Filomat 34:9 (2020), 2939–2951
https://doi.org/10.2298/FIL2009939C

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Variational Inequalities, Variational Inclusions and Common Fixed Point Problems in Banach Spaces

Lu-Chuan Ceng[a], Qing Yuan[b]

[a] *Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*
[b] *School of Mathematics and Statistics, Linyi University, Linyi 276000, China*

Abstract. In this paper, let X be a uniformly convex and q-uniformly smooth Banach space with $1 < q \leq 2$. We introduce and study modified implicit extragradient iterations for treating a common solution of a common fixed-point problem of a countable family of nonexpansive mappings, a general system of variational inequalities, and a variational inclusion in X.

1. Introduction

Multivalued monotone inclusion is an important optimization problem, which can be viewed as a real mathematical modelling for many engineering design, such as, transportation, single processing and image reconstruction; see, e.g., [1–3, 5, 14, 24, 27, 35]. There are a huge number of approximation methods for solving multivalued monotone inclusion problems; see, e.g., [6, 8, 16, 21, 25, 26]. Most efficient one is the resolvent method, which transfers the inclusion problem into a fixed point problem via an inverse problem; see, e.g., [12, 13, 26, 30]. Let X be a real Banach space with the dual space X^*. Both the norms of X and dual space X^* are presented by $\|\cdot\|$. Let C be a convex and closed set in X. Let $T : C \to C$ be a nonlinear single-valued mapping. From now on, one employs $\mathrm{Fix}(T)$ to represent the set of fixed points of T. Recall that T is said to be strictly contractive iff $\|Tx - Ty\| \leq \delta\|x - y\|$, $\forall x, y \in C$, where constant $\delta \in (0, 1)$. T is said to be nonexpansive iff $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. The theory of nonexpansive mappings, whose complementary mappings are monotone, is interesting and important in operator theory. Some efficient approximation methods were studied for fixed points of nonexpansive mappings and their extensions; see [15, 19, 22, 23, 32] and the references therein. Recall that $J(x) := \{\phi \in X^* : \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\}$ $\forall x \in X$, where $\langle\cdot,\cdot\rangle$ represents the generalized duality pairing between X and X^*. Recall a Banach space X is said to be smooth (it has Gâteaux differentiable norm) if $\lim_{t\to 0^+} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $\|x\| = \|y\| = 1$. J is norm-to-weak* continuous in such a space. Moreover, the norm of X is said to be Fréchet differentiable, if for each $\|x\| = 1$, the limit is attained uniformly for $\|y\| = 1$. The norm of X is said to be uniformly Fréchet

2010 *Mathematics Subject Classification*. Primary 49H05; Secondary 47H09, 47J20
Keywords. Modified implicit extragradient; Variational inclusions; Nonexpansive operators; Uniform convexity; Uniform smoothness.
Received: 16 September 2019; Revised: 12 February 2020; Accepted: 22 March 2020
Communicated by Adrain Petrusel
This research was funded by the Natural Science Foundation of Shandong Province of China (ZR2017LA001) and partially supported by NSF of China (Grant no. 11771196). The first author was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002) and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).
Email addresses: zenglc@shnu.edu.cn (Lu-Chuan Ceng), yuanqing@lyu.edu.cn (Qing Yuan)

differentiable, if the limit is attained uniformly. J is norm-to-norm uniformly continuous on bounded sets in such a space. A space X is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $\|x\| = \|y\| = 1$, $\|\frac{x+y}{2}\| > 1 - \delta \Rightarrow \|x - y\| < \varepsilon$. It is known that a uniformly convex Banach space is reflexive. Let $A_1, A_2 : C \to X$ be two nonlinear mappings. In the framework that J is single-valued, consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* + \mu_1 A_1 y^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle y^* + \mu_2 A_2 x^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1}$$

with constants $\mu_1, \mu_2 > 0$, which is called a general system of variational inequalities (GSVI). In particular, if $X = H$ a Hilbert space, then GSVI (1) reduces to the following GSVI of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* + \mu_1 A_1 y^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle y^* + \mu_2 A_2 x^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases}$$

with constants $\mu_1, \mu_2 > 0$. The literature on the system of variational inequalities is vast and gradient-like methods have received great attention; see, e.g., [9–11, 17, 20, 21, 28, 31] and references therein. In addition, if $A_1 = A_2 = A$ and $x^* = y^*$, then GSVI (1.1) reduces to the variational inequality of finding $x^* \in C$ such that $\langle Ax^*, J(x - x^*)\rangle \geq 0\ \forall x \in C$. In 2006, Aoyama, Iiduka and Takahashi [4] proposed an iterative scheme of finding its approximate solutions and proved the weak convergence of the sequences generated by the proposed scheme.

Recntly Ceng et al. [11] suggested and analyzed an implicit iterative algorithm by the two-step relaxed extragradient method in the setting of uniformly convex and 2-uniformly smooth Banach space X with 2-uniform smoothness coefficient κ_2. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $A_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1, 2$. Let $f : C \to C$ be a contraction with constant $\delta \in (0, 1)$. Let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on C such that $\Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \neq \emptyset$, where $\mathrm{GSVI}(C, A_1, A_2)$ is the fixed point set of the mapping $G := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} y_n = (1 - \alpha_n)\Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)x_n + \alpha_n f(y_n), \\ x_{n+1} = (1 - \beta_n)S_n y_n + \beta_n x_n, \quad \forall n \geq 0, \end{cases} \tag{2}$$

with $0 < \mu_i < \frac{2\alpha_i}{\kappa_2}$ for $i = 1, 2$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\liminf_{n\to\infty} \beta_n > 0$ and $\limsup_{n\to\infty} \beta_n < 1$. They proved the strong convergence of $\{x_n\}$ to $x^* \in \Omega$, which solved the variational inequality: $\langle (I - f)x^*, J(x^* - p)\rangle \leq 0\ \forall p \in \Omega$. Furthermore, let X be a uniformly convex and q-uniformly smooth Banach space with q-uniform smoothness coefficient κ_q, where $1 < q \leq 2$. Let $\Pi_C, A_1, A_2, G, \{S_n\}_{n=0}^{\infty}$ be the same mappings as above. Assume that $\Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \neq \emptyset$. Suppose that $F : C \to X$ is a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$, $f : C \to X$ is L-Lipschitzian mapping with constant $L \geq 0$. Assume $0 < \rho < (\frac{q\eta}{\kappa_q k^q})^{\frac{1}{q-1}}$, $0 < \mu_i < (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}, i = 1, 2$, and and $0 \leq \gamma L < \tau$, where $\tau = \rho(\eta - \frac{\kappa_q \rho^{q-1} k^q}{q})$. Song and Ceng [28] proposed and considered a general iterative scheme by the modified relaxed extragradient method, that is, for arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)x_n, \\ x_{n+1} = \Pi_C[\gamma_n x_n + ((1 - \gamma_n)I - \alpha_n \rho F)S_n y_n + \alpha_n \gamma f(x_n)] \quad \forall n \geq 0, \end{cases} \tag{3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfying the conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; (ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$; and (iii) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\liminf_{n\to\infty} \beta_n > 0$. They proved the strong convergence of $\{x_n\}$ to $x^* \in \Omega$, which solves the variational inequality: $\langle (\rho F - \gamma f)x^*, J(x^* - p)\rangle \leq 0\ \forall p \in \Omega$.

The purpose of this paper is to find a common solution of GSVI (1), a variational inclusion (VI) and a common fixed point problem (CFPP) of a countable family of nonexpansive mappings in a uniformly convex and q-uniformly smooth Banach space where $1 < q \leq 2$. We introduce the modified implicit extragradient iterations, which are based on Korpelevich's extragradient method, viscosity approximation method and Mann's iteration method. We then prove the strong convergence of the sequences generated by modified implicit extragradient iterations to a common solution of the GSVI, VI and CFPP, which solves a hierarchical variational inequality (HVI).

2. Preliminaries

Let X be a real Banach space with the dual X^*. For simplicity, the norms of X and X^* are denoted by the same symbol $\|\cdot\|$. Let C be a convex and closed set in X. We write $x_n \rightharpoonup x$ (respectively, $x_n \to x$) to indicate the weak (respectively, strong) convergence of the sequence $\{x_n\}$ to x. It is known that the normalized duality mapping J from X into the family of nonempty (by Hahn-Banach's theorem) weak* compact subsets of X^* satisfies $J(tx) = tJ(x)$ and $J(-x) = -J(x)$ for all $t > 0$ and $x \in X$.

Let $A : C \to 2^X$ be a set-valued operator with $Ax \neq \emptyset\ \forall x \in C$. Let $q > 1$. An operator A is said to be accretive if for each $x, y \in C$, there exists $j_q(x-y) \in J_q(x-y)$ such that $\langle u - v, j_q(x-y)\rangle \geq 0\ \ \forall u \in Ax, v \in Ay$. An accretive operator A is said to be α-inverse-strongly accretive of order q if for each $x, y \in C$, there exist $\alpha > 0$ and $j_q(x-y) \in J_q(x-y)$ such that $\langle u - v, j_q(x-y)\rangle \geq \alpha\|Ax - Ay\|^q\ \ \forall u \in Ax, v \in Ay$.

An accretive operator A is said to be m-accretive if and only if A is accretive and $(I + \lambda A)C = X$ for all $\lambda > 0$. For an accretive operator A, we define the mapping $J_\lambda^A : (I + \lambda A)C \to C$ by $J_\lambda^A = (I + \lambda A)^{-1}$ for each $\lambda > 0$. Such J_λ^A is called the resolvent of A for each $\lambda > 0$.

Lemma 1. [18] Let X be smooth and uniformly convex, and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbf{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2$ for all $x, y \in B_r = \{y \in X : \|y\| \leq r\}$.

Let $\rho_X : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of X defined by $\rho_X(t) = \sup\{(\|x+y\|+\|x-y\|)/2-1 : x \in U,\ \|y\| \leq t\}$. A Banach space X is said to be uniformly smooth if $\lim_{t\to 0^+} \rho_X(t)/t = 0$. Let $q \in (1, 2]$ be a fixed real number. A Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_X(t) \leq ct^q\ \forall t > 0$. It is well known that each Hilbert, L^p and ℓ_p spaces are uniformly smooth where $p > 1$.

Let $q > 1$. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) := \{\phi \in X^* : \langle x, \phi\rangle = \|x\|^q \ \text{ and } \ \|\phi\| = \|x\|^{q-1}\} \qquad \forall x \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between X and X^*. It is easy to see that $J_q(x) = J(x)\|x\|^{q-2}$, and if $X = H$, then $J_2 = J = I$ the identity mapping of H.

Lemma 2. [33] Let $q \in (1, 2]$ be a given real number and let X be q-uniformly smooth. Then, for any given $x, y \in X$ the inequality holds: $\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle\ \ \forall j_q(x + y) \in J_q(x + y)$. Moreover, $\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q\ \forall x, y \in X$, where κ_q is the q-uniformly smooth constant of X. In particular, if X is 2-uniformly smooth, then $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + \kappa_2\|y\|^2\ \forall x, y \in X$, where κ_2 is the 2-uniformly smooth constant of X.

Let D be a set in C and let $\Pi : C \to D$ be a mapping. Then Π is said to be sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a self mapping Π on C is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. Then the following are equivalent: (i) Π is sunny and nonexpansive; (ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle\ \forall x, y \in C$; (iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0\ \ \forall x \in C, y \in D$. It is known that the following statements hold: (i) the resolvent identity: $J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x)\ \ \forall \lambda, \mu > 0,\ x \in X$; (ii) if J_λ^A is a resolvent of A for $\lambda > 0$, then J_λ^A is a single-valued nonexpansive mapping with $\mathrm{Fix}(J_\lambda^A) = A^{-1}0$, where $A^{-1}0 = \{x \in C : 0 \in Ax\}$; (iii) in a Hilbert space H, an operator A is m-accretive if and only if A is maximal monotone.

Let $A : C \to X$ be an α-inverse-strongly accretive mapping of order q and $B : C \to 2^X$ be an m-accretive operator. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A)\ \ \forall \lambda > 0$. From [4], one has $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0\ \forall \lambda > 0$ and $\|T_\lambda x - T_\lambda y\| \leq \|x - y\|$ if $0 < \lambda \leq (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$.

Lemma 3. [4] Let X be q-uniformly smooth. Let the mapping $A : C \to X$ be α-inverse-strongly accretive of order q. Then the following inequality holds: for $\lambda > 0$,

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \le \|x - y\|^q - \lambda(q\alpha - \kappa_q \lambda^{q-1})\|Ax - Ay\|^q \quad \forall x, y \in C.$$

In particular, if $0 < \lambda \le (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Lemma 4. [28] Let X be q-uniformly smooth. Suppose that Π_C is a sunny nonexpansive retraction from X onto C. Let the mapping $A_i : C \to X$ be α_i-inverse-strongly accretive of order q for $i = 1, 2$. Let the mapping $G : C \to C$ be defined as $Gx := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)\ \forall x \in C$. If $0 < \mu_i \le (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ for $i = 1, 2$, then $G : C \to C$ is nonexpansive. For given $(x^*, y^*) \in C \times C$, (x^*, y^*) is a solution of GSVI (1.1) if and only if $x^* = \Pi_C(y^* - \mu_1 A_1 y^*)$ where $y^* = \Pi_C(x^* - \mu_2 A_2 x^*)$, that is, $x^* = Gx^*$.

Lemma 5. [7] Let X be strictly convex, and $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\cap_{n=0}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ for $x \in C$ is well defined, nonexpansive and $\mathrm{Fix}(S) = \cap_{n=0}^{\infty}\mathrm{Fix}(T_n)$ holds.

Lemma 6. [28] Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of self-mappings on C. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1}x\| : x \in C\} < \infty$. Then for each $x \in C$, $\{S_n x\}$ converges strongly to some point of C. Moreover, let S be a self-mapping on C defined by $Sx = \lim_{n\to\infty} S_n x\ \forall x \in C$. Then $\lim_{n\to\infty} \sup\{\|S_n x - Sx\| : x \in C\} = 0$.

Lemma 7. [34] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the conditions: $a_{n+1} \le (1 - \lambda_n)a_n + \lambda_n \gamma_n\ \forall n \ge 1$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$, and (ii) $\limsup_{n\to\infty} \gamma_n \le 0$ or $\sum_{n=1}^{\infty} |\lambda_n \gamma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

3. Iterative Algorithms and Convergence Criteria

Theorem 1. Let X be uniformly convex and q-uniformly smooth with $1 < q \le 2$. Let Π_C be a sunny nonexpansive retraction from X onto C. Assume that for $i = 1, 2$, the mappings $A, A_i : C \to X$ are α-inverse-strongly accretive of order q and α_i-inverse-strongly accretive of order q, respectively. Let $B : C \to 2^X$ be an m-accretive operator, and let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on C such that $\Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (A + B)^{-1}0 \neq \emptyset$ where $\mathrm{GSVI}(C, A_1, A_2)$ is the fixed point set of $G := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)$ with $0 < \mu_i < (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ for $i = 1, 2$. Let $f : C \to C$ be a δ-contraction with constant $\delta \in (0, 1)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} v_n = \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)x_n, \\ y_n = (1 - \alpha_n)S_n v_n + \alpha_n f(y_n), \\ x_{n+1} = (1 - \beta_n)J_{\lambda_n}^B(y_n - \lambda_n A y_n) + \beta_n x_n, \quad n \ge 0, \end{cases} \tag{4}$$

where $\{\lambda_n\} \subset (0, (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}})$, and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfy the following conditions:
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \beta_n > 0$;
(iii) $0 < \bar{\lambda} \le \lambda_n\ \forall n \ge 0$ and $\lim_{n\to\infty} \lambda_n = \lambda < (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$.
Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$ for any bounded subset D of C and let S be a self-mapping on C defined by $Sx = \lim_{n\to\infty} S_n x\ \forall x \in C$ and suppose that $\mathrm{Fix}(S) = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. Then $x_n \to x^* \in \Omega \Leftrightarrow x_n - y_n \to 0$, where $x^* \in \Omega$ is a unique solution to the variational inequality: $\langle (I - f)x^*, J(x^* - p)\rangle \le 0\ \ \forall p \in \Omega$.

Proof. Set $u_n = \Pi_C(x_n - \mu_2 A_2 x_n)$. It is easy to see that scheme (4) can be rewritten as

$$\begin{cases} y_n = (1 - \alpha_n)S_n G x_n + \alpha_n f(y_n), \\ x_{n+1} = (1 - \beta_n)T_n y_n + \beta_n x_n, \quad \forall n \ge 0, \end{cases} \tag{5}$$

where $T_n := J^B_{\lambda_n}(I - \lambda_n A)\ \forall n \geq 0$. Now, we claim that the necessity of the theorem is valid. Indeed, if $x_n \to x^* \in \Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (A+B)^{-1}0$, then $S_n x^* = x^*$, $Gx^* = x^*$ and $T_n x^* = x^*$. Also, according to Lemma 4, we know that the mapping $G : C \to C$ is nonexpansive. So it follows from (5) that

$$\begin{aligned}
\|y_n - x_n\| &\leq \|y_n - x^*\| + \|x^* - x_n\| \\
&\leq (1-\alpha_n)\|S_n G x_n - x^*\| + \alpha_n\|f(y_n) - x^*\| + \|x_n - x^*\| \\
&\leq \|f(x_n) - x^*\|) + (1-\alpha_n)\|x_n - x^*\| + \alpha_n(\|f(y_n) - f(x_n)\| + \|x_n - x^*\| \\
&\leq 2\|x_n - x^*\| + \alpha_n(\delta\|y_n - x_n\| + \|f(x_n) - x^*\|),
\end{aligned}$$

which immediately yields

$$\|y_n - x_n\| \leq \frac{2}{1-\alpha_n\delta}\|x_n - x^*\| + \frac{\alpha_n}{1-\alpha_n\delta}\|f(x_n) - x^*\| \to 0 \quad (n \to \infty)$$

(due to $x_n \to x^*,\ \alpha_n \to 0$ and the boundedness of $\{f(x_n)\}$).

Next we show the sufficiency of the theorem. To reach the aim, we assume $x_n - y_n \to 0$ and divide the proof of the sufficiency as follows. Pick a fixed $p \in \Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (A+B)^{-1}0$ arbitrarily. Then $S_n p = p,\ Gp = p$ and $T_n p = p$. Moreover, by Lemma 5 we have

$$\begin{aligned}
\|y_n - p\| &= \|(1-\alpha_n)(S_n G x_n - p) + \alpha_n(f(y_n) - p)\| \\
&\leq (1-\alpha_n)\|S_n G x_n - p\| + \alpha_n\|f(y_n) - f(p)\| + \alpha_n\|f(p) - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\delta\|y_n - p\| + \alpha_n\|f(p) - p\|,
\end{aligned}$$

which hence implies that

$$\|y_n - p\| \leq (1 - \frac{1-\delta}{1-\alpha_n\delta}\alpha_n)\|x_n - p\| + \frac{1}{1-\alpha_n\delta}\alpha_n\|f(p) - p\|. \tag{6}$$

Thus, from (5) and Lemma 5, one has

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1-\beta_n)\|T_n y_n - p\| + \beta_n\|x_n - p\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n\|x_n - p\| \\
&\leq (1-\beta_n)\{(1 - \tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)\|x_n - p\| + \tfrac{1}{1-\alpha_n\delta}\alpha_n\|f(p) - p\|\} + \beta_n\|x_n - p\| \\
&= [1 - \tfrac{(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}\alpha_n]\|x_n - p\| + \tfrac{(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}\alpha_n\tfrac{\|f(p)-p\|}{1-\delta}.
\end{aligned}$$

By induction, we get $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}, \{v_n\}, \{y_n\}, \{Gx_n\}, \{S_n v_n\}, \{T_n y_n\}$ due to (6) and the nonexpansivity of $I - \mu_1 A_1,\ I - \mu_2 A_2,\ G, S_n, T_n$. Using (5), we have

$$\begin{cases}
y_n = (1-\alpha_n)S_n G x_n + \alpha_n f(y_n), \\
y_{n-1} = (1-\alpha_{n-1})S_{n-1} G x_{n-1} + \alpha_{n-1} f(y_{n-1}) \quad \forall n \geq 1.
\end{cases}$$

Simple calculations show that

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq |\alpha_n - \alpha_{n-1}|\|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| + \alpha_n\|f(y_n) - f(y_{n-1})\| \\
&\quad + (1-\alpha_n)\|S_n G x_n - S_{n-1}Gx_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| + \alpha_n\delta\|y_n - y_{n-1}\| \\
&\quad + (1-\alpha_n)\|S_n G x_n - S_{n-1}Gx_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| + \alpha_n\delta\|y_n - y_{n-1}\| \\
&\quad + (1-\alpha_n)[\|S_n G x_n - S_n G x_{n-1}\| + \|S_n G x_{n-1} - S_{n-1}Gx_{n-1}\|] \\
&\leq |\alpha_n - \alpha_{n-1}|\|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| + \alpha_n\delta\|y_n - y_{n-1}\| \\
&\quad + (1-\alpha_n)[\|x_n - x_{n-1}\| + \|S_n G x_{n-1} - S_{n-1}Gx_{n-1}\|],
\end{aligned}$$

which hence yields

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq (1 - \tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)\|x_n - x_{n-1}\| + \tfrac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n\delta}\|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| \\
&\quad + \tfrac{1-\alpha_n}{1-\alpha_n\delta}\|S_n G x_{n-1} - S_{n-1}Gx_{n-1}\|.
\end{aligned} \tag{7}$$

From Lemma 2 and Lemma 5, one deduces that

$$\begin{aligned}
&\|T_n y_n - T_{n-1}y_{n-1}\| \le \|T_n y_n - T_n y_{n-1}\| + \|T_n y_{n-1} - T_{n-1}y_{n-1}\| \\
&\le \|y_n - y_{n-1}\| + \|J^B_{\lambda_n}(I-\lambda_n A)y_{n-1} - J^B_{\lambda_{n-1}}(I-\lambda_{n-1}A)y_{n-1}\| \\
&\le \|y_n - y_{n-1}\| + \|J^B_{\lambda_n}(I-\lambda_n A)y_{n-1} - J^B_{\lambda_{n-1}}(I-\lambda_n A)y_{n-1}\| \\
&\quad + \|J^B_{\lambda_{n-1}}(I-\lambda_n A)y_{n-1} - J^B_{\lambda_{n-1}}(I-\lambda_{n-1}A)y_{n-1}\| \\
&= \|y_n - y_{n-1}\| + \|J^B_{\lambda_{n-1}}(\tfrac{\lambda_{n-1}}{\lambda_n}I + (1-\tfrac{\lambda_{n-1}}{\lambda_n})J^B_{\lambda_n})(I-\lambda_n A)y_{n-1} - J^B_{\lambda_{n-1}}(I-\lambda_n A)y_{n-1}\| \\
&\quad + \|J^B_{\lambda_{n-1}}(I-\lambda_n A)y_{n-1} - J^B_{\lambda_{n-1}}(I-\lambda_{n-1}A)y_{n-1}\| \\
&\le \|y_n - y_{n-1}\| + |1-\tfrac{\lambda_{n-1}}{\lambda_n}|\|J^B_{\lambda_n}(I-\lambda_n A)y_{n-1} - (I-\lambda_n A)y_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Ay_{n-1}\| \\
&\le \|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1,
\end{aligned} \tag{8}$$

where $\sup_{n\ge1}\{\frac{1}{\lambda}\|J^B_{\lambda_n}(I-\lambda_n A)y_{n-1} - (I-\lambda_n A)y_{n-1}\| + \|Ay_{n-1}\|\} \le M_1$ for some $M_1 > 0$. This together with (7), implies that

$$\begin{aligned}
\|T_n y_n - T_{n-1}y_{n-1}\| &\le (1-\tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)\|x_n - x_{n-1}\| + \tfrac{|\alpha_n-\alpha_{n-1}|}{1-\alpha_n\delta}\|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| \\
&\quad + \tfrac{1-\alpha_n}{1-\alpha_n\delta}\|S_nGx_{n-1} - S_{n-1}Gx_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1 \\
&\le \|x_n - x_{n-1}\| + \tfrac{|\alpha_n-\alpha_{n-1}|}{1-\alpha_n\delta}\|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| \\
&\quad + \|S_nGx_{n-1} - S_{n-1}Gx_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1.
\end{aligned}$$

So it follows that

$$\begin{aligned}
&\|T_n y_n - T_{n-1}y_{n-1}\| - \|x_n - x_{n-1}\| \\
&\le \tfrac{|\alpha_n-\alpha_{n-1}|}{1-\alpha_n\delta}\|f(y_{n-1}) - S_{n-1}Gx_{n-1}\| + \|S_nGx_{n-1} - S_{n-1}Gx_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1.
\end{aligned}$$

Since $\sum_{n=1}^\infty \sup_{x\in D}\|S_nx - S_{n-1}x\| < \infty$ for bounded subset $D = \{Gx_n : n \ge 0\}$ of C (due to the assumption), we know that $\lim_{n\to\infty}\|S_nGx_{n-1} - S_{n-1}Gx_{n-1}\| = 0$. Note that $\alpha_n \to 0$ and $|\lambda_n - \lambda_{n-1}| \to 0$ as $n\to\infty$. Thus, from the boundedness of $\{f(y_n)\}$ and $\{S_nGx_n\}$ we get

$$\limsup_{n\to\infty}(\|T_n y_n - T_{n-1}y_{n-1}\| - \|x_n - x_{n-1}\|) \le 0.$$

Using Suzuki's lemma [29], one yields that $\lim_{n\to\infty}\|T_n y_n - x_n\| = 0$. Hence

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|T_n y_n - x_n\| = 0. \tag{9}$$

We next denote $\bar{p} := \Pi_C(I-\mu_2A_2)p$. Note that $u_n = \Pi_C(I-\mu_2A_2)x_n$ and $v_n = \Pi_C(I-\mu_1A_1)u_n$. Then $v_n = Gx_n$. From Lemma 4, we have

$$\begin{aligned}
\|u_n - \bar{p}\|^q &\le \|(I-\mu_2A_2)x_n - (I-\mu_2A_2)p\|^q \\
&\le \|x_n - p\|^q - \mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2x_n - A_2p\|^q,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\|v_n - p\|^q &\le \|(I-\mu_1A_1)u_n - (I-\mu_1A_1)\bar{p}\|^q \\
&\le \|u_n - \bar{p}\|^q - \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q.
\end{aligned} \tag{11}$$

Substituting (10) into (11), we obtain

$$\begin{aligned}
\|v_n - p\|^q &\le \|x_n - p\|^q - \mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2x_n - A_2p\|^q \\
&\quad - \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q.
\end{aligned} \tag{12}$$

According to Lemma 1, we have

$$\begin{aligned}
\|y_n - p\|^q &\le (1-\alpha_n)\|S_nv_n - p\|^q + q\alpha_n\langle f(p) - p, J_q(y_n - p)\rangle + \alpha_n\|f(y_n) - f(p)\|^q \\
&\le (1-\alpha_n)\|v_n - p\|^q + q\alpha_n\|f(p) - p\|\|y_n - p\|^{q-1} + \alpha_n\delta\|y_n - p\|^q,
\end{aligned}$$

which hence yields

$$\|y_n - p\|^q \le (1 - \frac{1-\delta}{1-\alpha_n\delta}\alpha_n)\|v_n - p\|^q + \frac{q\alpha_n}{1-\alpha_n\delta}\|f(p) - p\|\|y_n - p\|^{q-1}.$$

That together with (12) and the nonexpansivity of T_n leads to

$$\begin{aligned}
&\|x_{n+1} - p\|^q = \|\beta_n(x_n - p) + (1-\beta_n)(T_n y_n - p)\|^q \\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)\|y_n - p\|^q \\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)\{(1 - \tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)\|v_n - p\|^q + \tfrac{q\alpha_n}{1-\alpha_n\delta}\|f(p) - p\|\|y_n - p\|^{q-1}\} \\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)(1 - \tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[\|x_n - p\|^q \\
&\quad - \mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2x_n - A_2p\|^q - \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] + \alpha_n M_2 \\
&= (1 - \tfrac{(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}\alpha_n)\|x_n - p\|^q - (1-\beta_n)(1 - \tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2x_n - A_2p\|^q \\
&\quad + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] + \alpha_n M_2 \\
&\le \|x_n - p\|^q - (1-\beta_n)(1 - \tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2x_n - A_2p\|^q \\
&\quad + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] + \alpha_n M_2,
\end{aligned} \tag{13}$$

where $\sup_{n\ge 0}\{\frac{q(1-\beta_n)}{1-\alpha_n\delta}\|f(p) - p\|\|y_n - p\|^{q-1}\} \le M_2$ for some $M_2 > 0$. So it follows from (13) and Lemma 2 that

$$\begin{aligned}
&(1-\beta_n)(1 - \tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2x_n - A_2p\|^q + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] \\
&\le \|x_n - p\|^q - \|x_{n+1} - p\|^q + \alpha_n M_2 \\
&\le q\|x_n - x_{n+1}\|\|x_{n+1} - p\|^{q-1} + \kappa_q\|x_n - x_{n+1}\|^q + \alpha_n M_2.
\end{aligned}$$

Since $(\frac{q\alpha_1}{\kappa_q})^{(\frac{1}{q-1})} > \mu_1$ and $(\frac{q\alpha_2}{\kappa_q})^{(\frac{1}{q-1})} > \mu_2$, we get from conditions (i), (ii) and (9)

$$\lim_{n\to\infty}\|A_2x_n - A_2p\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|A_1u_n - A_1\bar{p}\| = 0. \tag{14}$$

Utilizing Lemma 4, we have

$$\begin{aligned}
&\|u_n - \bar{p}\|^2 = \|\Pi_C(I - \mu_2A_2)x_n - \Pi_C(I - \mu_2A_2)p\|^2 \\
&\le \langle (I - \mu_2A_2)x_n - (I - \mu_2A_2)p, J(u_n - \bar{p})\rangle \\
&= \langle x_n - p, J(u_n - \bar{p})\rangle + \mu_2\langle A_2p - A_2x_n, J(u_n - \bar{p})\rangle \\
&\le \tfrac{1}{2}[\|x_n - p\|^2 + \|u_n - \bar{p}\|^2 - g_1(\|x_n - u_n - (p - \bar{p})\|)] + \mu_2\|A_2p - A_2x_n\|\|u_n - \bar{p}\|,
\end{aligned}$$

which implies that

$$\|u_n - \bar{p}\|^2 \le \|x_n - p\|^2 - g_1(\|x_n - u_n - (p - \bar{p})\|) + 2\mu_2\|A_2p - A_2x_n\|\|u_n - \bar{p}\|. \tag{15}$$

In the same way, we derive

$$\begin{aligned}
2\|v_n - p\|^2 &\le 2\langle (I - \mu_1A_1)u_n - (I - \mu_1A_1)\bar{p}, J(v_n - p)\rangle \\
&\le \|u_n - \bar{p}\|^2 + \|v_n - p\|^2 - g_2(\|u_n - v_n + (p - \bar{p})\|) + 2\mu_1\|A_1\bar{p} - A_1u_n\|\|v_n - p\|,
\end{aligned}$$

which implies that

$$\|v_n - p\|^2 \le \|u_n - \bar{p}\|^2 - g_2(\|u_n - v_n + (p - \bar{p})\|) + 2\mu_1\|A_1\bar{p} - A_1u_n\|\|v_n - p\|. \tag{16}$$

Substituting (15) into (16), we get

$$\begin{aligned}
\|v_n - p\|^2 &\le \|x_n - p\|^2 - g_1(\|x_n - u_n - (p - \bar{p})\|) - g_2(\|u_n - v_n + (p - \bar{p})\|) \\
&\quad + 2\mu_2\|A_2p - A_2x_n\|\|u_n - \bar{p}\| + 2\mu_1\|A_1\bar{p} - A_1u_n\|\|v_n - p\|.
\end{aligned} \tag{17}$$

Furthermore,

$$\begin{aligned}\|y_n-p\|^2 &= \|\alpha_n(f(y_n)-f(p))+(1-\alpha_n)(S_nv_n-p)+\alpha_n(f(p)-p)\|^2\\ &\le \|\alpha_n(f(y_n)-f(p))+(1-\alpha_n)(S_nv_n-p)\|^2+2\alpha_n\langle f(p)-p,J(y_n-p)\rangle\\ &\le \alpha_n\|f(y_n)-f(p)\|^2+(1-\alpha_n)\|S_nv_n-p\|^2+2\alpha_n\langle f(p)-p,J(y_n-p)\rangle\\ &\le \alpha_n\delta\|y_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2+2\alpha_n\|f(p)-p\|\|y_n-p\|,\end{aligned}$$

which together with (17), leads to

$$\begin{aligned}&\|y_n-p\|^2 \le (1-\tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)\|v_n-p\|^2+\tfrac{2\alpha_n}{1-\alpha_n\delta}\|f(p)-p\|\|y_n-p\|\\ &\le (1-\tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[\|x_n-p\|^2-g_1(\|x_n-u_n-(p-\bar{p})\|)-g_2(\|u_n-v_n+(p-\bar{p})\|)\\ &\quad +2\mu_2\|A_2p-A_2x_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|]+\tfrac{2\alpha_n}{1-\alpha_n\delta}\|f(p)-p\|\|y_n-p\|\\ &\le (1-\tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[\|x_n-p\|^2-g_1(\|x_n-u_n-(p-\bar{p})\|)-g_2(\|u_n-v_n+(p-\bar{p})\|)]\\ &\quad +2\mu_2\|A_2p-A_2x_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\tfrac{2\alpha_n}{1-\alpha_n\delta}\|f(p)-p\|\|y_n-p\|.\end{aligned} \tag{18}$$

Thus, we obtain from (4) and (18) that

$$\begin{aligned}&\|x_{n+1}-p\|^2=\|\beta_n(x_n-p)+(1-\beta_n)(T_ny_n-p)\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\{(1-\tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[\|x_n-p\|^2-g_1(\|x_n-u_n-(p-\bar{p})\|)\\ &\quad -g_2(\|u_n-v_n+(p-\bar{p})\|)]+2\mu_2\|A_2p-A_2x_n\|\|u_n-\bar{p}\|\\ &\quad +2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\tfrac{2\alpha_n}{1-\alpha_n\delta}\|f(p)-p\|\|y_n-p\|\}\\ &\le (1-\tfrac{(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}\alpha_n)\|x_n-p\|^2-(1-\beta_n)(1-\tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[g_1(\|x_n-u_n-(p-\bar{p})\|)\\ &\quad +g_2(\|u_n-v_n+(p-\bar{p})\|)]+2\mu_2\|A_2p-A_2x_n\|\|u_n-\bar{p}\|\\ &\quad +2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\tfrac{2\alpha_n}{1-\alpha_n\delta}\|f(p)-p\|\|y_n-p\|\\ &\le \|x_n-p\|^2-(1-\beta_n)(1-\tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[g_1(\|x_n-u_n-(p-\bar{p})\|)+g_2(\|u_n-v_n+(p-\bar{p})\|)]\\ &\quad +2\mu_2\|A_2p-A_2x_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\tfrac{2\alpha_n}{1-\alpha_n\delta}\|f(p)-p\|\|y_n-p\|,\end{aligned}$$

which hence yields

$$\begin{aligned}&(1-\beta_n)(1-\tfrac{1-\delta}{1-\alpha_n\delta}\alpha_n)[g_1(\|x_n-u_n-(p-\bar{p})\|)+g_2(\|u_n-v_n+(p-\bar{p})\|)]\\ &\le 2\mu_2\|A_2p-A_2x_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|\\ &\quad +\tfrac{2\alpha_n}{1-\alpha_n\delta}\|f(p)-p\|\|y_n-p\|+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+2\mu_2\|A_2p-A_2x_n\|\|u_n-\bar{p}\|\\ &\quad +2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\tfrac{2\alpha_n}{1-\alpha_n\delta}\|f(p)-p\|\|y_n-p\|.\end{aligned}$$

Utilizing conditions (i), (ii), (9) and (14), we have

$$\lim_{n\to\infty} g_1(\|x_n-u_n-(p-\bar{p})\|)=0 \quad \text{and} \quad \lim_{n\to\infty} g_2(\|u_n-v_n+(p-\bar{p})\|)=0.$$

Utilizing the properties of g_1 and g_2, we deduce that

$$\lim_{n\to\infty}\|x_n-u_n-(p-\bar{p})\|=0 \quad \text{and} \quad \lim_{n\to\infty}\|u_n-v_n+(p-\bar{p})\|=0. \tag{19}$$

From (19) we obtain

$$\|x_n-Gx_n\|=\|x_n-v_n\|\le\|x_n-u_n-(p-\bar{p})\|+\|u_n-v_n+(p-\bar{p})\|\to 0 \quad \text{as } n\to\infty. \tag{20}$$

Next, we claim $\|x_n-Sx_n\|\to 0$, $\|x_n-T_\lambda x_n\|\to 0$ and $\|x_n-Wx_n\|\to 0$ as $n\to\infty$, where $Sx=\lim_{n\to\infty}S_nx\ \forall x\in C$, $T_\lambda=J_\lambda^B(I-\lambda A)$ and $Wx=\theta_1Sx+\theta_2Gx+\theta_3T_\lambda x\ \forall x\in C$ for constants $\theta_1,\theta_2,\theta_3\in(0,1)$ satisfying $\theta_1+\theta_2+\theta_3=1$. Indeed, since $y_n=\alpha_nf(y_n)+(1-\alpha_n)S_nGx_n$ leads to $\|S_nGx_n-y_n\|=\frac{\alpha_n}{1-\alpha_n}\|f(y_n)-y_n\|$, we deduce from (20), $\alpha_n\to 0$ and $x_n-y_n\to 0$ (due to the assumption of the sufficiency) that

$$\begin{aligned}\|S_nx_n-x_n\| &\le \|S_nx_n-S_nGx_n\|+\|S_nGx_n-y_n\|+\|y_n-x_n\|\\ &\le \|x_n-Gx_n\|+\tfrac{\alpha_n}{1-\alpha_n}\|f(y_n)-y_n\|+\|y_n-x_n\|\to 0 \quad (n\to\infty),\end{aligned}$$

which ensures that

$$\|Sx_n - x_n\| \le \|Sx_n - S_nx_n\| + \|S_nx_n - x_n\| \to 0 \quad (n \to \infty). \tag{21}$$

Furthermore, utilizing the similar arguments to those of (8), we obtain

$$\begin{aligned}\|T_ny_n - T_\lambda y_n\| &\le |1 - \tfrac{\lambda}{\lambda_n}|\|J^B_{\lambda_n}(I - \lambda_nA)y_n - (I - \lambda_nA)y_n\| + |\lambda_n - \lambda|\|Ay_n\| \\ &\le |1 - \tfrac{\lambda}{\lambda_n}|\|T_ny_n - (I - \lambda_nA)y_n\| + |\lambda_n - \lambda|\|Ay_n\|.\end{aligned}$$

Since $\lim_{n\to\infty} \lambda_n = \lambda$ and the sequences $\{y_n\}, \{T_ny_n\}, \{Ay_n\}$ are bounded, we get

$$\lim_{n\to\infty} \|T_ny_n - T_\lambda y_n\| = 0. \tag{22}$$

Taking into account condition (iii), i.e., $0 < \bar{\lambda} \le \lambda_n \ \forall n \ge 0$ and $\lim_{n\to\infty} \lambda_n = \lambda < (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$, we know that $0 < \bar{\lambda} \le \lambda < (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$. So it follows that $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0$ and $T_\lambda : C \to C$ is nonexpansive. Since $x_{n+1} = \beta_nx_n + (1 - \beta_n)T_ny_n$ leads to $\|T_ny_n - x_n\| = \frac{1}{1-\beta_n}\|x_{n+1} - x_n\|$, we deduce from (19), (22), $x_n - y_n \to 0$ and $\liminf_{n\to\infty}(1 - \beta_n) > 0$ that

$$\begin{aligned}\|T_\lambda x_n - x_n\| &\le \|T_\lambda x_n - T_\lambda y_n\| + \|T_\lambda y_n - T_ny_n\| + \|T_ny_n - x_n\| \\ &\le \|x_n - y_n\| + \|T_\lambda y_n - T_ny_n\| + \tfrac{1}{1-\beta_n}\|x_{n+1} - x_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{23}$$

We now define the mapping $Wx = \theta_1Sx + \theta_2Gx + \theta_3T_\lambda x \ \forall x \in C$ for constants $\theta_1, \theta_2, \theta_3 \in (0, 1)$ satisfying $\theta_1 + \theta_2 + \theta_3 = 1$. Then by Lemma 5 we know that $\mathrm{Fix}(W) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T_\lambda) = \Omega$. Observe that

$$\begin{aligned}\|x_n - Wx_n\| &= \|\theta_1(x_n - Sx_n) + \theta_2(x_n - Gx_n) + \theta_3(x_n - T_\lambda x_n)\| \\ &\le \theta_1\|x_n - Sx_n\| + \theta_2\|x_n - Gx_n\| + \theta_3\|x_n - T_\lambda x_n\|.\end{aligned} \tag{24}$$

From (20), (21), (23) and (24), we get

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{25}$$

Next, we focus on

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \le 0, \tag{26}$$

where $x^* =$s-$\lim_{n\to\infty} x_t$ with x_t being a fixed point of the contraction $x_t \mapsto tf(x) + (1 - t)Wx$ for each $t \in (0, 1)$. Indeed, one guarantees that for each $t \in (0, 1)$, x_t solves the fixed point equation $x_t = tf(x_t) + (1 - t)Wx_t$. Hence $\|x_t - x_n\| = \|(1 - t)(Wx_t - x_n) + t(f(x_t) - x_n)\|$. By Lemma 1, we conclude that

$$\begin{aligned}&\|x_t - x_n\|^2 = \|(1 - t)(Wx_t - x_n) + t(f(x_t) - x_n)\|^2 \\ &\le (1 - t)^2\|Wx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\ &\le (1 - t)^2(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\ &\le (1 - t)^2(\|x_t - x_n\| + \|Wx_n - x_n\|)^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\ &= (1 - t)^2[\|x_t - x_n\|^2 + 2\|x_t - x_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2] \\ &\quad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\langle x_t - x_n, J(x_t - x_n)\rangle \\ &= (1 - 2t + t^2)\|x_t - x_n\|^2 + f_n(t) + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2,\end{aligned} \tag{27}$$

where

$$f_n(t) = (1 - t)^2(2\|x_t - x_n\| + \|x_n - Wx_n\|)\|x_n - Wx_n\| \to 0 \quad (n \to \infty). \tag{28}$$

It follows from (27) that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{29}$$

Letting $n \to \infty$ in (29) and noticing (28), we derive

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}M_3, \tag{30}$$

where $\sup_{t\in(0,1),n\ge0}\|x_t - x_n\|^2 \le M_3$ for some $M_3 > 0$. Taking $t \to 0$ in (30), we have

$$\limsup_{t\to0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0.$$

On the other hand, we have

$$\begin{aligned}&\langle f(x^*) - x^*, J(x_n - x^*)\rangle = \langle f(x^*) - x^*, J(x_n - x^*)\rangle - \langle f(x^*) - x^*, J(x_n - x_t)\rangle\\ &\quad + \langle f(x^*) - x^*, J(x_n - x_t)\rangle - \langle f(x^*) - x_t, J(x_n - x_t)\rangle + \langle f(x^*) - x_t, J(x_n - x_t)\rangle\\ &\quad - \langle f(x_t) - x_t, J(x_n - x_t)\rangle + \langle f(x_t) - x_t, J(x_n - x_t)\rangle\\ &= \langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle + \langle x_t - x^*, J(x_n - x_t)\rangle\\ &\quad + \langle f(x^*) - f(x_t), J(x_n - x_t)\rangle + \langle f(x_t) - x_t, J(x_n - x_t)\rangle.\end{aligned}$$

So it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle &\le \limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle\\ &+(1+\delta)\|x_t - x^*\|\limsup_{n\to\infty}\|x_n - x_t\| + \limsup_{n\to\infty}\langle f(x_t) - x_t, J(x_n - x_t)\rangle.\end{aligned}$$

Taking into account that $x_t \to x^*$ as $t \to 0$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle &= \limsup_{t\to0}\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle\\ &\le \limsup_{t\to0}\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle.\end{aligned} \tag{31}$$

Since X is uniformly smooth, the normalized duality mapping J is norm-to-norm uniformly continuous on bounded subsets of X. Therefore, the two limits are interchangeable and hence (26) holds. According to the assumption $x_n - y_n \to 0$ of the sufficiency, we get $J(y_n - x^*) - J(x_n - x^*) \to 0$. Thus, we conclude from (26) that

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(x^*) - x^*, J(y_n - x^*)\rangle = \limsup_{n\to\infty}\{\langle f(x^*) - x^*, J(x_n - x^*)\rangle\\ &+\langle f(x^*) - x^*, J(y_n - x^*) - J(x_n - x^*)\rangle\} = \limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \le 0.\end{aligned} \tag{32}$$

Finally,

$$\begin{aligned}\|y_n - x^*\|^2 &= \|(1-\alpha_n)(S_nGx_n - x^*) + \alpha_n(f(x^*) - x^*) + \alpha_n(f(y_n) - f(x^*))\|^2\\ &\le \|\alpha_n(f(y_n) - f(x^*)) + (1-\alpha_n)(S_nGx_n - x^*)\|^2 + 2\alpha_n\langle f(x^*) - x^*, J(y_n - x^*)\rangle\\ &\le \alpha_n\|f(y_n) - f(x^*)\|^2 + (1-\alpha_n)\|S_nGx_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, J(y_n - x^*)\rangle\\ &\le \alpha_n\delta\|y_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, J(y_n - x^*)\rangle,\end{aligned}$$

which hence yields

$$\|y_n - x^*\|^2 \le (1 - \frac{\alpha_n(1-\delta)}{1-\alpha_n\delta})\|x_n - x^*\|^2 + \frac{\alpha_n(1-\delta)}{1-\alpha_n\delta}\cdot\frac{2\langle f(x^*) - x^*, J(y_n - x^*)\rangle}{1-\delta}. \tag{33}$$

By the convexity of $\|\cdot\|^2$, the nonexpansivity of T_n and (5), we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|T_ny_n - x^*\|^2\\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2,\end{aligned}$$

which together with (33) leads to

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\{(1-\tfrac{\alpha_n(1-\delta)}{1-\alpha_n\delta})\|x_n-x^*\|^2 \\ &\quad + \tfrac{\alpha_n(1-\delta)}{1-\alpha_n\delta}\cdot\tfrac{2\langle f(x^*)-x^*,J(y_n-x^*)\rangle}{1-\delta}\} \\ &= [1-\tfrac{\alpha_n(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}]\|x_n-x^*\|^2+\tfrac{\alpha_n(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}\cdot\tfrac{2\langle f(x^*)-x^*,J(y_n-x^*)\rangle}{1-\delta}. \end{aligned} \tag{34}$$

Since $\liminf_{n\to\infty}\frac{(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}>0$, $\{\frac{(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}\}\subset(0,1)$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$, we know that $\{\frac{\alpha_n(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}\}\subset(0,1)$ and $\sum_{n=0}^{\infty}\frac{\alpha_n(1-\beta_n)(1-\delta)}{1-\alpha_n\delta}=\infty$. Utilizing (32) and Lemma 7, we conclude from (34) that $\|x_n-x^*\|\to 0$ as $n\to\infty$. This completes the proof.

Remark 1. The problem of finding an element of $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GSVI}(C,A_1,A_2)$ in [20, Theorem 3.1] is extended to develop our problem of finding an element of $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GSVI}(C,A_1,A_2)\cap(A+B)^{-1}0$ where $(A+B)^{-1}0$ is the solution set of the VI: $0\in(A+B)x$. The implicit (two-step) relaxed extragradient method in [28], Theorem 1 is extended to develop our modified implicit extragradient method (4). That is, two iterative steps $y_n=\alpha_n f(y_n)+(1-\alpha_n)Gx_n$ and $x_{n+1}=\beta_n x_n+(1-\beta_n)S_n y_n$ in [20], Theorem 1 is extended to develop our two iterative steps $y_n=\alpha_n f(y_n)+(1-\alpha_n)S_nGx_n$ and $x_{n+1}=\beta_n x_n+(1-\beta_n)T_n y_n$, where $T_n=J^B_{\lambda_n}(I-\lambda_nA)$. In addition, the uniformly convex and 2-uniformly smooth Banach space X in [20], Theorem 1 is extended to the uniformly convex and q-uniformly smooth Banach space X in our Theorem 1, where $1<q\le 2$. The problem of finding an element of $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GSVI}(C,A_1,A_2)$ in [28], Theorem 1 is extended to develop our problem of finding an element of $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GSVI}(C,A_1,A_2)\cap(A+B)^{-1}0$ where $(A+B)^{-1}0$ is the solution set of the VI: $0\in(A+B)x$. The modified relaxed extragradient method in [28], Theorem 1 is extended to develop our modified implicit extragradient method (4). That is, two iterative steps $y_n=(1-\beta_n)x_n+\beta_nGx_n$ and $x_{n+1}=\Pi_C[\alpha_n\gamma f(x_n)+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\rho F)S_ny_n]$ in [28], Theorem 1 is extended to develop our two iterative steps $y_n=\alpha_n f(y_n)+(1-\alpha_n)S_nGx_n$ and $x_{n+1}=\beta_nx_n+(1-\beta_n)T_ny_n$, where $T_n=J^B_{\lambda_n}(I-\lambda_nA)$.

Let $g:H\to\mathbf{R}$ be a convex smooth function and $h:H\to\mathbf{R}$ be a proper convex and lower semicontinuous function. The convex minimization problem is to find $x^*\in H$ such that

$$g(x^*)+h(x^*)=\min_{x\in H}\{g(x)+h(x)\}.$$

By Fermat's rule, we know that the above problem is equivalent to the problem of finding $x^*\in H$ such that $0\in\nabla g(x^*)+\partial h(x^*)$, where ∇g is the gradient of g and ∂h is the subdifferential of h. It is also known that if ∇g is $\frac{1}{\alpha}$-Lipschitz continuous, then it is also α-inverse-strongly monotone. From Theorem 1, we can obtain the following result.

Theorem 2. Let $g:H\to\mathbf{R}$ be a convex and differentiable function with $\frac{1}{\alpha}$-Lipschitz continuous gradient ∇g and $h:H\to\mathbf{R}$ be a convex and lower semicontinuous function. Let the mapping $A_i:C\to H$ be α_i-inverse-strongly monotone for $i=1,2$. Let S be a nonexpansive self-mapping on C such that $\Omega=\mathrm{Fix}(S)\cap\mathrm{GSVI}(C,A_1,A_2)\cap(\nabla g+\partial h)^{-1}0\neq\emptyset$ where $(\nabla g+\partial h)^{-1}0$ is the set of minimizers attained by $g+h$, and $\mathrm{GSVI}(C,A_1,A_2)$ is the fixed point set of $G:=P_C(I-\mu_1A_1)P_C(I-\mu_2A_2)$ with $0<\mu_i<2\alpha_i$ for $i=1,2$. Let $f:C\to C$ be a δ-contraction with constant $\delta\in(0,1)$. For arbitrarily given $x_0\in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} v_n=P_C(I-\mu_1A_1)P_C(x_n-\mu_2A_2x_n),\\ y_n=(1-\alpha_n)Sv_n+\alpha_nf(y_n),\\ x_{n+1}=(1-\beta_n)J^{\partial h}_{\lambda_n}(y_n-\lambda_n\nabla g(y_n))+\beta_nx_n, \quad n\ge 0, \end{cases}$$

where $\{\lambda_n\}\subset(0,2\alpha)$, and $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty}\alpha_n=\infty$ and $\lim_{n\to\infty}\alpha_n=0$;

(ii) $\limsup_{n\to\infty}\beta_n<1$ and $\liminf_{n\to\infty}\beta_n>0$;

(iii) $0<\bar{\lambda}\le\lambda_n\ \forall n\ge 0$ and $\lim_{n\to\infty}\lambda_n=\lambda<2\alpha$.

Then $x_n\to x^*\in\Omega\iff x_n-y_n\to 0$, where $x^*\in\Omega$ is a unique solution to the variational inequality: $\langle(I-f)x^*,x^*-p\rangle\le 0\ \ \forall p\in\Omega$.

Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a linear bounded operator with its adjoint T^*. Consider the split feasibility problem (SFP) of finding a point x^* with the property that $x^* \in C \quad \text{and} \quad Tx^* \in Q$. The SFP can be used to model the intensity-modulated radiation therapy. It is clear that the set of solutions of the SFP is $C \cap T^{-1}Q$. To solve the SFP, we can rewrite it as the following convexly constrained minimization problem: $\min_{x \in C} g(x) := \frac{1}{2}\|Tx - P_Q Tx\|^2$. Note that the function g is differentiable convex and has a Lipschitz gradient given by $\nabla g = T^*(I - P_Q)T$. Further, ∇g is $\frac{1}{\|T\|^2}$-inverse-strongly monotone, where $\|T\|^2$ is the spectral radius of T^*T. Thus, x^* solves the SFP if and only if x^* solves the variational inclusion problem of finding $x^* \in H_1$ such that

$$\begin{aligned} 0 \in \nabla g(x^*) + \partial I_C(x^*) \quad & \Leftrightarrow \ x^* - \lambda \nabla g(x^*) \in (I + \lambda \partial I_C)x^* \\ & \Leftrightarrow \ J_\lambda^{\partial I_C}(x^* - \lambda \nabla g(x^*)) = x^* \\ & \Leftrightarrow \ P_C(x^* - \lambda \nabla g(x^*)) = x^*. \end{aligned}$$

From Theorem 1, we can obtain the following result.

Theorem 3. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator with its adjoint T^*. Let the mapping $A_i : C \to H_1$ be α_i-inverse-strongly monotone for $i = 1, 2$. Let S be a nonexpansive self-mapping on C such that $\Omega = \mathrm{Fix}(S) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (C \cap T^{-1}Q) \neq \emptyset$ where $\mathrm{GSVI}(C, A_1, A_2)$ is the fixed point set of $G := P_C(I - \mu_1 A_1)P_C(I - \mu_2 A_2)$ with $0 < \mu_i < 2\alpha_i$ for $i = 1, 2$. Let $f : C \to C$ be a δ-contraction with constant $\delta \in (0, 1)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} v_n = P_C(I - \mu_1 A_1)P_C(x_n - \mu_2 A_2 x_n), \\ y_n = (1 - \alpha_n)Sv_n + \alpha_n f(y_n), \\ x_{n+1} = (1 - \beta_n)P_C(y_n - \lambda_n T^*(I - P_Q)Ty_n) + \beta_n x_n, \quad n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset (0, \frac{2}{\|T\|^2})$, and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfy the following conditions:
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \beta_n > 0$;
(iii) $0 < \bar{\lambda} \leq \lambda_n \ \forall n \geq 0$ and $\lim_{n\to\infty} \lambda_n = \lambda < \frac{2}{\|T\|^2}$.
Then $x_n \to x^* \in \Omega \ \Leftrightarrow \ x_n - y_n \to 0$, where $x^* \in \Omega$ is a unique solution to the variational inequality: $\langle (I - f)x^*, x^* - p\rangle \leq 0 \ \ \forall p \in \Omega$.

References

[1] Q.H. Ansari, M. Islam, J.C. Yao, Nonsmooth variational inequalities on Hadamard manifolds, Appl. Anal. 99 (2020), 340-358.
[2] Q.H. Ansari, Regularization of proximal point algorithms in Hadamard manifolds, J. Fixed Point Theory Appl. 21 (2019), Article ID 25.
[3] N.T. An, N.M. Nam, Solving k-center problems involving sets based on optimization techniques, J. Glob. Optim. 67 (2020), 189–209.
[4] K. Aoyama, H. Iiduka, W. Takahashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces, Fixed Point Theory Appl. 2006 (2006), Article ID 35390.
[5] B.A. Bin Dehaish, Weak and strong convergence of algorithms for the sum of two accretive operators with applications, J. Nonlinear Convex Anal. 16 (2015), 1321-1336.
[6] B.A. Bin Dehaish, A regularization projection algorithm for various problems with nonlinear mappings in Hilbert spaces, J. Ineqal. Appl. 2015 (2015), Article ID 51.
[7] R.E. Bruck, Properties of fixed point sets of nonexpansive mappings in Banach spaces, Trans. Amer. Math. Soc. 179 (1973), 251-262.
[8] L.C. Ceng, Q.H. Ansari, J.C. Yao, Some iterative methods for finding fixed points and for solving constrained convex minimization problems, Nonlinear Anal. 74 (2011), 5286-5302.
[9] L.C. Ceng, A. Petrusel, J.C. Yao, Y. Yao, Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces, Fixed Point Theory 19 (2018), 487-501.
[10] L.C. Ceng, A. Petrusel, Y. Yao, J.C. Yao, Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions, Fixed Point Theory 20 (2019), 113-133.
[11] L.C. Ceng, M. Shang, Generalized Mann viscosity implicit rules for solving systems of variational inequalities with constraints of variational inclusions and fixed point problems, Mathematics, 7 (2019), 933.
[12] S.S. Chang, C.F. Wen, J.C. Yao, Zero point problem of accretive operators in Banach spaces, Bull. Malaysian Math. Sci. Soc. 42 (2019), 105-118.

[13] S.S. Chang, C.F. Wen, J.C. Yao, Common zero point for a finite family of inclusion problems of accretive mappings in Banach spaces, Optimization 67 (2019), 1183-1196.

[14] S.Y. Cho, S.M. Kang, Approximation of common solutions of variational inequalities via strict pseudocontractions, Acta Math. Sci. 32 (2012), 1607-1618.

[15] S.Y. Cho, Generalized mixed equilibrium and fixed point problems in a Banach space, J. Nonlinear Sci. Appl. 9 (2016), 1083-1092.

[16] S.Y. Cho, Strong convergence analysis of a hybrid algorithm for nonlinear operators in a Banach space, J. Appl. Anal. Comput. 8 (2018), 19-31.

[17] S.Y. Cho, On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems, Appl. Math. Comput. 235 (2014), 430-438.

[18] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002), 938-945.

[19] W. Li, S.M. Kang, Convergence analysis of an iterative algorithm for monotone operators, J. Inequal. Appl. 2013 (2013), Article ID 199.

[20] A. Latif, On solutions of a system of variational inequalities and fixed point problems in Banach spaces, Fixed Point Theory Appl. 2013 (2013), 176.

[21] L.V. Nguyen, Some results on strongly pseudomonotone quasi-variational inequalities, Set-Valued Var. Anal. (2019), 10.1007/s11228-019-00508-1.

[22] X. Qin, L. Wang, J.C. Yao, Inertial splitting method for maximal monotone mappings, J. Nonlinear Convex Anal. 21 (2020), in press.

[23] X. Qin, J.C. Yao, A viscosity iterative method for a split feasibility problem, J. Nonlinear Convex Anal. 20 (2019), 1497–1506.

[24] X. Qin, N.T. An, Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets, Comput. Optim. Appl. 74 (2019), 821–850.

[25] X. Qin, A regularization method for treating zero points of the sum of two monotone operators, Fixed Point Theory Appl. 2014 (2014), Article ID 75.

[26] X. Qin, L. Wang, Iterative algorithms with errors for zero points of m-accretive operators, Fixed Point Theory Appl. 2013 (2013), Article ID 148.

[27] D.R. Sahu, Convergence rate analysis of proximal gradient methods with applications to composite minimization problems, Optimization, (2020), 10.1080/02331934.2019.1702040

[28] Y.L. Song, L.C. Ceng, A general iteration scheme for variational inequality problem and common fixed point problems of nonexpansive mappings in q-uniformly smooth Banach spaces, J. Glob. Optim. 57 (2013), 327-1348.

[29] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305 (2005), 227-239.

[30] W. Takahashi, C.F. Wen, The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space, Fixed Point Theory 19 (2018), 407-419.

[31] W. Takahashi, J.C. Yao, The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces, J. Nonlinear Convex Anal. 20 (2019), 173-195.

[32] W. Takahashi, H.K. Xu, Iterative methods for generalized split feasibility problems in Hilbert spaces, Set-Valued Var. Anal. 23 (2015), 205-221.

[33] W. Takahashi, Nonlinear Functional Analysis -Fixed Point Theory and its Applications, Yokohama Publishers, 2009.

[34] Z. Xue, H. Zhou, Y.J. Cho, Iterative solutions of nonlinear equations for m-accretive operators in Banach spaces, J. Nonlinear Convex Anal. 1 (2000), 313-320.

[35] X. Zhao, et al., Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems, Appl. Math. Optim. 78 (2018), 613-641.