alpha\)-inverse-strongly monotone. From Theorem 1, we can obtain the following result.

**Theorem 2.** Let \(g : H \to \mathbb{R}\) be a convex and differentiable function with \(\frac{1}{2}\)-Lipschitz continuous gradient \(\nabla g\) and \(h : H \to \mathbb{R}\) be a convex and lower semicontinuous function. Let the mapping \(A_1 : C \to H\) be \(\alpha_1\)-inverse-strongly monotone for \(i = 1, 2\). Let \(S\) be a nonexpansive self-mapping on \(C\) such that \(\Omega = \text{Fix}(S) \cap \text{GSVI}(C, A_1, A_2) \cap (\nabla g + \partial h)^{-1}0 \neq \emptyset\) where \((\nabla g + \partial h)^{-1}0\) is the set of minimizers attained by \(g + h\), and \(\text{GSVI}(C, A_1, A_2)\) is the fixed point set of \(G := P_C (I - \mu_1 A_1) P_C (I - \mu_2 A_2)\) with \(0 < \mu_i < 2\alpha_i\) for \(i = 1, 2\). Let \(f : C \to C\) be a \(\delta\)-contraction with constant \(\delta \in (0,1)\). For arbitrarily given \(x_0 \in C\), let \(\{x_n\}\) be a sequence generated by
\[
\begin{align*}
v_n &= P_C (I - \mu_1 A_1) P_C (x_n - \mu_2 A_2 x_n), \\
y_n &= (1 - \alpha_n)Sv_n + \alpha_n f(y_n), \\
x_{n+1} &= (1 - \beta_n) \frac{\mu_n}{\lambda_n} (y_n - \lambda_n \nabla g(y_n)) + \beta_n x_n, \quad n \geq 0,
\end{align*}
\]

where \([\lambda_n] \subset (0,2\alpha),\) and \([\alpha_n], \{\beta_n\} \subset (0,1)\) satisfy the following conditions:
(i) \(\sum_{n=0}^{\infty} \alpha_n = \infty\) and \(\lim_{n \to \infty} \alpha_n = 0;\)
(ii) \(\limsup_{n \to \infty} \beta_n < 1\) and \(\liminf_{n \to \infty} \beta_n > 0;\)
(iii) \(0 < \lambda \leq \lambda_n \forall n \geq 0\) and \(\lim_{n \to \infty} \lambda_n = \lambda < 2\alpha\).

Then \(x_n \to x^* \in \Omega \iff x_n - y_n \to 0\), where \(x^* \in \Omega\) is a unique solution to the variational inequality: \((I - f)x^*, x^* - p) \leq 0 \forall p \in \Omega\).
Let C and Q be nonempty closed convex subsets of Hilbert spaces $H_1$ and $H_2$, respectively. Let $T: H_1 \to H_2$ be a linear bounded operator with its adjoint $T^*$. Consider the split feasibility problem (SFP) of finding a point $x^*$ with the property that $x^* \in C$ and $T^*x^* \in Q$. The SFP can be used to model the intensity-modulated radiation therapy. It is clear that the set of solutions of the SFP is $H \cap T^{-1}Q$. To solve the SFP, we can rewrite it as the following convexly constrained minimization problem: $\min_{x \in C} g(x) := \frac{1}{2} \|Tx - P_QT^*x\|^2$.

Note that the function $g$ is differentiable convex and has a Lipschitz gradient given by $\nabla g = T^*(I - P_Q)T$. Further, $\nabla g$ is $\frac{1}{\|T\|^2}$-inverse-strongly monotone, where $\|T\|$ is the spectral radius of $T^*T$. Thus, $x^*$ solves the SFP if and only if $x^*$ solves the variational inclusion problem of finding $x^* \in H$ such that

$$0 \in \nabla g(x^*) + \partial\mathcal{L}(x^*) \iff x^* - \lambda \nabla g(x^*) \in (I + \lambda \partial\mathcal{L})x^* \iff f^\mathcal{L}_\lambda(x^* - \lambda \nabla g(x^*)) = x^* \iff P_{\mathcal{C}}(x^* - \lambda \nabla g(x^*)) = x^*.$$

From Theorem 1, we can obtain the following result.

**Theorem 3.** Let $C$ and $Q$ be nonempty closed convex subsets of $H_1$ and $H_2$, respectively. Let $T: H_1 \to H_2$ be a bounded linear operator with its adjoint $T^*$. Let the mapping $A_i: C \to H_1$ be $\alpha_i$-inverse-strongly monotone for $i = 1, 2$. Let $S$ be a nonexpansive self-mapping on $C$ such that $\Omega = \text{Fix}(S) \cap \text{GSVI}(C, A_1, A_2) \cap (C \cap T^{-1}Q) \neq \emptyset$ where GSVI$(C, A_1, A_2)$ is the fixed point set of $G := P_C(I - \mu_1A_1)P_C(I - \mu_2A_2)$ with $0 < \mu_i < 2\alpha_i$ for $i = 1, 2$. Let $f: C \to C$ be a $\delta$-contraction with constant $\delta \in (0, 1)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{align*}
v_n &= P_C(I - \mu_1A_1)P_C(x_n - \mu_2A_2x_n), \\
y_n &= (1 - \alpha_n)sv_n + \alpha_nf(y_n), \\
x_{n+1} &= (1 - \beta_n)P_C(y_n - \lambda_nT(I - P_Q)Ty_n) + \beta_nx_n, \quad n \geq 0,
\end{align*}$$

where $[\lambda_n] \subset (0, \frac{2}{\|f\|^2})$, and $[\alpha_n], [\beta_n] \subset (0, 1)$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \to \infty} \alpha_n = 0$;

(ii) $\limsup_{n \to \infty} \beta_n < 1$ and $\liminf_{n \to \infty} \beta_n > 0$;

(iii) $0 < \lambda \leq \lambda_n \forall n \geq 0$ and $\lim_{n \to \infty} \lambda_n = \lambda < \frac{2}{\|f\|^2}$.

Then $x_n \to x^* \in \Omega \iff x_n - y_n \to 0$, where $x^* \in \Omega$ is a unique solution to the variational inequality: $\langle (I - f)x^*, x^* - p \rangle \leq 0$ $\forall p \in \Omega$.

**References**


