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Employing Kuratowski Measure of Non-compactness for Positive Solutions of System of Singular Fractional q-Differential Equations with Numerical Effects

Mohammad Esmael Samei[a]

[a]*Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan 6517838695, Iran*

Abstract. In this work, we investigate the existence of solutions for the system of two singular fractional q-differential equations under integral boundary conditions via the concept of Caputo fractional q-derivative and fractional Riemann-Liouville type q-integral. Some new sxistence results are obtained by applying Krattowski measure of non-compactness. Also, the Darbo's fixed point theorem and the Lebesgue dominated convergence theorem are the main tools in deriving our proofs. Lastly, we present an example illustrating the primary effects.

1. Introduction

Fractional calculus and Fractional q-calculus are the significant branches in mathematical analysis. The field of fractional calculus has countless applications (for instance, see [1–4]). Similarly, the subject of fractional differential equations ranges from the theoretical views of existence and uniqueness of solutions to the analytical and mathematical methods for finding solutions (for more details, consider [5–16]). Likewise, some researchers have been investigated the existence of solutions for some singular fractional differential equations (for example, see [17–25]).

In this article, motivated by among these achievements, we will stretch out the positive solutions for the singular system of q-differential equations

$$\begin{cases} D_q^{\alpha_1} u(t) + g_1(t, u(t), v(t)) = 0, \\ D_q^{\alpha_2} v(t) + g_2(t, u(t), v(t)) = 0, \end{cases} \tag{1}$$

under boundary conditions $u(0) = v(0) = 0$, for $i = 2, \dots, n-1$, $u^{(i)}(0) = v^{(i)}(0) = 0$ and

$$u(1) = \left[I_q^{\gamma_1}\Big(w_1(t)u(t)\Big)\right]_{t=1}, \qquad v(1) = \left[I_q^{\gamma_2}\Big(w_2(t)v(t)\Big)\right]_{t=1},$$

where $\alpha_j \in (n, n+1]$ with $n \geq 3$, $\gamma_j \geq 1$, $g_j \in C(E)$, g_j are singular at $t = 0$ which satisfy the local Carathéodory condition on $E = (0,1] \times (0,\infty) \times (0,\infty)$, and $w_j \in \overline{\mathcal{L}} = L^1[0,1]$ are non-negative somehow that

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Email address: mesamei@basu.ac.ir, mesamei@gmail.com (Mohammad Esmael Samei)

$[I_q^{\gamma_j}(w_j(t))]_{t=1} \in [0, \frac{1}{2})$ for $j = 1, 2$.

We recall some of the previous works briefly. In 1910, the subject of q-difference equations was introduced by Jackson [26]. After that, at the beginning of the last century, studies on q-difference equations, appeared in so many works, especially in Carmichael [27], Mason [28], Adams [29], Trjitzinsky [30], Agarwal [31]. An excellent account in the study of fractional differential and q-differential equations can be found in [32–34]. In 2012, Liu *et al.* [35] discussed the singular equation $D^\alpha x(t) + h(t, x(t)) = 0$ under boundary conditions $x(1) = 0$ and $[I^{2-\alpha}x(t)]'_{t=0} = 0$, where t belongs to $[0, 1]$, $1 < \alpha < 2$ and D^α is the Riemann-Liouville fractional derivative. In 2013, Zhai *et al.* [36] discussed about positive solutions for the fractional differential equation with conditions

$$\begin{cases} -D^\alpha x(t) = g(t, x(t)) + h(t, x(t)), \\ x(0) = x'(0) = x''(0) = x''(1) = 0, \\ \text{ or } x(0) = x'(0) = x''(0) = 0, \quad x''(1) = \beta x''(\eta), \end{cases}$$

where t and α belong to $(0, 1)$, $(3, 4]$, respectively, and D^α is the Riemann-Liouville fractional derivative. In the same year, the singular problem

$$D^\alpha x = g\Big(t, x(t), D^\beta x(t), D^\gamma x(t)\Big) + h\Big(t, x(t), D^\beta x(t), D^\gamma x(t)\Big),$$

under boundary conditions $x(0) = x'(0) = x''(0) = x'''(0) = 0$ is reviewed, where α, β, γ belong to $(3, 4)$, $(0, 1)$, $(1, 2)$, respectively, D^α is the Caputo fractional derivative and function g is a Carathéodory on $[0, 1] \times (0, \infty)^3$. Also, Wang in [37] investigated the existence of positive solution for the system

$$D^\alpha x_i(t) + h_i(t, x_1(t), x_2(t)) = 0,$$

for $i = 1, 2$, under boundary conditions $x_1(0) = x_1'(0) = 0$, $x_2(0) = x_2'(0) = 0$ and

$$x_1(1) = \int_0^1 x_1(t) d\eta(t), \qquad x_2(1) = \int_0^1 x_2(t) d\eta(t),$$

where $t \in [0, 1]$, $\alpha \in (2, 3]$, $h_1, h_2 \in C([0, 1] \times [0, \infty) \times [0, \infty), \mathbb{R})$, D^α is the Riemann-Liouville fractional derivative and $\int_0^1 x_i(t) d\eta(t)$ denotes the Riemann-Stieltjes integral. In 2014, Yan *et al.* [38] studied the boundary value problems

$${}^cD_{0^+}^\alpha u(t) = f(t, u(t), {}^cD_{0^+}^\beta u(t)),$$

for $t \in [0, 1]$ with boundary conditions $u(0) = u'(0) = y(u(t))$, $\int_0^1 t(s) dt = m$ and $u^{(k)}(0) = 0$ for $2 \le k \le n - 1$, where ${}^cD_{0^+}^\alpha$, ${}^cD_{0^+}^\beta$ are the Caputo fractional derivatives, $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $y : C([0, 1], \mathbb{R}) \to \mathbb{R}$ is a continuous function and $m \in \mathbb{R}$, $n - 1 < \alpha < n$, $(n \ge 2)$, $0 < \beta < 1$ is real number. In 2016, Jleli *et al.* [8] by using a measure of non-compactness argument combined with a generalized version of Darbo's theorem, provided sufficient conditions for the existence of at least one solution of the functional equation

$$u(t) = F\left(t, u(\mu(t)), \frac{f(t, u(\gamma(t)))}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} g(s, u(s))\, d_q s\right),$$

$t \in I = [0, 1]$, where $\alpha \in (1, \infty)$, $q \in (0, 1)$, $f, g : I \times \mathbb{R} \to \mathbb{R}$, $\mu, \gamma : I \to I$ and $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In 2019, Samei *et al.* [7] discussed the fractional hybrid q-differential inclusions

$${}^cD_q^\alpha \left(\frac{x}{f\left(t, x, I_q^{\alpha_1} x, \cdots, I_q^{\alpha_n} x\right)} \right) \in F\left(t, x, I_q^{\beta_1} x, \cdots, I_q^{\beta_k} x\right),$$

with the boundary conditions $x(0) = x_0$ and $x(1) = x_1$, where $1 < \alpha \le 2$, $q \in (0, 1)$, $x_0, x_1 \in \mathbb{R}$, $\alpha_i > 0$, for $i = 1, 2, \ldots, n$, $\beta_j > 0$, for $j = 1, 2, \ldots, k$, $n, k \in \mathbb{N}$, ${}^cD_q^\alpha$ denotes Caputo type q-derivative of order α, I_q^β denotes

Riemann-Liouville type q-integral of order β, $f : J \times \mathbb{R}^n \to (0, \infty)$ is continuous and $F : J \times \mathbb{R}^k \to P(\mathbb{R})$ is multifunction. Also, Ntouyas *et al.* [13] by applying definition of the fractional q-derivative of the Caputo type and the fractional q-integral of the Riemann–Liouville type, studied the existence and uniqueness of solutions for a multi-term nonlinear fractional q-integro-differential equations under some boundary conditions

$$^{c}D_q^{\alpha}x(t) = w\Big(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t), {}^{c}D_q^{\beta_1}x(t), {}^{c}D_q^{\beta_2}x(t), \dots, {}^{c}D_q^{\beta_n}x(t)\Big).$$

In 2020, Liang *et al.* [14] investigated the existence of solutions for a nonlinear problems regular and singular fractional q-differential equation

$$^{c}D_q^{\alpha}f(t) = w\Big(t, f(t), f'(t), {}^{c}D_q^{\beta}f(t)\Big),$$

with conditions $f(0) = c_1 f(1)$, $f'(0) = c_2 {}^{c}D_q^{\beta}f(1)$ and $f^{(k)}(0) = 0$ for $2 \le k \le n-1$, here $n-1 < \alpha < n$ with $n \ge 3$, $\beta, q, c_1 \in (0, 1)$, $c_2 \in (0, \Gamma_q(2-\beta))$, function w is a L^{κ}-Carathéodory, $w(t, x_1, x_2, x_3)$ may be singular and ${}^{c}D_q^{\alpha}$ the fractional Caputo type q-derivative. Similar results have been presented in other studies [9, 10, 12, 15, 16].

The rest of the paper is arranged as follows: in Section 2, we recall some preliminary concepts and fundamental results of q-calculus. Section 3 is devoted to the main results, while example illustrating the obtained results and algorithm for the problems are presented in Section 4.

2. Preliminaries

First, we point out some of the materials on the fractional q-calculus and fundamental results of it which needed in the next sections (for more information, consider [2, 3, 26]). Then, some well-known theorems of fixed point theorem and definition are expressed.

Assume that $q \in (0, 1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [26]. The power function $(x-y)_q^n$ with $n \in \mathbb{N}_0$ is defined by $(x-y)_q^{(n)} = \prod_{k=0}^{n-1}(x - yq^k)$ for $n \ge 1$ and $(x-y)_q^{(0)} = 1$, where x and y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [29]. Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have

$$(x-y)_q^{(\alpha)} = x^{\alpha} \prod_{k=0}^{\infty} (x - yq^k)/(x - yq^{\alpha+k}).$$

If $y = 0$, then it is clear that $x^{(\alpha)} = x^{\alpha}$ (Algorithm 1). The q-Gamma function is given by

$$\Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1},$$

where $z \in \mathbb{R} \backslash \{0, -1, -2, \cdots\}$ [26]. Note that, $\Gamma_q(z+1) = [z]_q \Gamma_q(z)$. The value of q-Gamma function, $\Gamma_q(z)$, for input values q and z with counting the number of sentences n in summation by simplifying analysis. For this design, we prepare a pseudo-code description of the technique for estimating q-Gamma function of order n which show in Algorithm 2. For any positive numbers α and β, the q-Beta function is defined by [34],

$$B_q(\alpha, \beta) = \int_0^1 (1-qs)_q^{(\alpha-1)} s^{\beta-1} \, d_q s. \tag{2}$$

The q-derivative of function f, is defined by $(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x \to 0} (D_q f)(x)$ which is shown in Algorithm 3 [29]. Also, the higher order q-derivative of a function f is defined by $(D_q^n f)(x) = D_q(D_q^{n-1} f)(x)$ for all $n \ge 1$, where $(D_q^0 f)(x) = f(x)$ [1, 29]. The q-integral of a function f defined on $[0, b]$ is defined by

$$I_q f(x) = \int_0^x f(s) d_q s = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k)$$

for $0 \le x \le b$, provided the series is absolutely converges [1, 29]. The q-derivative of function f, is defined by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$ which is shown in Algorithm 3 [1, 29]. If a in $[0, b]$, then

$$\int_a^b f(u)d_q u = I_q f(b) - I_q f(a) = (1-q)\sum_{k=0}^{\infty} q^k \left[b f(bq^k) - a f(aq^k) \right],$$

whenever the series exists. The operator I_q^n is given by $(I_q^0 h)(x) = h(x)$ and $(I_q^n h)(x) = (I_q(I_q^{n-1}h))(x)$ for $n \ge 1$ and $g \in C([0, b])$ [1, 29]. It has been proved that $(D_q(I_q f))(x) = f(x)$ and $(I_q(D_q f))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ [1, 29]. The fractional Riemann-Liouville type q-integral of the function f on J for $\alpha \ge 0$ is defined by $(I_q^0 f)(t) = f(t)$ and

$$(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s,$$

for $t \in J$ and $\alpha > 0$ [39]. Also, the Caputo fractional q-derivative of a function f is defined by

$$\begin{aligned} \left({}^c D_q^\alpha f\right)(t) &= \left(I_q^{[\alpha]-\alpha}(D_q^{[\alpha]} f)\right)(t) \\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)} \int_0^t (t-qs)^{([\alpha]-\alpha-1)} \left(D_q^{[\alpha]} f\right)(s) d_q s, \end{aligned} \tag{3}$$

where $t \in J$ and $\alpha > 0$ [39]. It has been proved that $(I_q^\beta(I_q^\alpha f))(x) = (I_q^{\alpha+\beta} f)(x)$ and $(D_q^\alpha(I_q^\alpha f))(x) = f(x)$, where $\alpha, \beta \ge 0$ [39]. By using Algorithm 2, we can calculate $(I_q^\alpha f)(x)$ which is shown in Algorithm 4.

Now, we present some necessary notations. Let $\bar{J} = [0, 1]$. We denote $L^1(\bar{J})$, $C_{\mathbb{R}}(\bar{J})$, $C^1_{\mathbb{R}}(\bar{J})$ by $\overline{\mathcal{L}}$, $\overline{\mathcal{A}}$, $\overline{\mathcal{B}}$, respectively. We say that h satisfies the local Carathéodory condition on $\bar{J} \times (0,\infty) \times (0,\infty)$ and denote it by $\mathrm{Car}(\bar{J} \times (0,\infty) \times (0,\infty))$ whenever has the following properties.

C1) For all $(x_1, x_2) \in (0,\infty) \times (0,\infty)$, $h(., x_1, x_2) : \bar{J} \to \mathbb{R}$ is measurable.

C2) For almost all $t \in \bar{J}$, $h(t, ., .) : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is continuous.

C3) For each compact subset C of $(0,\infty)\times(0,\infty)$ there exists a function $\psi_C \in \overline{\mathcal{L}}$ such that $|h(t, x_1, x_2)| \le \psi_C(t)$ for each $t \in \bar{J}$ and all $(x_1, x_2) \in C$.

We denote the set of all bounded subsets of Banach space A by $\mathcal{F}_A$.

Definition 2.1. *[40] The positive real-valued function μ define on $\mathcal{F}_A$ is measure of non-compactness whenever $\mu(C) = 0$ if and only if C is relatively compact and satisfies the following conditions:*

1) If $C_1 \subset C_2$ then $\mu(C_1) \le \mu(Q_2)$.

2) $\mu(\overline{Conv\ (C)}) = \mu(C)$.

3) $\mu(C_1 \cup C_2) = \max\{\mu(C_1), \mu(C_2)\}$.

4) $\mu(C_1 + C_2) \le \mu(C_1) + \mu(C_2)$.

5) $\mu(\lambda C) = |\lambda|\mu(C)$ for all scalar λ.

Assume that the sets $S_1, S_2, \ldots, S_n$ be a cover for $C \in \mathcal{F}_A$. The Kuratowski measure of non-compactness of C is defined by

$$K(C) = \inf_{\mathrm{diam}\,(S_i)<\epsilon} \epsilon$$

and denoted by $K(C)$ [40]. Take $K(C) = \infty$, $K(C) = 0$ whenever C is unbounded, C is empty set, respectively [40]. Also, for all $C \in \mathcal{F}_A$, we have $K(C) \le \mathrm{diam}\ (C)$ [40]. We need next results.

Lemma 2.2. *[41] If $x \in \overline{\mathcal{A}} \cap \overline{\mathcal{L}}$ with $D_q^\alpha x \in \mathcal{A} \cap \mathcal{L}$, then $I_q^\alpha D_q^\alpha x(t) = x(t) + \sum_{i=1}^n c_i t^{\alpha - i}$, where $[\alpha] \leq n < [\alpha] + 1$ and c_i is some real number.*

Theorem 2.3. *[40] Let a nonempty subset C of a Banach space A is bounded, closed and convex. The self-continuous operator Θ define on C has a fixed point whenever there exists a constant $0 \leq \lambda < 1$ such that $K(\Theta(Q)) \leq r.K(Q)$ for all $Q \subset C$, where K is the Kuratowski measure of non-compactness on A.*

3. Main results

In this part, first we provide some lemmas.

Lemma 3.1. *The solution of the problem $D_q^\alpha u(t) + v(t) = 0$ for $\alpha \geq 3$ and $g \in J$), under boundary conditions*

$$u(0) = u^{(2)}(0) = \cdots = u^{(n-1)}(0) = 0$$

and $u(1) = [I_q^\gamma(w(t)u(t))]_{t=1}$ is $u(t) = \int_0^1 G(t, qs) v(s)\, d_q s$ where $v, w \in \overline{\mathcal{L}}$, $\gamma \geq 1$ and

$$G(t, qs) = a_1(t, s, \alpha) + \frac{t}{\mu(\gamma)} \int_0^1 (1 - qt)^{(\gamma - 1)} w(t) a_1(t, s, \alpha)\, d_q t, \tag{4}$$

whenever $t \leq s$,

$$G(t, qs) = a_2(t, s, \alpha) + \frac{t}{\mu(\gamma)} \int_0^1 (1 - qt)^{\gamma - 1} w(t) a_2(t, s, \alpha)\, d_q t, \tag{5}$$

whenever $s \leq t$ for $s, t \in \overline{J}$, here $\mu(\gamma) = \Gamma_q(\gamma) - \int_0^1 t(1 - qt)^{(\gamma - 1)} w(t)\, d_q t$, and

$$a_1(t, s, \alpha) = \frac{t(1 - qs)^{(\alpha - 1)}}{\Gamma_q(\alpha)},$$
$$a_2(t, s, \alpha) = \frac{t(1 - qs)^{(\alpha - 1)} - (t - qs)^{(\alpha - 1)}}{\Gamma_q(\alpha)}.$$

Proof. First, note that Lemma 2.2 implies $u(t) = -I_q^\alpha v(t) + \sum_{i=0}^n c_i t^i$, for some real constants c_i. Also, By using the condition $u(0) = u^{(i)}(0) = 0$ for $i \geq 2$, we obtain $c_i = 0$ for $0 \leq i \leq n$. Thus, $u(t) = -I_q^\alpha v(t) + c_1 t$. Since

$$\left[I_q^\gamma (w(t)u(t))\right]_{t=1} = \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1 - qs)^{(\gamma - 1)} w(s)\, d_q s,$$

by using the boundary condition at $t = 1$ we have $-I_q^\alpha v(1) + c_1 = I_q^\gamma w(1)$. Therefore $c_1 = I_q^\alpha v(1) + I_q^\gamma w(1)$. Hence $u(t) = -I_q^\alpha v(t) + I_q^\alpha v(1) + I_q^\gamma w(1)$ and so $u(t)$ is equal to

$$\int_0^1 a_1(t, s, \alpha) v(s)\, d_q s + t \left[I_q^\gamma (w(t)u(t))\right]_{t=1}$$

and

$$\int_0^1 a_2(t, s, \alpha) v(s)\, d_q s + t \left[I_q^\gamma (w(t)u(t))\right]_{t=1},$$

when $t \leq s$ and $s \leq t$, respectively. This implies that

$$\begin{aligned}
\left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} &= \frac{1}{\Gamma_q(\gamma)} \int_0^1 \left(\int_0^1 (1-qt)^{(\gamma-1)} w(t) a_1(t,s,\alpha) v(s)\, d_q s \right) d_q t \\
&+ \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1-qt)^{(\gamma-1)} t w(t) \left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} \, d_q t, \\
\left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} &= \frac{1}{\Gamma_q(\gamma)} \int_0^1 \left(\int_0^1 (1-qt)^{(\gamma-1)} w(t) a_2(t,s,\alpha) v(s)\, d_q s \right) d_q t \\
&+ \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1-qt)^{(\gamma-1)} t w(t) \left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} \, d_q t,
\end{aligned}$$

for $t \leq s$, $s \leq t$, respectively. On the other hand

$$\left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} = \int_0^1 \left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} d_q t,$$

then we have

$$\begin{aligned}
\int_0^1 \left(1 - \frac{1}{\Gamma_q(\gamma)} (1-qt)^{(\gamma-1)} t w(t)\right) \left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} \, d_q t &= I_q^{\gamma}\left(w(1) \int_0^1 a_1(t,s,\alpha) v(s)\, d_q s\right), \\
\int_0^1 \left(1 - \frac{1}{\Gamma_q(\gamma)} (1-qt)^{(\gamma-1)} t w(t)\right) \left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} \, d_q t &= I_q^{\gamma}\left(w(1) \int_0^1 a_2(t,s,\alpha) v(s)\, d_q s\right),
\end{aligned}$$

for $t \leq s$, $s \leq t$, respectively. Hence,

$$\begin{aligned}
\left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} \left(1 - \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1-qt)^{(\gamma-1)} t w(t)\, d_q t\right) &= I_q^{\gamma}\left(w(1) \int_0^1 a_1(t,s,\alpha) v(s)\, d_q s\right), \\
\left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} \left(1 - \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1-qt)^{(\gamma-1)} t w(t)\, d_q t\right) &= I_q^{\gamma}\left(w(1) \int_0^1 a_2(t,s,\alpha) v(s)\, d_q s\right),
\end{aligned}$$

whenever $t \leq s$, $s \leq t$, respectively, and so

$$\begin{aligned}
\left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} &= \frac{1}{\Gamma_q(\gamma)\left[1 - \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1-qt)^{(\gamma-1)} t w(t)\, d_q t\right]} I_q^{\gamma}\left(w(1) \int_0^1 a_1(t,s,\alpha) v(s)\, d_q s\right), \\
\left[I_q^{\gamma}(w(t)u(t))\right]_{t=1} &= \frac{1}{\Gamma_q(\gamma)\left[1 - \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1-qt)^{(\gamma-1)} t w(t)\, d_q t\right]} I_q^{\gamma}\left(w(1) \int_0^1 a_2(t,s,\alpha) v(s)\, d_q s\right),
\end{aligned}$$

for $t \leq s$, $s \leq t$, respectively. This implies that $u(t)$ is equal to

$$\begin{aligned}
&\int_0^1 a_1(t,s,\alpha) v(s)\, d_q s + \frac{t}{\Gamma_q(\gamma) - \int_0^1 (1-qt)^{(\gamma-1)} t w(t)\, d_q t} I_q^{\gamma}\left(w(1) \int_0^1 a_1(t,s,\alpha) v(s)\, d_q s\right), \\
&\int_0^1 a_2(t,s,\alpha) v(s)\, d_q s + \frac{t}{\Gamma_q(\gamma) - \int_0^1 (1-qt)^{(\gamma-1)} t w(t)\, d_q t} I_q^{\gamma}\left(w(1) \int_0^1 a_2(t,s,\alpha) v(s)\, d_q s\right),
\end{aligned}$$

for $t \leq s$, $s \leq t$, respectively, which are same as (4) and (5), respectively. So the proof is complete. □

By employing simple calculations for $G(t, qs)$ in (4) and (5), we conclude that

$$G(t, qs) \in \left[0, \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha-1)}\left(1+\frac{1}{\mu(\gamma)}\int_0^1 (1-qt)^{(\gamma-1)} w(t)\, d_qt\right)\right],$$

for all $t, s \in \overline{J}$. At present, for $n \geq 1$, consider the map $g_{i,n}(t, u, v) = g_i(t, \chi_n(u), \chi_n(v))$, where

$$\chi_n(x) = \begin{cases} x, & x \geq \frac{1}{n}, \\ \frac{1}{n}, & x < \frac{1}{n}. \end{cases}$$

Here, we first investigate the regular system

$$\begin{cases} D_q^{\alpha_1} u + g_{1,n}(t, u, v) = 0, \\ D_q^{\alpha_2} u + g_{2,n}(t, u, v) = 0, \end{cases} \tag{6}$$

under some conditions in the problem (1). For $i = 1, 2$ and each n belongs to $\mathbb{N}$, define the function

$$F_{n,i}(u, v)(t) = \int_0^1 G_{\alpha_i}(t, qs) g_{n,i}(s, u(s), v(s))\, d_qs,$$

where $G_{\alpha_i}(t, qs)$ is the q-Green function in Lemma 3.1 which replaced α and γ by α_i and γ_i, respectively. Also, we take $\Theta_n(u, v)(t) = (F_{n,1}(u, v)(t), F_{n,2}(u, v)(t))$ and

$$\|\Theta_n(u, v)(t)\|_* = \max\{F_{n,1}(u, v)(t), F_{n,2}(u, v)(t)\}.$$

Since g_1 and $g_2 \in \text{Car}\,(\overline{J} \times \mathbb{R}^2)$, by simple review we conclude that $g_{n,1}$, $g_{n,2} \in \text{Car}\,(\overline{J} \times \mathbb{R}^2)$ and so there exist ψ_1 and $\psi_2 \in \overline{\mathcal{L}}$ such that $|g_{n,i}(t, u(t), v(t))| \leq \psi_i(t)$ for $n \in \mathbb{N}$, t belongs to $\overline{J}$ and $i = 1, 2$. We denote the set of all $(u, v) \in \overline{\mathcal{A}}^2$ such that $\|(u, v)\|_* \leq \|\psi\|_\infty^*$ by $\mathcal{D}$, where $\|\psi\|_\infty^* = \max\{\|\psi_1\|_\infty, \|\psi_2\|_\infty\}$. One can check that, $\mathcal{D}$ is closed, bounded and convex.

Lemma 3.2. *Let $n \in \mathbb{N}$. For each bounded subset of $C(\overline{J}, \mathbb{R}) \times C(\overline{J}, \mathbb{R})$, the self-map Θ_n defined on $\mathcal{D}$ is equi-continuous.*

Proof. Assume that $(u, v) \in \mathcal{D}$ be given, $i = 1, 2$ and $n \geq 1$. We can see that,

$$F_{n,i}(u, v)(t) \leq \int_0^1 \frac{(1-qs)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i-1)}\left[1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\, d_qt\right] g_{n,i}(s, u(s), v(s))\, d_qs.$$

Thus,

$$F_{n,i}(u, v)(t) \leq \int_0^1 \frac{(1-qs)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i-1)}\left[1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{(\gamma_i-1)} w_i(t)\, d_qt\right] \varphi_i(s)\, d_qs. \tag{7}$$

On the other hand, $[I_q^{\gamma_i}(w_i(t))]_{t=1} \in [0, \frac{1}{2})$, then $\frac{1}{\Gamma_q(\gamma_i)}\int_0^1 (1-qt)^{(\gamma_i-1)} w_i(t)\, d_qt \in [0, \frac{1}{2})$. Also, we get

$$\frac{1}{\Gamma_q(\gamma_i)}\int_0^1 (1-qt)^{(\gamma_i-1)} t w_i(t)\, d_qt \leq \frac{1}{\Gamma_q(\gamma_i)}\int_0^1 (1-qt)^{(\gamma_i-1)} w_i(t)\, d_q.$$

Therefore,

$$\frac{1}{\Gamma_q(\gamma_i)}\int_0^1 (1-qt)^{(\gamma_i-1)} t w_i(t)\, d_qt \in \left[0, \frac{1}{2}\right)$$

and so $1 - \frac{1}{\Gamma_q(\gamma_i)} \int_0^1 (1-qt)^{(\gamma_i - 1)} t w_i(t)\, d_q t \in [0, \frac{1}{2})$. Indeed,

$$\frac{1}{\mu(\gamma_i)} \int_0^1 (1-qt)^{(\gamma_i-1)} w_i(t)\, d_q t = \frac{\int_0^1 (1-qt)^{(\gamma_i-1)} w_i(t)\, d_q t}{\Gamma_q(\gamma_i) - \int_0^1 (1-qt)^{(\gamma_i-1)} t w_i(t)\, d_q t}$$
$$= \frac{\frac{1}{\Gamma(p_i)} \int_0^1 (1-qt)^{(\gamma_i-1)} w_i(t)\, d_q t}{1 - \frac{1}{\Gamma_q(\gamma_i)} \int_0^1 (1-qt)^{(\gamma_i-1)} t w_i(t)\, dt} \in [0, 1)$$

and so $1 + \frac{1}{\mu(\gamma_i)} \int_0^1 (1-qt)^{(\gamma_i-1)} w_i(t)\, d_q t \leq 2$. By applying the previous inequality and (7), we obtain

$$\begin{aligned} F_{n,i}(u,v)(t) &\leq \frac{2}{\Gamma_q(\alpha_i - 1)} \int_0^1 (1-qs)^{(\alpha_i-2)} \varphi_i(s)\, d_q s \\ &\leq \frac{2\|\varphi_i\|_\infty}{\Gamma_q(\alpha_i - 1\,)} \int_0^1 (1-qs)^{(\alpha_i-2)}\, d_q s \\ &= \frac{2}{\Gamma_q(\alpha_i)} \|\varphi_i\|_\infty \leq \|\varphi_i\|_\infty \leq \|\varphi\|_\infty^* \end{aligned}$$

and so $\|\Theta_n(u,v)\|_* \leq \|\varphi\|_\infty^*$. Hence, Θ_n maps $\mathcal{D}$ into $\mathcal{D}$. Assume that $B \subset C(\bar{J}, \mathbb{R}) \times C(\bar{J}, \mathbb{R})$ is bounded. Also, let $\{(u_k, v_k)\}_{k=1}^\infty$ be a bounded sequence in B and $t_1, t_2 \in \bar{J}$ with $t_1 < t_2$. Then, we have

$$\begin{aligned} &|F_{i,n}(u_k, v_k)(t_2) - F_{i,n}(u_k, v_k)(t_1)| \\ &\quad \leq \frac{1}{\Gamma_q(\alpha_i)} \Big[\int_0^{t_1} \left[(t_2 - qs)^{(\alpha_i - 1)} - (t_1 - qs)^{(\alpha_i-1)} \right] g_{n,i}(s, u_k(s), v_k(s))\, d_q s \\ &\qquad + \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha_i-1)} g_{n,i}(s, u_k(s), v_k(s)) d_q s \Big] \\ &\qquad + (t_2 - t_1) \int_0^1 \left[\frac{(1-qs)^{(\alpha_i-1)}}{\Gamma_q(\alpha_i)} + G_{2,i}(s) \right] g_{n,i}(s, u_k(s), v_k(s))\, d_q s \\ &\quad \leq \frac{1}{\Gamma_q(\alpha_i)} \Big[\int_0^1 \left[(t_2 - qs)^{(\alpha_i-1)} - (t_1 - qs)^{(\alpha_i - 1)} \right] \varphi_i(s)\, d_q s \\ &\qquad + (t_2 - qt_1)^{(\alpha_i - 1)} \|\varphi_i\|_1 + (t_2 - t_1) \|\varphi_i\|_1 \\ &\qquad \times \Big(\frac{1}{\Gamma_q(\alpha_i)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1-qt)^{\gamma_i - 1} w_i(t)\, d_q t \Big) \Big], \end{aligned}$$

where for $i = 1, 2$, $G_{1,i}(t,s)$ is equal to $a_1(t,s,\alpha)$, $a_2(t,s,\alpha)$ whenever $t \leq s$, $s \leq t$, respectively which is obtained by replacing α_i by α in (4) and (5), respectively, and $G_{2,i}(s)$ is equal to

$$\frac{1}{\mu(p_i)} \int_0^1 (1-qt)^{\gamma_i-1} w_i(t) a_1(t,s,\alpha_i)\, d_q t, \qquad \frac{1}{\mu(p_i)} \int_0^1 (1-qt)^{\gamma_i-1} w_i(t) a_2(t,s,\alpha_i)\, d_q t,$$

for $t \leq s$, $s \leq t$, respectively. Let $\epsilon \in J$ be given, $t_1, t_2 \in \bar{J}$ such that $t_1 < t_2$ and $s \in [0, t_1]$. We choose $\delta > 0$ such that $t_1 - t_2 < \delta$ implies $(t_2 - s)^{\alpha_i - 1} - (t_1 - s)^{\alpha_i - 1} < \epsilon$. Also, suppose that $k \in [1, \infty)$ and $0 \leq t_1 < t_2 \leq 1$ with $t_1 - t_2 < \min\{\delta, \epsilon\}$ be given. Then we get

$$\left| F_{i,n}(u_k, v_k)(t_2) - F_{i,n}(u_k, v_k)(t_1) \right| \leq \epsilon \|\varphi_i\|_1 \Big(\frac{3}{\Gamma_q(\alpha_i)} + \frac{1}{\mu(\gamma_i)} \int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\, d_q t \Big)$$

and so $\lim_{t_2\to t_1} \|\Theta_n(u_k,v_k)(t_2)-\Theta_n(u_k,v_k)(t_1)\|_* = 0$. Also, we have

$$\begin{aligned}\|\Theta_n(u_k,v_k)(t)\|_* &\le \max\Big\{\int_0^1 \frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1-1)}\,(1+G_{2,1}(s))\,\varphi_1(s)\,d_qs,\\ &\qquad \int_0^1 \frac{(1-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2-1)}\Big(1+G_{2,2}(s)\Big)\varphi_2(s)\,d_qs\Big\}\\ &\le \max\Big\{\frac{\|\varphi_1\|_1}{\Gamma_q(\alpha_1-1)}\Big(1+\frac{1}{\mu(\gamma_1)}\int_0^1(1-qt)^{\gamma_1-1}w_1(t)\,d_qt\Big),\\ &\qquad \frac{\|\varphi_2\|_1}{\Gamma_q(\alpha_2-1)}\Big(1+\frac{1}{\mu(\gamma_2)}\int_0^1(1-qt)^{\gamma_2-1}w_2(t)\,d_qt\Big)\Big\}.\end{aligned}$$

Let $\{(u_k,v_k)\}_{k=1}^{\infty}$ be sequence in B and $(u_k,v_k)\to(u,v)$. Hence, $u_k\to u$, $v_k\to v$. Note that,

$$\begin{aligned}\|\Theta_n(u_k,v_k)(t)-&\Theta_n(u,v)(t)\|_*\\ &\le \max\Big\{\int_0^1 G_{\alpha_1}(t,qs)|g_{1,n}(s,u_k(s),v_k(s))-g_{1,n}(s,u(s),v(s))|\,d_qs,\\ &\qquad \int_0^1 G_{\alpha_2}(t,qs)|g_{2,n}(s,u_k(s),v_k(s))-g_{2,n}(s,u(s),v(s))|\,d_qs\Big\}\\ &\le 2\|\varphi\|_1^*\Big(\frac{1+\Lambda_M}{\Gamma_q(\alpha_m-1)}\Big),\end{aligned}$$

where $\alpha_m=\min\{\alpha_1,\alpha_2\}$ and

$$\Lambda_M=\max_{i=1,2}\Big\{\frac{1}{\mu(\gamma_i)}\int_0^1(1-qt)^{\gamma_i-1}w_i(t)\,d_qt\Big\}.$$

Since for $i=1,2$, $|g_{i,n}(s,u_k(s),v_k(s))-g_{i,n}(s,u(s),v(s))|\to 0$ and by employing the theorem of Lebesgue dominated convergence, we conclude that Θ_n is equi-continuous on B for each $n\in\mathbb{N}$. □

Theorem 3.3. *Assume that $g_1,g_2\in Car\,(\overline{J}\times(0,\infty)^2)$ and for $n\ge 3$, $\alpha_1,\alpha_2\in(n,n+1]$. Then for each $n\ge 1$ the system*

$$\begin{cases} D_q^{\alpha_1}u+g_{1,n}(t,u,v)=0\\ D_q^{\alpha_2}v+g_{2,n}(t,u,v)=0\end{cases}\tag{8}$$

under conditions $u(0)=v(0)=0$, $u^{(i)}(0)=v^{(i)}(0)=0$ for $i=2,\dots,n-1$, $u(1)=[I_q^{\gamma_1}(w_1(t)u(t))]_{t=1}$ and $v(1)=[I_q^{\gamma_2}(w_2(t)v(t))]_{t=1}$ has a solution, whenever the following assumptions hold.

1) *There exist $\gamma_1,\gamma_2\ge 1$ and nonnegative functions $w_1,w_2\in\overline{\mathcal{L}}$ such that*

$$[I_q^{\gamma_1}(w_1(t))]_{t=1},[I_q^{\gamma_2}(w_2(t))]_{t=1}\in[0,\frac{1}{2}).$$

2) *There exist $h_1,h_2\in\overline{\mathcal{L}}$ such that $2\|h_i\|_1<\Gamma_q(\alpha_i-1)$ for almost all $t\in\overline{J}$ and $i=1,2$.*

3) *For any bounded subset Q of $\overline{\mathcal{A}}^2$, $K(g_i(t,Q))\le h_i(t)K(Q)$ for $i=1,2$.*

Proof. Let Q be a bounded subset of $\overline{\mathcal{A}}^2$ for $n\ge 1$ and $i=1$ or 2. We choose bounded subsets A and B of $\overline{\mathcal{A}}$ such that $Q=(A,B)$. We take the sets A_1 and B_1 of all $u\in A$ and $u\in B$, respectively, such that $u\ge\frac{1}{n}$. Then, we get

$$\begin{aligned}K(g_{i,n}(t,Q)) &= K\left(g_{i,n}(t,A,B)\right) = K\left(g_i(t,\chi_n(A),\chi_n(B))\right) \leq K\left(\chi_n(A),\chi_n(B)\right)\\ &= K\Big(A_1 \cup \Big\{\frac{1}{n}\Big\}, B_1 \cup \Big\{\frac{1}{n}\Big\}\Big)\\ &= K\Big((A_1,B_1) \cup \Big(\frac{1}{n},A_1\Big) \cup \Big(B_1,\frac{1}{n}\Big)\Big)\\ &= \max\Big\{K\Big(A_1,B_1\Big), K\Big(A_1,\frac{1}{n}\Big), K\Big(B_1,\frac{1}{n}\Big)\Big\}.\end{aligned}$$

Let $K(B_1) = d$. Then there exist $C_i \subset \overline{\mathcal{A}}$ and $m \in \mathbb{N}$ such that $B_1 \subset \bigcup_{i=1}^{m} C_i$ and $\operatorname{diam}(C_i) < d$. Hence, $\left(\frac{1}{n}, B_1\right) \subset \bigcup_{i=1}^{m}\left(\frac{1}{n}, C_i\right)$,

$$\operatorname{diam}\left(\frac{1}{n}, C_i\right) = \sup_{x,y\in C_i}\left\|\left(\frac{1}{n},x\right)-\left(\frac{1}{n},y\right)\right\|_* = \sup_{x,y\in C_i}|x-y| = \operatorname{diam}(C_i),$$

and $K(\frac{1}{n}, B_1) \leq K(B_1)$. By employing a similar technique, we will have $K(B_1) \leq K(\frac{1}{n}, B_1)$. Thus, $K(B_1) = K(\frac{1}{n}, B_1)$ and $K(A_1) = K(A_1, \frac{1}{n})$. Hence, there exist $m_0 \geq 1$ and $(Y_i, Z_i) \subset \overline{\mathcal{A}}^2$ such that $(A_1, SB_1) \subset \bigcup_{i=1}^{m_0}(Y_i, Z_i)$ and $\operatorname{diam}(Y_i, Z_i) \leq d_0$ whenever $K(A_1, B_1) = d_0$. This implies that

$$\sup\left\{\|(y,z)-(y',z')\|_* : (y,z),(y',z') \in (Y_i,Z_i)\right\} \leq d_0$$

and so

$$\sup\{\max\{|y-y'|,|z-z'|\} \;:\; y,y' \in Y_i,\ z,z' \in Z_i\} \leq d_0.$$

Hence, $\sup_{y,y'\in Y_i}|y-y'| \leq d_0$ and

$$\sup_{z,z'\in Z_i}|z-z'| \leq d_0.$$

Thus, $A_1 \subset \bigcup_{i=1}^{m_0} Y_i$ with $\operatorname{diam}(Y_i) \leq d_0$ and $B_1 \subset \bigcup_{i=1}^{m_0} Z_i$ with $\operatorname{diam}(Z_i) \leq d_0$ for each i. Indeed, $K(A_1) \leq K(A_1, B_1)$ and $K(B_1) \leq K(A_1, B_1)$. Hence,

$$\max\Big\{K(A_1,B_1), K\Big(A_1,\frac{1}{n}\Big), K\Big(\frac{1}{n},B_1\Big)\Big\} = K(A_1,B_1)$$

and so for $i = 1, 2$, we get $K(g_{i,n}(t,Q)) \leq h_i(t)K(A_1,B_1) \leq h_i(t)K(Q)$. As well, we obtain

$$K(\Theta_n(Q)) = K\Big(\int_0^1 G_{\alpha_1}(t,qs)g_{1,n}(s,Q)\,d_qs, \int_0^1 G_{\alpha_2}(t,qs)g_{2,n}(s,Q)\,d_qs\Big).$$

For each $s \in \overline{J}$, $n \geq 1$ and $i = 1, 2$, we take $d_i(s) := K(g_{i,n}(s,Q)) \leq h_i(s)K(Q)$. Choose $k_0 \in \mathbb{N}$ and bounded subsets $X_{i,j}$ of $\overline{\mathcal{A}}^2$ for $i = 1, 2$ somehow that $g_{i,n}(s,Q) \subseteq \bigcup_{j=1}^{k_0} X_{i,j}$. Then, we have $\operatorname{diam}(h_{i,j}) \leq d_i(s) \leq h_i(s)K(Q)$ and

$$\begin{aligned}G_{\alpha_i}(t,qs)g_{i,n}(s,Q) &\subseteq \int_0^1 \bigcup_{j=1}^{k_0} \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1(1-qt)^{\gamma_i-1}w_i(t)\,d_qt\Big)X_{i,j}\,d_qs\\ &= \bigcup_{j=1}^{k_0}\int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1(1-qt)^{\gamma_i-1}w_i(t)\,d_qt\Big)X_{i,j}\,d_qs\end{aligned}$$

for $i = 1, 2$, here

$$\int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1(1-qt)^{\gamma_i-1}w_i(t)\,d_qt\Big)X_{i,j}\,d_qs$$

is the set of all

$$\int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)x(s)\,d_qs$$

where $x \in X_{i,j}$. Thus,

$$\begin{aligned}
&\operatorname{diam}\Big(\frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)X_{i,j}\,d_qs\Big)\\
&= \sup_{x,x'\in X_{i,j}}\Big|\int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)x(s)\,d_qs\\
&\quad -\int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)x'(s)\,d_qs\Big|\\
&= \sup_{x,x'\in X_{i,j}}\int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)|x(s)-x'(s)|\,d_qs\\
&\le \int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)\operatorname{diam}(X_{i,j})\,d_qs\\
&\le \int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)d_i(s)\,d_qs
\end{aligned}$$

and so

$$\begin{aligned}
&K\Big(\int_0^1 G_{\alpha_i}(t,qs)g_{i,n}(s,Q)\,d_qs\Big)\\
&\qquad\le \int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)K(g_{i,n}(s,Q))\,d_qs\\
&\qquad\le \int_0^1 \frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)h_i(s)K(Q)\,d_qs\\
&\qquad\le K(Q)\Big\|\frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)\Big\|_\infty \|h_i\|_1.
\end{aligned}$$

By simple review, we can conclude that

$$\lambda_i = \Big\|\frac{(1-qs)^{\alpha_i}}{\Gamma_q(\alpha_i-1)}\Big(1+\frac{1}{\mu(\gamma_i)}\int_0^1 (1-qt)^{\gamma_i-1} w_i(t)\,d_qt\Big)\Big\|_\infty \|h_i\|_1 \in [0,1)$$

for $i=1,2$. So, by applying last result, we obtain

$$\max_{i=1,2}\Big\{K(\int_0^1 G_{\alpha_i}(t,qs)g_{i,n}(s,Q)\,d_qs)\Big\} \le \lambda K(Q),$$

here $\lambda = \max\{\lambda_1,\lambda_2\}$. At present, consider the space $\overline{\mathcal{A}}^2$ endowed with norm

$$\|(.,.)\|_{**}\|(y_1,y_2)\|_{**} = \max\{\|y_1\|_*, \|y_2\|_*\}.$$

It is proved in first part, if Y and $Y' \subset \overline{\mathcal{A}}^2$ then $K(Y), K(Y') \le K(Y,Y')$, where Y, Y' are bounded sets. We know that $(\overline{\mathcal{A}}^2, \|(.,.)\|_{**})$ is a Banach space. Suppose that $K(Y)$, $K(Y')$ are equal to r, r', respectively and $r := \max\{r, r'\}$. We choose $n, n' \ge 1$ such that $Y \subset \bigcup_{i=1}^n Z_i$ and $Y' \subset \bigcup_{j=1}^{n'} Z'_j$, where $Z_i, Z'_j \subset \overline{\mathcal{A}}^2$, $\operatorname{diam}(Z_i) < r$

and diam $(Z'_j) < r'$ for $i = 1, \dots, n$ and $j = 1, \dots, n'$. Let $n \geq n'$. Put $Z'_{n'+1} = Z'_{n'+2} = \cdots = Z'_n := Z_n$. Then, $(Y, Y') \subset \bigcup_{i=1}^{n}(Z_i, Z'_i)$ and for each $i = 1, \dots, n$, we have

$$\begin{aligned}
\operatorname{diam}(Z_i, Z'_i) &= \sup_{z_1,z_2\in Z_i, z'_1,z'_2\in Z'_i} \left\|(z_1, z'_1) - (z_2, z'_2)\right\|_{**} \\
&= \sup_{z_1,z_2\in Z_i, z'_1,z'_2\in Z'_i} \left\|(z_1 - z'_1, z_2 - z'_2)\right\|_{**} \\
&= \sup_{z_1,z_2\in Z_i, z'_1,z'_2\in V_i} \left\{\max\left\{\|(z_1 - z_2)\|_*, \|(z'_1 - z'_2)\|_*\right\}\right\} \\
&\leq \max\{r, r'\} = r.
\end{aligned}$$

Hence, $K(Y, Y') \leq \max\{K(Y), K(Y')\}$ and so $K(Y, Y') = \max\{K(Y), K(Y')\}$. Thus,

$$\begin{aligned}
K(\Theta_n(Q)) &= K\Big(\int_0^1 G_{\alpha_1}(t, qs)g_{1,n}(s, Q)\, d_qs, \int_0^1 G_{\alpha_2}(t, qs)g_{2,n}(s, Q)\, d_qs\Big) \\
&= \max_{i=1,2}\left\{\int_0^1 G_{\alpha_i}(t, qs)g_{i,n}(s, Q)\, d_qs\right\} \\
&\leq \lambda K(Q).
\end{aligned}$$

Therefore, by using the Darbo's fixed point theorem, Θ_n has a fixed point in $\mathcal{D}$ for all n. This implies that the system has a solution $(u_n, v_n) \in \mathcal{D}$, that is,

$$u_n(t) = \int_0^1 G_{\alpha_1}(t, qs)g_{1,n}\Big(s, u_n(s), v_n(s)\Big)\, d_qs, \qquad v_n(t) = \int_0^1 G_{\alpha_2}(t, qs)g_{2,n}\Big(s, u_n(s), v_n(s)\Big)\, d_qs.$$

Then the proof is complete. □

Now, we provide result for the singular system.

Theorem 3.4. *Let $g_1, g_2 \in Car\,(\overline{J} \times (0, \infty)^2)$, $\alpha_1, \alpha_2 \in (n, n + 1]$ with $n \geq 3$. Then the singular system*

$$\begin{cases} D_q^{\alpha_1}u + g_1(t, u, v) = 0 \\ D_q^{\alpha_2}v + g_2(t, u, v) = 0 \end{cases} \tag{9}$$

with boundary conditions $u(0) = v(0) = 0$, $u^{(i)}(0) = v^{(i)}(0) = 0$ for $i = 2, \dots, n - 1$, $u(1) = [I_q^{\gamma_1}(w_1(t)u(t))]_{t=1}$ and $v(1) = [I_q^{\gamma_2}(w_2(t)v(t))]_{t=1}$ has a solution, whenever the following assumptions hold.

1) *There exist $\gamma_1, \gamma_2 \geq 1$ and non-negative functions $w_1, w_2 \in \overline{\mathcal{L}}$ such that*

$$[I_q^{\gamma_1}(w_1(t))]_{t=1}, [I_q^{\gamma_2}(w_2(t))]_{t=1} \in [0, \frac{1}{2}).$$

2) *There exist $h_1, h_2 \in \overline{\mathcal{L}}$ such that $2\|h_i\|_1 < \Gamma_q(\alpha_i - 1)$ for each $t \in \overline{J}$ and $i = 1, 2$.*

3) *For any bounded subset Q of $\overline{\mathcal{A}}^2$, $K(g_i(t, Q)) \leq h_i(t)K(Q)$ where $i = 1, 2$ and K is the Kuratowski measure of non-compactness.*

Proof. By applying Theorem 3.3, we conclude that the problem (1) has a solution $(u_n, v_n) \in \mathcal{D}$ for all n. Also, there is $(u, v) \in \mathcal{D}$ such that $\lim_{n\to\infty}(u_n, v_n) = (u, v)$, because $\mathcal{D}$ is closed. By simple check, we conclude that (u, v) satisfies the boundary condition of the problem (1). On the other hand, we obvious that $\lim_{n\to\infty} g_{i,n}(t, u_n(t), v_n(t)) = g_i(t, u(t), v(t))$ for almost all $t \in \overline{J}$ and $i = 1, 2$. Thus, we obtain

$$G_{\alpha_i}(t, qs)g_{i,n}(s, u_n(s), v_n(s)) \leq \frac{1}{\Gamma(\alpha_i - 1)}\Big(\frac{1}{\mu(\gamma_i)}\int_0^1 (1 - qt)^{\gamma_i-1}w_i(t)\, d_qt\Big)\varphi_i(s),$$

for $i = 1, 2$, each n and all $(t, s) \in \overline{J}^2$. Now, by applying the Lebesgue dominated convergence theorem, we get

$$u(t) = \int_0^1 G_{\alpha_1}(t, qs) g_{1,n}\Big(s, u(s), v(s)\Big)\, d_q s, \qquad v(t) = \int_0^1 G_{\alpha_2}(t, qs) g_{2,n}\Big(s, u(s), v(s)\Big)\, d_q s.$$

This implies that, (u, v) is a solution for the problem (1). $\square$

4. Example illustrative for the problem with algorithms

Here, we provide an example to illustrate our main result. In this way, we give a computational technique for checking the problem (1) in Theorem 3.4. We need to present a simplified analysis could be executed values of the q-Gamma function. To this aim, we consider a pseudo-code description of the method for calculation of the q-Gamma function of order n in Algorithm 2 (for more details, see the link https://en.wikipedia.org/wiki/Q-gamma_function).

Table 1 shows that when q is constant, the q-Gamma function is an increasing function. Also, for smaller values of x, an approximate result is obtained with less values of n. It has been shown by underlined rows. Table 2 shows that the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns. They have been underlined in line 8 of the first column, line 17 of the second column and line 29 of third columns of Table 2. Also, Table 3 is the same as Table 2, but x values increase in 3. Similarly, the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns. Furthermore, we provided algorithms 3 and 5 which calculated $D_q^\alpha f(x)$ and $I_q^\alpha f(x)$, respectively.

Example 4.1. *We define the singular fractional system similar to the problem* (1) *by*

$$\begin{cases} D_q^{\frac{7}{2}} u(t) + \frac{1}{5\sqrt{t}}\Big(\frac{1}{2}u(t) + \frac{1}{3}v(t)\Big) = 0, \\ D_q^{\frac{10}{3}} u(t) + \frac{3}{10\sqrt[3]{t}}\Big(\frac{1}{4}u(t) + \frac{3}{5}v(t)\Big) = 0, \end{cases} \tag{10}$$

under boundary conditions $u(0) = v(0) = u'(0) = v'(0) = u''(0) = v''(0) = 0$ *and*

$$u(1) = \Big[I_q^{\frac{17}{3}}(tu(t))\Big]_{t=1}, \qquad v(1) = \Big[I_q^{\frac{16}{3}}(t^{\frac{1}{2}}v(t))\Big]_{t=1}.$$

By comparison with problem (1), *we can consider the maps*

$$g_1(t, u, v) = \frac{1}{5\sqrt{t}}\left(\frac{1}{2}u + \frac{1}{3}v\right),$$
$$g_2(t, u, v) = \frac{3}{10\sqrt[3]{t}}\left(\frac{1}{4}u + \frac{3}{5}v\right).$$

Also, by definition of functions g_1 *and* g_2, *we consider* $h_1(t) = \frac{1}{5\sqrt{t}}$, $h_2(t) = \frac{3}{10\sqrt[3]{t}}$, $x(u, v) = \frac{1}{2}u + \frac{1}{3}v$ *and* $y(u, v) = \frac{1}{4}u + \frac{3}{5}v$. *Put* $\alpha_1 = \frac{7}{2}$, $\alpha_2 = \frac{10}{3}$, $\gamma_1 = \frac{17}{3}$, $\gamma_2 = \frac{16}{3}$, $w_1(t) = t$, $w_2(t) = \sqrt{t}$. *It can be seen that* $g_1, g_2 \in Car\,(\overline{J} \times (0, \infty)^2)$, $h_1, h_2 \in \overline{\mathcal{L}}$ *are non-negative and* $w_1, w_2 \in \overline{\mathcal{L}}$. *Also, we have*

$$\begin{aligned} \Big[I_q^{\gamma_1}(w_1(t))\Big]_{t=1} &= \Big[I_q^{\frac{17}{3}}(t)\Big]_{t=1} = \frac{1}{\Gamma_q(\frac{17}{3})}\int_0^1 (1 - qs)^{(\frac{14}{3})} s\, d_q s = \frac{1}{\Gamma_q(\frac{17}{3})}\frac{\Gamma_q(2)\Gamma_q(\frac{17}{3})}{\Gamma_q(2 + \frac{17}{3})} \in \Big[0, \frac{1}{2}\Big), \\ \Big[I_q^{\gamma_2}(w_2(t))\Big]_{t=1} &= \Big[I_q^{\frac{16}{3}}(\sqrt{t})\Big]_{t=1} = \frac{1}{\Gamma_q(\frac{16}{3})}\int_0^1 (1 - qs)^{(\frac{13}{3})} \sqrt{s}\, d_q s = \frac{1}{\Gamma_q(\frac{16}{3})}\frac{\Gamma_q(\frac{3}{2})\Gamma_q(\frac{16}{3})}{\Gamma_q(\frac{3}{2} + \frac{16}{3})} \in \Big[0, \frac{1}{2}\Big) \end{aligned} \tag{11}$$

and

$$\begin{aligned} \|h_1\|_1 &= \int_0^1 \frac{1}{5\sqrt{t}}\,dt = 0.4 < \frac{1}{2}\Gamma_q\Big(\frac{7}{2}-1\Big) = \frac{1}{2}\Gamma_q(\alpha_1 - 1), \\ \|h_2\|_1 &= \int_0^1 \frac{3}{10\sqrt[3]{t}}\,dt = 0.18 < \frac{1}{2}\Gamma_q\Big(\frac{10}{3}-1\Big) = \frac{1}{2}\Gamma_q(\alpha_2 - 1). \end{aligned} \tag{12}$$

Tables 4 and 5 show the values of $\left[I_q^{\gamma_1}(w_1(t))\right]_{t=1}$ *and* $\left[I_q^{\gamma_2}(w_2(t))\right]_{t=1}$ *in inequalities (11) for some different values of* q*, respectively. Also, we get*

$$\begin{aligned} K(x(Q)) = K(x((M,N))) &= K\Big(\frac{1}{2}M + \frac{1}{3}N\Big) \\ &= \max\{K(M), K(N)\}\Big(\frac{5}{6}\Big) = K(Q)\Big(\frac{5}{6}\Big) \le K(Q). \end{aligned}$$

for each $Q = (M,N) \subset \overline{\mathcal{A}}^2$*. Since* $g_1(t,u,v) = h(t)x(u,v)$*, we conclude that*

$$K(g_1(t,Q)) = K(h_1(t)x(Q)) = h_1(t)K(x(Q)) \le h_1(t)K(Q).$$

Therefore, by employing a similar technique, we have

$$K(g_2(t,Q)) = K(h_2(t)y(Q)) = h_1(t)K(y(Q)) \le h_1(t)K(Q).$$

Theorem 3.4 implies that the system (1) has a solution.

Ethics approval and consent to participate

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References

[1] C. Adams, Note on the integro-q-difference equations, Transactions of the American Mathematical Society 31 (4) (1929) 861–867.
[2] P. Rajković, S. Marinković, M. Stanković, Fractional integrals and derivatives in q-calculus, Applicable Analysis and Discrete Mathematics 1 (2007) 311–323.
[3] M. Annaby, Z. Mansour, q-Fractional Calculus and Equations, Springer Heidelberg, Cambridge, 2012. doi.org/10.1007/978-3-642-30898-7
[4] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, 2000.
[5] B. Ahmad, J. Nieto, Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, Boundary Value Problems 2011 (2011) 36. doi.org/10.1186/1687-2770-2011-36
[6] R. Agarwal, M. Belmekki, M. Benchohra, A survey on semilinear differential equations and inclusions invovling Riemann-Liouville fractional derivative, Advances in Difference Equations 2009 (2009) 981728. doi.org/10.1155/2009/981728
[7] M. Samei, G. Khalilzadeh Ranjbar, Some theorems of existence of solutions for fractional hybrid q-difference inclusion, Journal of Advanced Mathematical Studies 12 (1) (2019) 63–76.

[8] M. Jleli, M. Mursaleen, B. Sanet, q-integral equations of fractional orders, Electronic Journal of Differential Equations 2016 (17) (2016) 1–14.

[9] M. Samei, V. Hedayati, S. Rezapour, Existence results for a fraction hybrid differential inclusion with Caputo–Hadamard type fractional derivative, Advances in Difference Equations 2019 (2019) 163. doi.org/10.1186/s13662-019-2090-8

[10] V. Hedayati, M. Samei, Positive solutions of fractional differential equation with two pieces in chain interval and simultaneous Dirichlet boundary conditions, Boundary Value Problems 2019 (2019) 141. doi.org/10.1186/s13661-019-1251-8

[11] V. Kalvandi, M. E. Samei, New stability results for a sum-type fractional q-integro-differential equation, Journal of Advanced Mathematical Studies 12 (2) (2019) 201–209.

[12] M. E. Samei, G. K. Ranjbar, V. Hedayati, Existence of solutions for equations and inclusions of multi-term fractional q-integro-differential with non-separated and initial boundary conditions, Journal of Inequalities and Applications 2019 (2019) 273. doi.org/10.1186/s13660-019-2224-2

[13] S. K. Ntouyas, M. E. Samei, Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus, Advances in Difference Equations 2019 (2019) 475. doi.org/10.1186/s13662-019-2414-8

[14] S. Liang, M. E. Samei, New approach to solutions of a class of singular fractional q-differential problem via quantum calculus, Advances in Difference Equations 2020 (2020) 14. doi.org/10.1186/s13662-019-2489-2

[15] M. E. Samei, V. Hedayati, G. K. Ranjbar, The existence of solution for k-dimensional system of Langevin Hadamard-type fractional differential inclusions with $2k$ different fractional orders, Mediterranean Journal of Mathematics 17 (2020) 1–22. doi.org/10.1007/s00009-019-1471-2

[16] M. E. Samei, G. Khalilzadeh Ranjbar, V. Hedayati, Existence of solutions for a class of Caputo fractional q-difference inclusion on multifunctions by computational results, Kragujevac Journal of Mathematics 45 (4) (2021) 543–570.

[17] Y. Liu, P. Wong, Global existence of solutions for a system of singular fractional differential equations with impulse effects, Journal of Applied Mathematics and Informatics 33 (3-4) (2015) 327–342. doi.org/10.14317/jami.2015.327

[18] M. Shabibi, S. Rezapour, S. Vaezpour, A singular fractional integro-differential equation, University Politehnica of Bucharest Scientific Bulletin, Series A 79 (1) (2017) 109–118.

[19] N. Nyamoradi, T. Bashiri, S. Vaezpour, D. Baleanu, Uniqueness and existence of positive solutions for singular fractional differential equations, Electronic Journal of Differential Equations 2014 (130) (2014) 1–13.

[20] S. Rezapour, M. Shabibi, A singular fractional differential equation with Riemann-Liouville integral boundary condition, Journal of Advanced Mathematical Studies 8 (1) (2015) 80–88.

[21] Y. Wang, L. Liu, Necessary and sufficient condition for the existence of positive solution to singular fractional differential equations, Advances in Difference Equations 2015 (2015) 207. doi.org/10.1186/s13662-015-0540-5

[22] R. Agarwal, D. O'regan, S. Stanek, Positive solutions for Dirichlet problem of singular nonlinear fractional differential equations, Journal of Mathematical Analysis and Applications 371 (1) (2010) 57–68. doi.org/10.1016/j.jmaa.2010.04.034

[23] Z. Bai, W. Sun, Existence and multiplicity of positive solutions for singular fractional boundary value problems, Computers & Mathematics with Applications 63 (2012) 1369–1381. doi.org/10.1016/j.camwa.2011.12.078

[24] Y. Tian, Y. Zhou, Positive solution for multi-point boundary value problem of fractional differential equations, Journal of Applied Mathematics and Computing 38 (1-2) (2012) 417–427. doi.org/10.1007/s12190-011-0487-y

[25] S. W. Vong, Positive solutions of singular fractional differential equations with integral boundary conditions, Mathematical and Computer Modelling 37 (5-6) (2013) 1053–1059. doi.org/10.1016/j.mcm.2012.06.024

[26] F. Jackson, q-difference equations, American Journal of Mathematics 32 (1910) 305–314. doi.org/10.2307/2370183

[27] R. Carmichael, The general theory of linear q-difference equations, American Journal of Mathematics 34 (1912) 147–168.

[28] T. Mason, On properties of the solution of linear q-difference equations with entire function coefficients, American Journal of Mathematics 37 (1915) 439–444.

[29] C. Adams, The general theory of a class of linear partial q-difference equations, Transactions of the American Mathematical Society 26 (1924) 283–312.

[30] W. Trjitzinsky, Analytic theory of linear q-difference equations, Acta Mathematica 61 (1933) 1–38. doi.org/10.1007/BF02547785

[31] R. Agarwal, Certain fractional q-integrals and q-derivatives, Proceedings of the Cambridge Philosophical Society 66 (1969) 365–370. doi.org/10.1017/S0305004100045060

[32] I. Podlubny, Fractional Differential Equations, Academic Press, 1999.

[33] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, Mathematics Studies, Elsevier Science, North-Holland, 2006.

[34] V. Kac, P. Cheung, Quantum Calculus, Universitext, Springer, New York, 2002. doi.org/10.1007/978-1-4613-0071-7-1

[35] Y. Liu, H. Shi, Existence of unbounded positive solutions for BVPS of singular fractional differential equations, Journal of Nonlinear Science and Applications 5 (4) (2012) 281–293. doi.org/10.22436/jnsa.005.04.04

[36] C. Zhai, W. Yan, C. Yang, A sum operator method for the existence and uniqueness of positive solutions to Riemann-Liouville fractional differential equation boundary value problems, Communications in Nonlinear Science and Numerical Simulation 18 (14) (2013) 858–866. doi.org/10.1016/j.cnsns.2012.08.037

[37] Y. Wang, Positive solution for a system of fractional integral boundary value problem, Boundary Value Problems 2013 (1) (2013) 256. doi.org/10.1186/1687-2770-2013-256

[38] R. Yan, S. Sun, H. Lu, Y. Zhao, Existence of solutions for fractional differential equations with integral boundary conditions, Advances in Difference Equations 2014 (2014) 25.

[39] R. Ferreira, Nontrivials solutions for fractional q-difference boundary value problems, Electronic journal of qualitative theory of differential equations 70 (2010) 1–101.

[40] E. Malkowsky, V. Rakocevi, An introduction into the theory of sequence spaces and measure of non-compactness, Zb. rad. Beogr 17 (2000) 143–234.

[41] N. Tatar, An impulsive nonlinear singular version of the Gronwall-Bihari inequality, Journal of Inequalities and Applications

2006 (2006) 1–12. doi.org/10.1155/JIA/2006/84561

Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

Input: a, b, α, n, q
$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * (a - b * a^k)/(a - b * q^{\alpha+k})$
  end for
  $p \leftarrow a^\alpha * s$
end if
Output: $(a-b)^{(\alpha)}$

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

Input: $n, q \in (0,1), x \in \mathbb{R}\backslash\{0,-1,2,\cdots\}$
$p \leftarrow 1$
for $k = 0$ to n **do**
  $p \leftarrow p(1-q^{k+1})(1-q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$
Output: $\Gamma_q(x)$

Algorithm 3 The proposed method for calculated $(D_q f)(x)$

Input: $q \in (0,1), f(x), x$
syms z
if $x = 0$ **then**
  $g \leftarrow \lim((f(z) - f(q*z))/((1-q)z), z, 0)$
else
  $g \leftarrow (f(x) - f(q*x))/((1-q)x)$
end if
Output: $(D_q f)(x)$

Table 1: Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$ and $n = 1, 2, \ldots, 15$ of Algorithm 2.

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	2.340263	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	11.257095	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	64.350881
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

Algorithm 4 The proposed method for calculated $(I_q^{\alpha} f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x
$s \leftarrow 0$
for $i = 0$ to n **do**
  $pf \leftarrow (1 - q^{i+1})^{\alpha-1}$
  $s \leftarrow s + pf * q^i * f(x * q^i)$
end for
$g \leftarrow (x^{\alpha} * (1-q) * s)/(\Gamma_q(x))$
Output: $(I_q^{\alpha} f)(x)$

Algorithm 5 The proposed method for calculated $\int_a^b f(r) d_q r$

Input: $q \in (0,1)$, α, n, $f(x)$, a, b
$s \leftarrow 0$
for $i = 0 : n$ **do**
  $s \leftarrow s + q^i * (b * f(b * q^i) - a * f(a * q^i))$
end for
$g \leftarrow (1-q) * s$
Output: $\int_a^b f(r) d_q r$

Table 2: Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \ldots, 35$ of Algorithm 2.

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	2.853295	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	8.470578
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	4.921893	8.479713	34	2.853224	4.921875	8.470517

Table 3: Some numerical results for calculation of $\Gamma_q(x)$ with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 40$ of Algorithm 2.

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	$\underline{11.257095}$	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	$\underline{49.065751}$	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	$\underline{259.967394}$
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

Table 4: Some numerical results of $\left[I_q^{\gamma_i}(t)\right]_{t=1}$ inequality (11) in Example 4.1 for $q \in \left\{\frac{1}{8}, \frac{1}{2}, \frac{8}{9}\right\}$. One can check that $\left[I_q^{\frac{17}{3}}(t)\right]_{t=1} \in [0, \frac{1}{2})$

	$q = \frac{1}{8}$			$q = \frac{1}{2}$			$q = \frac{8}{9}$		
n	$\Gamma_q(2)$	$\Gamma_q(2+\gamma_1)$	$\left[I_q^{\gamma_1}(t)\right]_{t=1}$	$\Gamma_q(2)$	$\Gamma_q(2+\gamma_1)$	$\left[I_q^{\gamma_1}(t)\right]_{t=1}$	$\Gamma_q(2)$	$\Gamma_q(2+\gamma_1)$	$\left[I_q^{\gamma_1}(t)\right]_{t=1}$
1	1.002	2.0979	0.4776	1.1429	38.3805	0.0298	3.3594	140964.0908	0
2	1.0002	2.0938	0.4777	1.0667	33.6243	0.0317	2.6617	61731.7617	0
3	1	2.0933	$\underline{0.4777}$	1.0323	31.5422	0.0327	2.2468	32423.6282	0.0001
4	1	2.0932	0.4777	1.0159	30.5659	0.0332	1.9734	19319.8718	0.0001
5	1	2.0932	0.4777	1.0079	30.0929	0.0335	1.7808	12631.2336	0.0001
6	1	2.0932	0.4777	1.0039	29.8601	0.0336	1.6387	8865.5569	0.0002
7	1	2.0932	0.4777	1.002	29.7446	0.0337	1.5301	6579.6665	0.0002
8	1	2.0932	0.4777	1.001	29.6871	0.0337	1.445	5107.0357	0.0003
9	1	2.0932	0.4777	1.0005	29.6584	0.0337	1.3769	4111.7549	0.0003
10	1	2.0932	0.4777	1.0002	29.6441	$\underline{0.0337}$	1.3216	3412.1729	0.0004
11	1	2.0932	0.4777	1.0001	29.6369	0.0337	1.276	2904.1757	0.0004
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	1	2.0932	0.4777	1	29.6297	0.0337	1.0072	943.649	0.0011
41	1	2.0932	0.4777	1	29.6297	0.0337	1.0064	939.9897	$\underline{0.0011}$
42	1	2.0932	0.4777	1	29.6297	0.0337	1.0056	936.7508	0.0011
43	1	2.0932	0.4777	1	29.6297	0.0337	1.005	933.8826	0.0011

Table 5: Some numerical results of $\left[I_q^{\gamma_i}(t)\right]_{t=1}$ inequality (11) in Example 4.1 for $q \in \left\{\frac{1}{8}, \frac{1}{2}, \frac{8}{9}\right\}$. One can check that $\left[I_q^{\frac{16}{3}}(t)\right]_{t=1} \in [0, \frac{1}{2})$

	$q = \frac{1}{8}$			$q = \frac{1}{2}$			$q = \frac{8}{9}$		
n	$\Gamma_q(2)$	$\Gamma_q(2+\gamma_1)$	$\left[I_q^{\gamma_1}(t)\right]_{t=1}$	$\Gamma_q(2)$	$\Gamma_q(2+\gamma_1)$	$\left[I_q^{\gamma_1}(t)\right]_{t=1}$	$\Gamma_q(2)$	$\Gamma_q(2+\gamma_1)$	$\left[I_q^{\gamma_1}(t)\right]_{t=1}$
1	0.9687	1.877	0.5161	0.9965	21.6657	0.046	1.6936	25796.1141	0.0001
2	0.9675	1.8733	0.5165	0.9565	18.9992	0.0503	1.4923	11873.769	0.0001
3	0.9673	1.8728	<u>0.5165</u>	0.9382	17.8313	0.0526	1.3628	6503.4478	0.0002
4	0.9673	1.8728	0.5165	0.9294	17.2835	0.0538	1.2721	4015.4293	0.0003
5	0.9673	1.8728	0.5165	0.9251	17.0181	0.0544	1.205	2706.2688	0.0004
6	0.9673	1.8728	0.5165	0.923	16.8875	0.0547	1.1535	1949.717	0.0006
7	0.9673	1.8728	0.5165	0.9219	16.8227	0.0548	1.1128	1479.9943	0.0008
8	0.9673	1.8728	0.5165	0.9214	16.7904	0.0549	1.0801	1171.4155	0.0009
9	0.9673	1.8728	0.5165	0.9211	16.7743	0.0549	1.0533	959.2878	0.0011
10	0.9673	1.8728	0.5165	0.921	16.7662	<u>0.0549</u>	1.031	807.9574	0.0013
11	0.9673	1.8728	0.5165	0.9209	16.7622	0.0549	1.0124	696.637	0.0015
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
45	0.9673	1.8728	0.5165	0.9209	16.7582	0.055	0.8945	245.149	0.0036
46	0.9673	1.8728	0.5165	0.9209	16.7582	0.055	0.8943	244.6677	<u>0.0037</u>
47	0.9673	1.8728	0.5165	0.9209	16.7582	0.055	0.8941	244.2408	0.0037