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Uniqueness Part of Schwarz Lemma for Driving Point Impedance Functions

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Abstract. In this paper, a boundary version of the uniqueness part of the Schwarz lemma for driving point impedance functions has been investigated. Also, more general results have been obtained for a different version of the Burns-Krantz uniqueness theorem. In these results, as different from the Burns-Krantz theorem, only the boundary points have been used as the conditions on the function. Also, more general majorants will be taken instead of power majorants in (1.1).

1. Introduction

Positive real functions are frequently utilized in electrical engineering as driving point impedance (DPI) functions. In circuit synthesis and analysis, DPI functions are used to determine the characteristic properties of electrical circuits. Satisfying the properties of positive real functions is important for DPI functions since this makes them physically realizable. The properties of positive real functions are given as follows [24]:

1-) $Z(s)$ is analytic and single valued in $\Re s \geq 0$ except possibly for poles on the axis of imaginaries,

2-) $Z(\overline{s}) = \overline{Z(s)}$

3-) $\Re Z(s) \geq 0$, in $\Re s \geq 0$

In electrical engineering, derivative of DPI functions is used for network analysis or synthesis. While some of the studies mathematically consider the derivative of DPI functions, some use it to implement practical applications. For example, in [9], derivative of DPI function was used to design a gyrator. On the other hand, its novel mathematical properties were investigated in [11]. Here, as in [24, 25], we are aiming to give a bound for the derivative of positive real functions. A similar study which we have evaluated the the derivative of driving point impedance functions at the origin is given in [21, 22].

2. Preliminary Considerations

The most classical version of the Schwarz Lemma examines the behavior of a bounded, analytic function mapping the origin to the origin in the unit disc $D = \{\lambda : |\lambda| < 1\}$. It is possible to see its effectiveness in the

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proofs of many important theorems. The Schwarz Lemma, which has broad applications and is the direct application of the maximum modulus principle, is given in the most basic form as follows ([6], p.329):

Let us consider a function $f(\lambda)$ an analytic in the unit disc D with $f(D) \subset D$. The Schwarz lemma asserts that $\left|f(\lambda)\right| \leq |\lambda|$, for every $\lambda \in D$ and $\left|f'(0)\right| \leq 1$. In addition, if the equality $\left|f(\lambda)\right| = |\lambda|$ holds for any $\lambda \neq 0$, or $\left|f'(0)\right| = 1$, then f is a rotation, that is, $f(\lambda) = \lambda e^{i\theta}$, θ is a real. For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to [13–16].

In recent years, a boundary version of Schwarz lemma was investigated in S.G. Krantz [10], D. Burns and S.G. Krantz [5], Dov Chelts [6], H. P. Boas [4], M. Mateljevi´c [12], Ornek-Akyel [19, 20] and a few other authors' papers. They studied the uniqueness portion of the Schwarz lemma. Also, in the last 15 years, there have been tremendous studies on Schwarz lemma at the boundary (see, [2–4, 10, 12, 17, 18, 23, 26] and references therein). Some of them are about the below boundary of modulus of the functions derivation at the points (contact points) which satisfies condition of the boundary of the unit circle.

D.M.Burns and S.G.Krantz [5] established exact terms in which the rigidity at the boundary can be stated and they proved the following result.

Theorem 2.1. *Let $f : D \to D$ be an analytic function from the unit disc to itself such that*

$$f(\lambda) = \lambda + O\left((\lambda - 1)^4\right) \tag{1.1}$$

as $\lambda \to 1$. Then $f(\lambda) = \lambda$ on the disc.

In the same paper, it is shown that the exponent 4 is optimal by giving the function $f(\lambda) = \lambda + \frac{1}{10}(\lambda - 1)^3$ as an example. Also, it is noted that it is enough to take the condition $\left(o(\lambda - 1)^3\right)$ instead of $O\left((\lambda - 1)^4\right)$.

The Burns-Krantz Theorem was improved in 1995 by Thomas L. Kriete and Barbara D. MacCluer [26], who replaced f with its real part and considered the radial limit in $\left(o(\lambda - 1)^3\right)$ instead of the unrestricted limit. Here is a more precise statement of their result.

Theorem 2.2. *Let $f : D \to D$ be an analytic function with radial limit $f(1) = 1$ and angular derivative $f'(1) = 1$. If*

$$\liminf_{r \to 1^-} \frac{\Re\left(f(r) - r\right)}{(1 - r)^3} = 0,$$

then $f(\lambda) = \lambda$.

Conditions on the local behavior of f near a finite set of boundary points which ensure that f is a finite Blaschke product were established by Dov Chelst [6].

Theorem 2.3. *Let $f : D \to D$ be an analytic function from the unit disc to itself.In addition, let $\phi : D \to D$ be a finite Blaschke product which equals $\tau \in \partial D$ on a finite set $A_f \subset \partial D$. If (i) for a given $\gamma_0 \in A_f$,*

$$f(\lambda) = \phi(\lambda) + O\left((\lambda - \gamma_0)^4\right), \text{ as } \lambda \to \gamma_0,$$

and (ii) for all $\gamma \in A_f - \{\gamma_0\}$,

$$f(\lambda) = \phi(\lambda) + O\left((\lambda - \gamma_0)^{k_\gamma}\right), \text{ for some } k_\gamma \geq 2 \text{ as } \lambda \to \gamma,$$

then $f(\lambda) = \phi(\lambda)$ on the disc.

In 2015, Miodrag Mateljevic obtained the following result by using inner function instead of Blaschke product [12].

Theorem 2.4. *Let $f : D \to D$ be an analytic function. Let B be an inner function function which equals 1 precisely on a set $A \subset \partial D$. Suppose the following condition are satisfied (a) for all $a \in A$*

$$f(e^{it}) = B(e^{it}) + o\left(\left(e^{it} - a\right)^2\right),\ e^{it} \in \partial D,\ e^{it} \to a,$$

(b) there is a $a_0 \in A$ such that

$$f(e^{it}) = B(e^{it}) + o\left(\left(e^{it} - a_0\right)^3\right),\ e^{it} \in \partial D,\ e^{it} \to a_0.$$

Then $f \equiv B$ on all of D.

Let $\mathfrak{N}$ be a class of functions $\mu : (0, +\infty) \to (0, +\infty)$ for each of which $\log \mu(x)$ is concave with respect to $\log x$. For each function $\mu \in \mathfrak{N}$ the limit

$$\mu_0 = \lim_{x \to 0} \frac{\log \mu(x)}{\log x},$$

exists, and $-\infty < \mu_0 \leq +\infty$. Here, the function $\mu \in \mathfrak{N}$ is called bilogaritmic concave majorant [27]. Obviously $x^\alpha \in \mathfrak{N}$ for any $\alpha > 0$.

Let $U(\lambda, r)$ be an open disc with centre λ and radius r. We propose the following assertion for the proof of our theorem [1].

Lemma 2.5. *Let $u = u(\lambda)$ be a positive harmonic function on the open disc $U(\lambda, r_0)$, $r_0 > 0$. Assume that for $\theta_0 \in [0, 2\pi)$, $\lim\limits_{r \to r_0} u(re^{i\theta_0})$ is satisfied. Then*

$$\liminf_{r \to r_0} \frac{u(re^{i\theta_0})}{r_0 - r} > 0.$$

3. Main Results

In this section, more general majorants will be taken instead of power majorants in condition (1.1). In the theorem given below, the uniqueness (rigidity) part of Schwarz lemma is considered for positive real functions defined on the right half plane. Let be $\mathrm{H} = \{s \in \mathbb{C} : \Re s > 0\}$.

Theorem 3.1. *Let $Z(s)$ be a positive real function that is continuous $\overline{\mathrm{H}} \cap U(\infty, \delta_0)$ for some $\delta_0 > 0$. Suppose that $\mu \in \mathfrak{N}$ and $\mu_0 > 3$. Also, it satisfies condition*

$$Z(s) = s + O\left(\mu(\frac{1}{|s|})\right),\ s \in \partial\mathrm{H},\ s \to \infty.$$

Then $Z(s) = s$ on H.

Proof. Consider the function

$$f(\lambda) = \frac{Z(s) - 1}{Z(s) + 1},\ \lambda = \frac{s - 1}{s + 1}.$$

Here, $f(\lambda)$ is an analytic function in D and $\left|f(\lambda)\right| < 1$ for $\lambda \in D$. From the hypothesis, we have

$$f(\lambda) = \frac{Z(s) - 1}{Z(s) + 1} = \frac{s - 1 + O\left(\mu(\frac{1}{s})\right)}{s + 1 + O\left(\mu(\frac{1}{s})\right)} = \frac{s - 1 + O\left(\mu(\frac{2}{s+1})\right)}{s + 1 + O\left(\mu(\frac{2}{s+1})\right)}$$

and for $s = \frac{1+\lambda}{1-\lambda}$

$$f(\lambda) = \frac{\frac{1+\lambda}{1-\lambda} - 1 + O\left(\mu(\frac{2}{\frac{1+\lambda}{1-\lambda}+1})\right)}{\frac{1+\lambda}{1-\lambda} + 1 + O\left(\mu(\frac{2}{\frac{1+\lambda}{1-\lambda}+1})\right)} = \frac{\frac{2\lambda}{1-\lambda} + O\left(\mu(1-\lambda)\right)}{\frac{2}{1-\lambda} + O\left(\mu(1-\lambda)\right)} = \lambda + O\left(\mu(1-\lambda)\right).$$

There exists a number $b_1 > 0$ such that

$$\left|f(\lambda) - \lambda\right| \leq b_1\mu(|1-\lambda|), \ \ \forall\lambda \in \partial D \cap U(1, \delta_0).$$

Let k and b_2 are represented as follows

$$k = \sup_{\substack{|\lambda-1|=\delta_0 \\ \lambda\in D}} \left|f(\lambda) - \lambda\right|, \ \ b_2 = \max\left\{\frac{k}{\mu(\delta_0)}, b_1\right\}.$$

It can be easliy seen that for all boundary points ofthe set $D \cap U(1, \delta_0)$, the inequality

$$\left|f(\lambda) - \lambda\right| \leq b_2\mu(|1-\lambda|)$$

is satisfied. Applying Theorem 3 in [27] to the set $D \cap U(1, \delta_0)$ and the function $f(\lambda) - \lambda$, one receives

$$\left|f(\lambda) - \lambda\right| \leq b_2\mu(|1-\lambda|), \ \ \forall\lambda \in D \cap U(1, \delta_0). \tag{2.1}$$

By $\mu_0 > 3$, follows that there exist some positive constant $\varepsilon > 0$ and $\sigma < \min(\delta_0, 1)$ such that

$$\frac{\log\mu(x)}{\log x} \geq 3 + \varepsilon, \quad \forall x \in (0, \sigma)$$

and

$$\log\mu(x) \leq (3+\varepsilon)\log x, \quad \forall x \in (0, \sigma). \tag{2.2}$$

In other words

$$\mu(x) \leq x^{3+\varepsilon}, \quad \forall x \in (0, \sigma).$$

From (2.1) and (2.2), we take

$$\left|f(\lambda) - \lambda\right| \leq b_2\left|1-\lambda\right|^{3+\varepsilon}, \quad \forall\lambda \in D \cap U(1, \sigma). \tag{2.3}$$

Consider the harmonic function g defined as

$$g(\lambda) = \Re\left(\frac{1+f(\lambda)}{1-f(\lambda)}\right) - \Re\left(\frac{1+\lambda}{1-\lambda}\right).$$

The function

$$m(\lambda) = \frac{1+f(\lambda)}{1-f(\lambda)}$$

maps the disc D to the right half plane and hence, the first term of $g(\lambda)$ is nonnegative, the second term is zero on $\partial D - \{1\}$. Consequently, when taking liminfs to any boundary point in $\partial D - \{1\}$, one always obtains a nonnegative value. With straightforward calculations, we obtain

$$g(\lambda) = \Re\left(\frac{2\left(f(\lambda) - \lambda\right)}{\left(1 - f(\lambda)\right)\left(1-\lambda\right)}\right).$$

From (2.3), we have

$$\lim_{\lambda\to 1,\ \lambda\in D}\frac{1-f(\lambda)}{1-\lambda}=1.$$

Therefore, there exist $\delta_1\in(0,\sigma)$ such that

$$\left|1-f(\lambda)\right|\geq\frac{|1-\lambda|}{2},\quad \forall\lambda\in D\cap U(1,\delta_1).$$

Thus, we obtain

$$\left|(1-f(\lambda))(1-\lambda)\right|\geq\frac{|1-\lambda|^2}{2},\quad \forall\lambda\in D\cap U(1,\delta_1)$$

and

$$\left|\frac{f(\lambda)-\lambda}{(1-f(\lambda))(1-\lambda)}\right|\leq b_2\frac{|1-\lambda|^{3+\varepsilon}}{\frac{|1-\lambda|^2}{2}}=2b_2|1-\lambda|^{1-\varepsilon},\quad \forall\lambda\in D\cap U(1,\delta_1). \tag{2.4}$$

Consequently, the function $g(\lambda)$ satisfies the following relation

$$\lim_{\lambda\to 1,\ \lambda\in D}g(\lambda)=0. \tag{2.5}$$

Applying the maximum principle [8] to the harmonic function $g(\lambda)$, we conclude either $g(\lambda)>0$, $\forall\lambda\in D$ or $g\equiv 0$. If g is not a constant, taking $\lambda=r$ in (2.4) gives us

$$\lim_{r\to 1}\frac{g(r)}{1-r}=0. \tag{2.6}$$

(2.4) and (2.6) contradict with assertion Lemma 2.5 statement. Consequently, $g\equiv 0$. So, we obtain

$$f(\lambda)=\lambda,$$

$$\frac{Z\left(\frac{1+\lambda}{1-\lambda}\right)-1}{Z\left(\frac{1+\lambda}{1-\lambda}\right)+1}=\lambda$$

and

$$Z\left(\frac{1+\lambda}{1-\lambda}\right)=\frac{1+\lambda}{1-\lambda}.$$

As a results, we obtain $Z(s)=s$. □

4. An Exemplary Application of Presented Theorem

In this section, an exemplary application of the presented theorem will be discussed. The function obtained in the previous section, corresponds to driving point impedance (DPI) function in electrical engineering. Accordingly, DPI functions represent the spectral properties of networks containing resistor-inductor (RL), resistor-capacitor (R-C) and resistor-inductor-capacitor (R-L-C) circuits. The s parameter used in DPI functions is actually the complex frequency parameter, therefore it is possible to say that DPI functions show the behaviour of the circuits in frequency domain.

In electrical engineering, the resistors are equal to a constant number in s-domain since they do not depend on frequency. However, inductors and capacitors are functions of s parameter and their values in

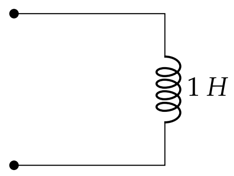

Figure 1: Circuit model for the driving point impedance function $Z(s) = s$.

frequency domain are given as sL and $\frac{1}{sC}$, respectively, where L is the value of inductor and C is the value of capacitor.

As it can be seen from Theorem 3.1, the extremal function has been obtained as $Z(s) = s$ and now, it is possible to say that this function corresponds to an inductor with the value of $1H$. This is shown in Fig. 1.

For an inductor, the magnitude of the DPI function linearly increases with the frequency since $Z(s) = sL$. Then, it is intuitive to expect a linearly varying line in frequency domain for the magnitude of the DPI function, $|Z(s)|$. In addition, it is well known in electrical engineering that the inductors causes a constant phase difference which is given as 90°. The magnitude and phase of the obtained DPI function, $Z(s) = s$, are shown in Fig. 2 where blue and red lines show the magnitude and phase responses, respectively.

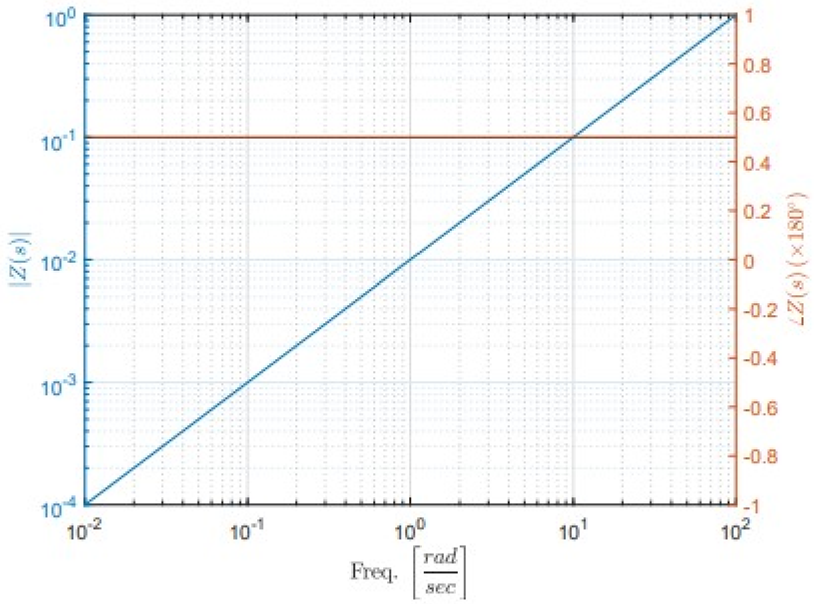

Figure 2: Frequency and phase responses for the circuit given in Fig. 1 where the corresponding DPI function is $Z(s) = s$.

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