Filomat 34:9 (2020), 3045–3064
https://doi.org/10.2298/FIL2009045A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Continuous Action of Enriched Lattice-Valued Convergence Groups: Some Examples

T. M. G. Ahsanullah*[a], Fawzi Al-Thukair[a], Jawaher Al-Mufarrij[b]

[a] *Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia*
[b] *Department of Mathematics, Women Section, King Saud University, Riyadh 12372, Saudi Arabia*

Abstract. Starting with a category **S**L-**CONVGRP**, of stratified enriched *cl*-premonoid-valued convergence groups as introduced earlier, we present a category **S**L-**CONVTGRP**, of stratified enriched *cl*-premonoid-valued convergence transformation groups, the idea behind this category is crept in the notion of convergence transformation group - a generalization of topological transformation group. In this respect, we are able to provide natural examples in support to our endeavor; these examples, however, stem from the action of convergence approach groups on convergence approach spaces, and the action of probabilistic convergence groups under triangular norm on probabilistic convergence spaces. Based on the category of enriched lattice-valued convergence spaces, a Cartesian closed category that enjoys lattice-valued convergence structure on function space, we look into among others, the lattice-valued convergence structures on the group of homeomorphisms of enriched lattice-valued convergence spaces, generalizing a concept of convergence transformation groups on convergence spaces, obtaining a characterization.

1. Introduction

Considering the notion of lattice-valued filter as introduced in [28], Jäger studied the category of lattice-valued convergence spaces, L-**CONV**, where it is pointed out, one may put it as: the notion of lattice-valued convergence space is an extension of $\{0,1\}$-valued convergence space which can be identified with classical convergence space (cf. Remark in pp. 6 [30], and [45]). He showed among other results that L-**CONV** is a Cartesian closed category [30] (see also, [1, 32]), stimulating interest among many researchers to work on lattice-valued convergence spaces, and quite a good number of papers surfaced in recent years, we quote here a few of them, cf. [2, 3, 6, 16, 19, 20, 25, 31–33, 35, 41, 51]. It may be mentioned here that the importance of various types of convergence, particularly, filter-theoretic convergence structures contributed immensely for the development of set-theoretic topology in general, and functional analysis in particular [8, 14, 15, 29, 50].
In 1988, Lowen and Lowen introduced a category of convergence approach spaces, **CAP** [37], which is also

2010 *Mathematics Subject Classification*. 54H11, 54E15, 18B30, 54C05
Keywords. Enriched lattice, topology, convergence group, continuous action of convergence group, topological group, topological transformation group, function space, group of homeomorphisms, convergence approach group, action of convergence approach group, action of probabilistic convergence groups under triangular norm, category
Received: 30 September 2019; Revised: 19 January 2020; Accepted: 15 March 2020
Communicated by Dijana Mosić
This article is dedicated to Professor Eva Colebunders on her 70th Birthday
Email addresses: tmga1@ksu.edu.sa (T. M. G. Ahsanullah*), thukair@ksu.edu.sa (Fawzi Al-Thukair), jmufarij@ksu.edu.sa (Jawaher Al-Mufarrij)

a Cartesian closed category. In [31], it is proved that **CAP** is simultaneously a reflective and coreflective subcategory of the category L-**CONV**. Considering the category **CAP**, Colebunders et al in [17] brought to light a category **CAM**, of convergence approach monoids, and studied the continuous action of convergence approach monoids on convergence approach spaces; with the help of the notion of convergence approach groups [38], they also studied the continuous action of convergence approach groups on convergence approach spaces. We, on the other hand, in [3] (see also [2]) introduced and studied the notion of stratified enriched *cl*-premonoid-valued convergence groups with the help of the notion of lattice-valued convergence structure introduced in [30].

The theory of topological transformation groups is quite old, and forms a fascinating and wide-ranging topic in the realm of mathematics having enormous applications, such as, topological dynamics, abstract harmonic analysis, ergodic theory, geometry, Lie groups, differential equations to name a few (see f.i. [13, 23, 40]). Although the theory of convergence transformation groups is relatively new, but possesses appealing characteristic, and thus, we believe that there is ample opportunity to do research in this direction including our efforts to their extensions to lattice-valued cases. We refer to [48] for further explanations of the importance of studying classical convergence structures to group action.

In [21] (see also [13, 43]), Gevorgyan while studying the notion of so-called G-spaces in connection to the action of topological groups on topological spaces, pointed out that the group of self-homeomorphisms cannot usually be made into a topological group unless the topological spaces under consideration are locally compact and locally connected. A similar argument made by Park in [42] who originally studied a notion of an action of convergence group on a convergence space leading to a notion of convergence transformation group on a convergence space, argued that a meaningful results in topological transformation groups could be achieved when one considers topological spaces as locally compact and Hausdorff, cf. [13]. He further argued that if one considers the category of convergence spaces [9–12, 44, 46, 47], then situation turned out to be very simple, needs no mention of an extra property. Note that the investigation of classical notion of group of homeomorphisms among other notable authors goes back to Arens [7], see also [18, 39, 40].

The motive behind the present article is, first, to introduce a notion of action of stratified enriched *cl*-premonoid-valued convergence group on enriched *cl*-premonoid-valued convergence space, and provide two important classes of natural examples. Secondly, we investigate stratified lattice-valued convergence structure on group of homeomorphisms, and provide a characterization. Thirdly, given an arbitrary group and an enriched lattice-valued convergence structure on it, we give a procedure to construct an enriched lattice-valued convergence transformation group on the given enriched lattice-valued convergence space. Furthermore, we look into categorical connection between the concepts of stratified enriched lattice-valued convergence transformation groups and the category of convergence approach transformation groups, and also, category of probabilistic convergence transformation groups under triangular norms. We arrange our work as follows.

In Preliminary Section 2, we give a general view of enriched *cl*-premonoid lattice structures including the well-known notions of enriched lattice-valued convergence structure that will be needed in the sequel. In Section 3, we provide the notion of the continuous action of enriched lattice-valued convergence groups on enriched lattice-valued convergence spaces leading to the notion of enriched lattice-valued convergence transformation groups on enriched lattice-valued convergence spaces, generalizing the notion of convergence transformation groups on convergence spaces - a classical notion that was introduced for the first time by Park in [42]; here we study group of homeomorphisms of enriched lattice-valued convergence spaces. Moreover, we present here a characterization theorem on lattice-valued convergence transformation groups. Section 4 deals with a construction of a stratified convergence transformation group while in Section 5, we bring the idea of convergence approach transformation group on convergence approach space - an idea which has not been mentioned explicitly in [17]; in this section we explore a possible link between the categories **S**L-**CONVTGRP**, of stratified enriched lattice-valued convergence transformation groups, and **CAPTGRP**, of convergence approach transformation groups. In Section 6, introducing a notion of probabilistic convergence transformation group under triangular norm [34], we provide another class of examples of enriched lattice-valued convergence transformation groups, here we explore again the relation between the categories **S**L-**CONVTGRP** and **PCONVTGRP***, of probabilistic convergence transformation groups under triangular norm.

2. Preliminaries

Throughout the text we consider $L = (L, \leq)$ a complete lattice with $\top$, the top element and $\bot$, the bottom element of L, for further details cf. [22, 49].

Definition 2.1. *[24] A triple $(L, \leq, *)$, where $* : L \times L \to L$ is a binary operation on L, is called a* **G***L-monoid if and only if the following holds:*
(GLM1) $(L, *)$ is a commutative semigroup;
(GLM2) $\forall \alpha \in L$: $\alpha * \top = \alpha$;
(GLM3) $*$ is distributive over arbitrary joins:
$\gamma * (\bigvee_{k \in K} \alpha_k) = \bigvee_{k \in K} (\gamma * \alpha_k)$, for $k \in K$, $\alpha_k, \gamma \in L$;
(GLM4) for every $\gamma \leq \alpha$ there exists $\beta \in L$ such that $\gamma = \alpha * \beta$ (divisibility).

Note that (GM1), (GM2) and (GM3) mean that we have a commutative and *integral* quantale. If $* = \wedge$, then the triple $(L, \leq, \wedge)$ is called a frame or a complete Heyting algebra.
For a commutative quantale, the implication operator $\to$, also known as residuum, is given by: $\to: L \times L \to L$, $\alpha \to \beta = \bigvee \{\gamma \in L | \alpha * \gamma \leq \beta\}$.

Definition 2.2. *[34] A triangular norm or t-norm in short, is a triple $(L = [0,1], \leq, *)$, where $* : [0,1] \times [0,1] \to [0,1]$ satisfying conditions (GLM1), (GLM2) and that $\forall \alpha, \beta, \gamma \in [0,1]$: $\alpha \leq \beta$ implies $\alpha * \gamma \leq \beta * \gamma$.*
A t-norm is continuous if it is continuous as a mapping from $[0,1] \times [0,1]$ to $[0,1]$. Among the important t-norms, we name a few, such as, minimum t-norm $\alpha * \beta = \alpha \wedge \beta$ and product t-norm $\alpha * \beta = \alpha\beta$.

Note that for a continuous *t*-norm, $([0,1], \leq, *)$ is a **G***L*-monoid.

Definition 2.3. *[24, 27] A triple $(L, \leq, \otimes)$, where $\otimes : L \times L \to L$ is a binary operation on L, is called a cl-premonoid if and only if the following conditions are fulfilled:*
(CP1) $\forall \alpha_1, \alpha_2, \beta_1, \beta_2 \in L$: $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$ implies $\alpha_1 \otimes \alpha_2 \leq \beta_1 \otimes \beta_2$;
(CP2) $\forall \alpha \in L$: $\alpha \leq \alpha \otimes \top$ and $\alpha \leq \top \otimes \alpha$;
(CP3) $\gamma \otimes (\bigvee_{k \in K} \alpha_k) = \bigvee_{k \in K} (\gamma \otimes \alpha_k)$, and
$(\bigvee_{k \in K} \alpha_k) \otimes \gamma = \bigvee_{k \in K} (\alpha_k \otimes \gamma)$ for $K \neq \emptyset$, $k \in K$, $\alpha_k, \gamma \in L$, is satisfied.

Definition 2.4. *[24, 28] The quadruple $(L, \leq, *, \otimes)$ is called an enriched cl-premonoid if and only if the following are fulfilled:*
(CLP1) $(L, \leq, *)$ is a **G***L-monoid;*
(CLP2) $(L, \leq, \otimes)$ is a cl-premonoid;
(CLP3) $*$ is dominated by $\otimes$: $\forall \alpha, \beta, \gamma, \delta \in L$, $(\alpha \otimes \beta) * (\gamma \otimes \delta) \leq (\alpha * \gamma) \otimes (\beta * \delta)$.

An enriched *cl*-premonoid $L = (L, \leq, *, \otimes)$ is said to be pseudo-bisymmetric if it satisfies the following axiom: $(\alpha * \beta) \otimes (\gamma * \delta) = ((\alpha \otimes \gamma) * (\beta \otimes \delta)) \bigvee ((\alpha \otimes \bot) * (\beta \otimes \top)) \bigvee ((\bot \otimes \gamma) * (\top \otimes \delta))$, $\forall \alpha, \beta, \gamma, \delta \in L$.

Remark 2.5. *[28] (a) If $\otimes = *$, then the quadruple $(L, \leq, *, \otimes = *)$, is a pseudo-bisymmetric enriched cl-premonoid.*
(b) If $* = \wedge$ and $\otimes = \wedge$, then the quadruple $(L, \leq, \wedge, \wedge)$ is a frame, which is a special case of (a).

Proposition 2.6. *[27, 28] Let $(L, \leq, *)$ be a GL-monoid. Then the following are fulfilled $\forall \alpha, \beta, \gamma, \delta, \beta_j \in L$:*
(1) $\alpha \leq \beta \to \gamma \Leftrightarrow \alpha * \beta \leq \gamma$;
(2) $\alpha * (\alpha \to \beta) \leq \beta$;
(3) $\alpha \leq \beta \Rightarrow \alpha \to \gamma \geq \beta \to \gamma$;
(4) $\alpha \leq \beta \Rightarrow \gamma \to \alpha \leq \gamma \to \beta$;
(5) $(\alpha \to \beta) \to \beta \geq \alpha$;
(6) $\alpha * (\beta \to \gamma) \leq \beta \to (\alpha * \gamma)$;
(7) $\alpha \to (\bigwedge_{j \in J} \beta_j) = \bigwedge_{j \in J} (\alpha \to \beta_j)$;
(8) $(\alpha \to \gamma) * (\beta \to \delta) \leq \alpha * \beta \to \gamma * \delta$;
(9) $\alpha \leq \beta \Leftrightarrow \alpha \to \beta = \top$;
(10) $\alpha \to \top = \top$, $\top \to \alpha = \alpha$, and $\bot \to \alpha = \top$.

In what follows, the quadruple $L = (L, \leq, *, \wedge)$ (or simply L) is assumed to be an enriched cl-premonoid, unless otherwise specified.
If $\alpha \in L$ and $A \subseteq X$, then the map $\alpha_A \colon X \to L$ is defined by

$$\alpha_A(x) = \begin{cases} \alpha, & \text{if } x \in A; \\ \perp, & \text{otherwise.} \end{cases}$$

In particular, $\top_X(x) = \top$, the characteristic function of X and $\perp_X(x) = \perp$, the zero function.
If $A := \{x\}$, then the characteristic function of the singleton is denoted by $\top_x := x$.
The set of all L-sets is denoted by $L^X (= \{\nu : X \to L\})$.
If $\cdot$ is a binary operation on a set X, then $\odot$ is a binary operation on L^X. Thus, for any $\nu_1, \nu_2 \in L^X$ and $z \in X$, $\nu_1 \odot \nu_2 : X \to L$ is defined by: $\nu_1 \odot \nu_2(z) = \bigvee\{\nu_1(x) * \nu_2(y) | x, y \in X, x \cdot y = z\}$; sometime, we write xy instead of $x \cdot y$. In particular, if $(X, \cdot)$ is a group and $x \in X$, then for any $\nu \in L^X$ and $z \in X$, $x \odot \nu(z) = \bigvee_{ab=z} \top_x(a) * \nu(b) = \bigvee_{xb=z} \top * \nu(b) = \nu(x^{-1}z)$ (since $\top$ is the unit element of L by Definition 2.1(GLM2)). Throughout the text we assume e as the identity element of the group $(X, \cdot)$.
If $\nu_1, \nu_2 \in L^X$, and $\to, *, \otimes$ are operations on L as explained before, then these operations are carried over to L^X point-wise:
(i) $(\nu_1 \to \nu_2)(x) = \nu_1(x) \to \nu_2(x)$;
(ii) $(\nu_1 * \nu_2)(x) = \nu_1(x) * \nu_2(x)$;
(iii) $(\nu_1 \otimes \nu_2)(x) = \nu_1(x) \otimes \nu_2(x), \forall x \in X$.

Definition 2.7. *[28] A map $\mathcal{F} : L^X \to L$ is called an L-filter on X if and only if the conditions below are satisfied:*
(LF1) $\mathcal{F}(\top_X) = \top, \mathcal{F}(\perp_X) = \perp$;
(LF2) if $\nu_1, \nu_2 \in L^X$ with $\nu_1 \leq \nu_2$, then $\mathcal{F}(\nu_1) \leq \mathcal{F}(\nu_2)$;
(LF3) $\mathcal{F}(\nu_1) \otimes \mathcal{F}(\nu_2) \leq \mathcal{F}(\nu_1 \otimes \nu_2), \forall \nu_1, \nu_2 \in L^X$.
(SL) An L-filter $\mathcal{F}$ is called a stratified L-filter if $\forall \alpha \in L, \forall \mu \in L^X, \alpha * \mathcal{F}(\mu) \leq \mathcal{F}(\alpha * \mu)$.

The set of all ordinary filters is denoted by $\mathbb{F}(X)$, and the set of all stratified L-filters on X is denoted by $\mathcal{F}^s_L(X)$. On $\mathcal{F}^s_L(X)$, partial ordering $\leq$ is defined by: if $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$, then $\mathcal{F} \leq \mathcal{G} \Leftrightarrow \mathcal{F}(\nu) \leq \mathcal{G}(\nu), \forall \nu \in L^X$. If $x \in X$, then $[x] \in \mathcal{F}^s_L(X)$, called point stratified L-filter on X, and is defined as $[x](\nu) = \nu(x)$, for all $\nu \in L^X$.
If $f \colon X \to Y$ is a function, then $f^{\leftarrow} : L^Y \to L^X$ is defined for any $\mu \in L^Y$ by $f^{\leftarrow}(\mu) = \mu \circ f$; and $f^{\rightarrow} : L^X \to L^Y$ is defined by $f^{\rightarrow}(\nu)(y) = \bigvee\{\nu(x) | f(x) = y\}$, for all $\nu \in L^X, y \in Y$.
Moreover, if $\mathcal{F} \in \mathcal{F}^s_L(X)$, then the stratified L-filter $f^{\Rightarrow}(\mathcal{F}) : L^Y \to L$ on Y is defined for any $\mu \in L^Y$ by $[f^{\Rightarrow}(\mathcal{F})](\mu) = \mathcal{F}(f^{\leftarrow}(\mu)) = \mathcal{F}(\mu \circ f)$.
If $\mathcal{F} \in \mathcal{F}^s_L(Y)$, then $f^{\Leftarrow}(\mathcal{F}) : L^X \to L$ is defined by $[f^{\Leftarrow}(\mathcal{F})](\nu) = \bigvee\{\mathcal{F}(\mu) | \mu \in L^Y, f^{\leftarrow}(\mu) \leq \nu\}$, for all $\nu \in L^X$, is a stratified L-filter on X if and only if for all $\mu \in L^Y, f^{\leftarrow}(\mu) = \perp_X \Rightarrow \mathcal{F}(\mu) = \perp$.
If $\nu \in L^X$ and $\mu \in L^Y$, then the product $\nu \times \mu \in L^{X \times Y}$ is defined by $\nu \times \mu = \nu \circ pr_1 * \mu \circ pr_2$, where $pr_1 : X \times Y \to X, (x, y) \mapsto x$ and $pr_2 : X \times Y \to Y, (x, y) \mapsto y$ are usual projections. Note that operation above on $*$ holds only for finite cases; otherwise, we need to take $* = \wedge$. However, for our case, this does not create any problem with $*$ as we work for finite case.

Proposition 2.8. *[24] If $(L, \leq, *, \otimes = *)$ is an enriched cl-premonoid, then for stratified L-filters $\mathcal{F}_1$ and $\mathcal{F}_2$, the supremum $\mathcal{F}_1 \vee \mathcal{F}_2$ exists if and only if $\mathcal{F}_1(\nu_1) * \mathcal{F}_2(\nu_2) = \perp$ $\forall \nu_1, \nu_2 \in L^X$ such that $\nu_1 * \nu_2 = \perp_X$. In particular, the supremum is the stratified L-filter defined for all $\nu \in L^X$ by $\mathcal{F}_1 \vee \mathcal{F}_2(\nu) = \bigvee\{\mathcal{F}_1(\nu_1) * \mathcal{F}_2(\nu_2) | \nu_1, \nu_2 \in L^X, \nu_1 * \nu_2 \leq \nu\}$.*

Remark 2.9. *[16](see also, [30]) If $(L, \leq, *, \otimes)$ is a pseudo-bisymmetric enriched cl-premonoid, then in view of the Definition 3.5[16], for $\mathcal{F} \in \mathcal{F}^s_L(X)$ and $\mathcal{G} \in \mathcal{F}^s_L(Y)$, their product $\mathcal{F} \times \mathcal{G} \in \mathcal{F}^s_L(X \times Y)$, where the product is given by*

$$\mathcal{F} \times \mathcal{G} = pr_1^{\Leftarrow}(\mathcal{F}) \vee pr_2^{\Leftarrow}(\mathcal{G}).$$

In particular, for a frame L, if $\mathcal{F} \in \mathcal{F}^s_L(X)$ and $\mathcal{G} \in \mathcal{F}^s_L(Y)$ and $\nu \in L^{X \times Y}$, we have

$$\mathcal{F} \times \mathcal{G}(\nu) = \bigvee\{\mathcal{F}(\nu_1) \wedge \mathcal{G}(\nu_2) \colon \nu_1 \times \nu_2 \leq \nu\}.$$

If $L = (L, \leq, *, \otimes)$ is a pseudo-bisymmetric enriched *cl*-premonoid, $(X, \cdot)$ is a group and $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, then the map $\mathcal{F} \odot \mathcal{G}\colon L^X \to L$ is defined for any $\nu \in L^X$ by: $\mathcal{F} \odot \mathcal{G}(\nu) = \bigvee\{\mathcal{F}(\nu_1) * \mathcal{G}(\nu_2) | \nu_1, \nu_2 \in L^X, \nu_1 \odot \nu_2 \leq \nu\}$.
If $(X, \cdot)$ is a group and $\mathcal{F} \in \mathcal{F}_L^s(X)$, then $\mathcal{F}^{-1}$ is defined by $\mathcal{F}^{-1}(\nu) = \mathcal{F}(\nu^{-1})$, where $\nu^{-1} : X \to L, x \mapsto \nu(x^{-1})$. Clearly, $\mathcal{F}^{-1} \in \mathcal{F}_L^s(X)$, since for any $\nu \in L^X$, $\jmath^{\Rightarrow}(\mathcal{F})(\nu) = \mathcal{F}(\jmath^{\leftarrow}(\nu)) = \mathcal{F}(\nu^{-1}) = \mathcal{F}^{-1}(\nu)$, where $\jmath : X \to X, x \mapsto x^{-1}$. Also, if $m : X \times X \to X, (x, y) \mapsto xy$, then for any $\nu_1, \nu_2 \in L^X$ and $z \in X$, $m^{\rightarrow}(\nu_1 \times \nu_2)(z) = \bigvee_{m(x,y)=z}(\nu_1 \times \nu_2)(x, y) = \bigvee_{xy=z}(\nu_1 \circ pr_1 * \nu_2 \circ pr_2)(x, y) = \bigvee_{xy=z} \nu_1 \circ pr_1(x, y) * \nu_2 \circ pr_2(x, y) = \bigvee_{xy=z} \nu_1(x) * \nu_2(y) = \nu_1 \odot \nu_2(z)$. Furthermore, for a group $(G, \cdot)$, if $\mathbb{F}, \mathbb{G} \in \mathbb{F}(G)$, then $\mathbb{F} \odot \mathbb{G}$ is defined as a filter generated by the sets $F \cdot G = \{pq : p \in F, q \in G\}$, where $F \in \mathbb{F}$ and $G \in \mathbb{G}$; usually, we denote it by $\mathbb{F} \odot \mathbb{G} = [\{F \cdot G : F \in \mathbb{F}, G \in \mathbb{G}\}]$.

Lemma 2.10. *[3] Let $L = (L, \leq, *, \otimes = *)$ be a **G**L-monoid and $(X, \cdot)$ be a group. Then for any $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $m^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) = \mathcal{F} \odot \mathcal{G}$.*

Note that if $L = ([0, 1], \leq, *, \otimes = \wedge)$ with a *t*-norm $*$, then the above lemma is always true. But if $L = (L, \leq, *, \otimes)$, then since we do not have an explicit formula for the product *L*-filter, we cannot say in the perspective of the preceding lemma that this lemma holds.

3. Continuous action of enriched lattice-valued convergence groups on enriched lattice-valued convergence spaces

Definition 3.1. *[30, 41] Let $L = (L, \leq, *, \otimes = \wedge)$ be an enriched* cl*-premonoid and $\lim\colon \mathcal{F}_L^s(X) \longrightarrow L^X$ be a mapping such that the following are satisfied:*
(C1) $\forall x \in X$, $\lim[x](x) = \top$;
(C2) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ with $\mathcal{F} \leq \mathcal{G}$, and $\forall x \in X$, $\lim \mathcal{F}(x) \leq \lim \mathcal{G}(x)$;
(C3) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $\forall x \in X$, $\lim \mathcal{F}(x) * \lim \mathcal{G}(x) \leq \lim(\mathcal{F} \wedge \mathcal{G})(x)$;
(C3s) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $\forall x \in X$, $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(x) \leq \lim(\mathcal{F} \wedge \mathcal{G})(x)$.
Then the pair $(X, \lim)$ is called an stratified enriched *cl*-premonoid-valued convergence space *(or simply by* enriched latticed-valued convergence space *if conditions (C1)-(C3) hold, if the conditions (C1),(C2) and (C3s) hold, then we speak of* stratified enriched *cl*-premonoid-valued strong convergence space or simply by enriched strong lattice-valued convergence space*.*
A mapping $f\colon (X, \lim) \longrightarrow (X', \lim')$ between enriched lattice-valued convergence spaces (resp. between enriched lattice-valued strong convergence spaces) is called continuous *if and only if $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X$, $\lim \mathcal{F}(x) \leq \lim' f^{\Rightarrow}(\mathcal{F})(f(x))$.*
The category of all stratified enriched cl*-premonoid-valued convergence spaces and continuous mappings is denoted by **S**L-**CONV** (resp. the category of all stratified enriched* cl*-premonoid-valued strong convergence spaces is denoted by **SS**L-**CONV**).*

Proposition 3.2. *[16, 41] (see also [30]) The category **S**L-**CONV** (resp. **SS**L-**CONV**) is topological over the category **SET**. In particular, if $\left(f_j\colon X \longrightarrow \left(Y_j, \lim_j\right)\right)_{j\in J}$ is a source, then the initial structure $\lim\colon \mathcal{F}_L^s(X) \longrightarrow L^X$ on X is given for any $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X$ by*

$$\lim \mathcal{F}(x) = \bigwedge_{j\in J} \lim_j f_j^{\Rightarrow}(\mathcal{F})(f_j(x)).$$

For the convenience of the reader, we however, recall the product structure of lattice-valued convergence spaces as given in [16, 41] (see [30] for the case of frame), although we will be using this for some specific lattices.
Let $L = (L, \leq, *, \otimes)$ be a pseudo-bisymmetric enriched *cl*-premonoid. Let $(X, \lim)$, $\left(Y, \overline{\lim}\right)$ be **S**L-convergence spaces, then their product $\lim \times \overline{\lim}\colon \mathcal{F}_L^s(X \times Y) \longrightarrow L^{X\times Y}$ defined for an $\mathcal{F} \in \mathcal{F}_L^s(X \times Y)$ by

$$\left(\lim \times \overline{\lim}\right)\mathcal{F} = pr_1^{\leftarrow}\left(\lim pr_1^{\Rightarrow}(\mathcal{F})\right) \wedge pr_2^{\leftarrow}\left(\overline{\lim} pr_2^{\Rightarrow}(\mathcal{F})\right).$$

The pair $\left(X \times Y, \lim \times \overline{\lim}\right)$ is a product stratified *L*-convergence space.

Definition 3.3. *[2, 3] Let $L = (L, \leq, *, \wedge)$ be an enriched* cl*-premonoid, $(G, \cdot) \in |\mathbf{GRP}|$, and $(G, \lim) \in |\mathbf{SL\text{-}CONV}|$ (resp. $(G, \lim) \in |\mathbf{SSL\text{-}CONV}|$). Then the triple $(G, \cdot, \lim)$ is called a* stratified L-convergence group *(resp.* stratified strong L-convergence group*) if and only if the following axioms are satisfied:*
(CGM) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(G)$, $\forall x, y \in G$: $\lim \mathcal{F}(x) * \lim \mathcal{G}(y) \leq \lim(\mathcal{F} \odot \mathcal{G})(xy)$(resp. $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y) \leq \lim(\mathcal{F} \odot \mathcal{G})(xy)$), whenever $\mathcal{F} \odot \mathcal{G}$ is stratified L-filter;
(CGI) $\forall \mathcal{F} \in \mathcal{F}_L^s(G)$ $\forall x \in X$, $\lim \mathcal{F}(x) \leq \lim \mathcal{F}^{-1}(x^{-1})$.
The category of all stratified L-convergence groups and continuous group homomorphisms is denoted by **SL-CONVGRP**. *Similarly, the category* **SSL-CONVGRP** *is described.*

Remark 3.4. *[2] If $(L, \leq, *, \otimes) = (L, \leq, * = \wedge, \otimes = \wedge)$, i.e., when L is a frame, then (CGM) is equivalent to the continuity of multiplication $m\colon G \times G \longrightarrow G, (x, y) \longmapsto xy$. In fact, viewing product stratified strong L-convergence structure for frame L as in the case of [30], one can say that a mapping $m\colon (G, \lim) \times (G, \lim) \longrightarrow (G, \lim), (x, y) \longmapsto xy$ is continuous if and only if for all stratified L-filters $\mathcal{F}$, $\mathcal{G}$, and for all (x, y), $\lim pr_1^{\Rightarrow}(\mathcal{F} \times \mathcal{G})(x) \wedge \lim pr_2^{\Rightarrow}(\mathcal{F} \times \mathcal{G})(y) \leq \lim m^{\Rightarrow}(\mathcal{F} \times \mathcal{G})(m(x, y))$ which in turn yields that $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y) = \lim pr_1^{\Rightarrow}(\mathcal{F} \times \mathcal{G})(x) \wedge \lim pr_2^{\Rightarrow}(\mathcal{F} \times \mathcal{G})(y) \leq \lim m^{\Rightarrow}(\mathcal{F} \times \mathcal{G})(m(x, y))$. In view of Lemma 2.10, this amounts to say that $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y) \leq \lim(\mathcal{F} \odot \mathcal{G})(xy)$.*

Definition 3.5. *Let $L = (L, \leq, *, \otimes = \wedge)$ be an enriched* cl*-premonoid. Let $(G, \cdot, \lim_G) \in |\mathbf{SL\text{-}CONVGRP}|$ (resp. $(G, \cdot, \lim_G) \in |\mathbf{SSL\text{-}CONVGRP}|$), and $(X, \lim_X) \in |\mathbf{SL\text{-}CONV}|$ (resp. $(X, \lim_X) \in |\mathbf{SSL\text{-}CONV}|$). Then the triple $((G, \cdot, \lim_G), (X, \lim_X), \varphi)$ is called a* stratified L-convergence transformation group *(resp.* stratified strong L-convergence transformation group*) on a stratified (resp. stratified strong) L-convergence space with respect to $\varphi\colon G \times X \longrightarrow X, (g, x) \longmapsto \varphi(g, x)$ if the following axioms are satisfied:*
(CTG1) $\forall \mathcal{H} \in \mathcal{F}_L^s(G)$, $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$, $\forall g \in G$ and $x \in X$:

$$\lim_G \mathcal{H}(g) * \lim_X \mathcal{F}(x) \leq \lim_X (\varphi^{\Rightarrow}(\mathcal{H} \times \mathcal{F}))(\varphi(g, x))$$
$$\text{(resp. } \lim_G \mathcal{H}(g) \wedge \lim_X \mathcal{F}(x) \leq \lim_X (\varphi^{\Rightarrow}(\mathcal{H} \times \mathcal{F}))(\varphi(g, x)));$$

(CTG2) $\varphi(e, x) = x$, $\forall x \in X$;
(CTG3) $\varphi(g \cdot h, x) = \varphi(g, \varphi(h, x))$, $\forall g, h \in G$ and $\forall x \in X$.
Here the function φ is called continuous action of G on X, *the enriched* cl*-premonoid-valued stratified convergence group or in short, stratified L-convergence group $(G, \cdot, \lim_G)$ is called a* phase group, *and the stratified L-convergence space $(X, \lim_X)$ is called* phase space.
The category of stratified L-convergence transformation groups (resp. stratified strong L-convergence transformation groups) denoted by **SL-CONVTGRP** *(resp.* **SSL-CONVTGRP***) consists of all stratified L-convergence transformation groups (resp. stratified strong L-convergence transformation groups) as objects, and all pairs $(k, f)\colon (G, X, \varphi) \longrightarrow (G', X', \varphi')$ as morphism, where*
(TG1) $G \xrightarrow{k} G'$ is a **SL-CONVGRP***-morphism (resp.* **SSL-CONVGRP***-morphism), i.e., continuous group homomorphism;*
(TG2) $X \xrightarrow{f} X'$ is a **SL-CONV***-morphism (resp.* **SSL-CONV***-morphism), i.e., a continuous mapping such that $\varphi' \circ (k \times f) = f \circ \varphi$.*

Remark 3.6. *We would like to point out briefly that in the introduction we already mentioned that the notion of lattice-valued convergence is an extension of $\{0, 1\}$-valued convergence which can be identified with classical convergence as mentioned in a remark in [30]. Considering this point, and exploiting the relationship between classical filters and stratified L-filters (see f.i. Section 6 [41] and Section 3 [31], see also [24]), one can view that our present study of lattice-valued convergence transformation group is an extension of classical convergence transformation group.*

Example 3.7. *Any stratified L-convergence groups $(G, \cdot, \lim)$ can be made into a stratified L-convergence transformation group $((G, \cdot, \lim), (G, \lim), \varphi)$ on itself in the following way: $((G, \cdot, \lim), (G, \lim), \varphi)$, where $\varphi\colon G \times G \longrightarrow G$ defined by $\varphi(t, s) = ts$, the left translation in G.*

Example 3.8. *Let $((G, \cdot, \lim_G), (X, \lim_X), \varphi) \in |\mathbf{SL\text{-}CONVTGRP}|$ on a stratified L-convergence group $(X, \lim_X)$ with respect to φ. Then $((G, \cdot, \lim_G), (X \times X, \lim_X \times \lim_X), \varphi')$, where $\varphi'\colon G \times (X \times X) \longrightarrow (X \times X)$ is defined by*

$\varphi'(g,(x,x')) = (\varphi(g,x),\varphi(g,x'))$, for all $g \in G$ and $(x,x') \in X \times X$, is a stratified L-convergence transformation group on the product stratified L-convergence space.

Example 3.9. *Let $((G,\cdot,\lim),(X,\lim),\varphi) \in |$**SL-CONVTGRP**$|$ on a stratified L-convergence space $(X,\lim)$ with respect to φ, and let $((G,\cdot,\lim),(Y,\lim'),\psi) \in |$**SL-CONVTGRP**$|$ on a stratified L-convergence space $(Y,\lim')$ with respect to ψ. Then $((G,\cdot,\lim),(X\times Y,\lim\times\lim'),\gamma) \in |$**SL-CONVTGRP**$|$ on a stratified L-convergence space $(X\times Y,\lim\times\lim')$ with respect to $\gamma\colon G\times(X\times Y)\longrightarrow(X\times Y)$, is defined by $\gamma(g,(x,y)) = (\varphi(g,x),\psi(g,y))$.*
In fact, for $g,h \in G$, and $(x,y) \in X\times Y$,

$$\gamma(g\cdot h,(x,y)) = (\varphi(g\cdot h,x),\psi(g\cdot h,y)) = (\varphi(g,\varphi(h,x)),\psi(g,\psi(h,y))) = \gamma(g,(\varphi(h,x),\psi(h,y))) = \gamma(g,\gamma(h,(x,y))).$$

The continuity of γ follows from the continuity of φ and ψ.

Definition 3.10. *[30] Let $(L,\leq,\wedge)$ be a frame, $(X,\lim)$, $(Y,\lim') \in |$**SSL-CONV**$|$ and $C(X,Y) = \{f\colon (X,\lim)\longrightarrow(Y,\lim') : f$ is continuous$\}$. If $c-\lim\colon \mathcal{F}^s_L(C(X,Y))\longrightarrow L^{C(X,Y)}$ is defined for any $\Phi \in \mathcal{F}^s_L(C(X,Y))$ and $f \in C(X,Y)$ by*

$$c-\lim\Phi(f) = \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(X)}\bigwedge_{x\in X}[\lim\mathcal{F}(x)\to\lim ev^{\Rightarrow}(\Phi\times\mathcal{F})(f(x))],$$

where $ev\colon C(X,Y)\times X\longrightarrow Y, (f,x)\longmapsto f(x)$ is called an evaluation mapping *whence limit $c-\lim$ is called* stratified L-convergence of continuous convergence.

Lemma 3.11. *[30, 33] Let $f\colon X\longrightarrow Y$ be a mapping and $ev\colon C(X,Y)\times X\longrightarrow Y$ be the evaluation mapping. Then for all $\mathcal{F}\in\mathcal{F}^s_L(X)$, $f^{\Rightarrow}(\mathcal{F}) = ev^{\Rightarrow}([f]\times\mathcal{F})$.*

Lemma 3.12. *[30] If L is a frame, then $(C(X,Y),c-\lim) \in |$**SSL-CONV**$|$.*

Definition 3.13. *Let $(X,\lim) \in |$**SSL-CONV**$|$ and $\mathcal{H}(X) = \{f\colon X\longrightarrow X\colon\ f$ is a homeomorphism, i.e., a bijective and both f, f^{-1} are continuous$\}$. Define a stratified strong L-convergence structure $\mathfrak{c}-\lim\colon\mathcal{F}(\mathcal{H}(X))\longrightarrow L^{\mathcal{H}(X)}$ on $\mathcal{H}(X)$ the group of homeomorphisms under composition: for any $\Phi\in\mathcal{F}^s_L(\mathcal{H}(X))$ and $f\in\mathcal{H}(X)$ by*
$\mathfrak{c}-\lim(\Phi)(f) = c-\lim(\Phi)(f)\ \wedge\ c-\lim^{(-1)}(\Phi)(f)$, *where*
$c-\lim(\Phi)(f) = \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(X)}\bigwedge_{x\in X}[\lim\mathcal{F}(x)\to\lim ev^{\Rightarrow}(\Phi\times\mathcal{F})(f(x))]$ (3.1),
and
$c-\lim^{(-1)}(\Phi)(f) = \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(X)}\bigwedge_{x\in X}[\lim\mathcal{F}(x)\to\lim ev^{\Rightarrow}\left(\Phi^{-1}\times\mathcal{F}\right)(f^{-1}(x))]$ (3.2).

Remark 3.14. *Note that in view of the Proposition 2.6, and upon combining preceding items* (3.1) *and* (3.2)*, one can write the above definition as follows:*
$\mathfrak{c}-\lim(\Phi)(f) = \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(X)}\bigwedge_{x\in X}\ \lim\mathcal{F}(x)\to[\lim ev^{\Rightarrow}(\Phi\times\mathcal{F})(f(x))\wedge\lim ev^{\Rightarrow}\left(\Phi^{-1}\times\mathcal{F}\right)(f^{-1}(x))]$, *where* $\Phi^{-1} = [\{\phi^{-1}\colon\phi\in\Phi\}]$, $\phi^{-1} = [\{f^{-1}\in\mathcal{H}(X)\colon f\in\mathcal{H}(X)\}]$.

Theorem 3.15. *Let $L = (L,\leq,\wedge)$ be a frame, and $\lim$ be a stratified strong L-convergence structure on X, then $(\mathcal{H}(X),\cdot,\mathfrak{c}-\lim) \in |$**SSL-CONVGRP**$|$, where $\mathfrak{c}-\lim = c-\lim\ \wedge c-\lim^{(-1)}$.*

Proof. (C1) follows upon using Lemma 3.11 ([30]: Lemma 8.2), while (C2) follows from Proposition 8.3 [30].
(C3s) Upon using exactly the similar rout as in Lemma 4.3 [32], one can proof this part. We prove only (CGM) and (CGI):
(CGM) Let $\Phi,\Psi\in\mathcal{F}^s_L(\mathcal{H}(X))$ and $f,g\in\mathcal{H}(X)$. We show that
$\mathfrak{c}-\lim\Phi(f)\wedge\mathfrak{c}-\lim\Psi(g)\leq\mathfrak{c}-\lim(\Phi\odot\Psi)(f\cdot g)$.
We have $c-\lim\Phi(f)\wedge(c-\lim\Psi(g)\wedge\mathcal{F}(x))$
$\leq c-\lim\Phi(f)\wedge\lim ev^{\Rightarrow}(\Psi\times\mathcal{F})(g(x))$
$\leq\lim ev^{\Rightarrow}(\Phi\times ev^{\Rightarrow}(\Psi\times\mathcal{F}))(f\cdot g(x)) = \lim ev^{\Rightarrow}((\Phi\odot\Psi)\times\mathcal{F}))(f\cdot g(x))$.
We give here for the reader an explicit proof upon using evaluation mapping for the validity of the inequality $ev^{\Rightarrow}(\Phi\times ev^{\Rightarrow}(\Psi\times\mathcal{F}))\leq ev^{\Rightarrow}((\Phi\odot\Psi)\times\mathcal{F}))$ for better understanding. For that take $\nu\in L^{\mathcal{H}(X)\times(\mathcal{H}(X)\times X)}$, then

for any $f, g \in \mathcal{H}(X)$,
$ev^{\Rightarrow}(\Phi \times ev^{\Rightarrow}(\Psi \times \mathcal{F}))(\nu) = [\Phi \times ev^{\Rightarrow}(\Psi \times \mathcal{F})](ev^{\leftarrow}(\nu))$
$= \bigvee_{\nu_1 \times \nu_2 \leq ev^{\leftarrow}(\nu), \nu_1 \in L^{\mathcal{H}(X)}, \nu_2 \in L^{\mathcal{H}(X)\times X}} [\Phi(\nu_1) \wedge \left(\bigvee_{\rho_1 \times \rho_2 \leq ev^{\leftarrow}(\nu_2), \rho_1 \in L^{\mathcal{H}(X)}, \rho_2 \in L^X} \Psi(\rho_1) \wedge \mathcal{F}(\rho_2)\right)]$
$= \bigvee_{\nu_1 \times \nu_2 \leq ev^{\leftarrow}(\nu); \rho_1 \times \rho_2 \leq ev^{\leftarrow}(\nu_2)} (\Phi(\nu_1) \wedge \Psi(\rho_1)) \wedge \mathcal{F}(\rho_2)$
$\leq \bigvee_{\nu_1(f) \wedge \nu_2(ev(g,x)) \leq \nu(ev(f, ev(g,x))); \rho_1(g) \wedge \rho_2(x) \leq \nu_2(ev(g,x))} (\Phi(\nu_1) \wedge \Psi(\rho_1)) \wedge \mathcal{F}(\rho_2)$
$\leq \bigvee_{\nu_1(f) \wedge \rho_1(g) \wedge \rho_2(x) \leq ev^{\leftarrow}(\nu)(f \cdot g, x)} (\Phi(\nu_1) \wedge \Psi(\rho_1)) \wedge \mathcal{F}(\rho_2)$
$\leq \bigvee_{\nu_1(f) \wedge \rho_1(g) \wedge \rho_2(x) \leq ev^{\leftarrow}(\nu)(f \cdot g, x)} \left(\bigvee_{\nu_1 \odot \rho_1 \leq \xi} (\Phi \odot \Psi)(\xi)\right) \wedge \mathcal{F}(\rho_2)$
$= \bigvee_{\nu_1(f) \wedge \rho_1(g) \wedge \rho_2(x) \leq ev^{\leftarrow}(\nu)(f \cdot g, x); \nu_1(f) \wedge \rho_1(g) \leq \xi(f \cdot g)} (\Phi \odot \Psi)(\xi) \wedge \mathcal{F}(\rho_2)$
$\leq \bigvee_{\xi(f \cdot g) \wedge \rho_2(x) \leq \varphi^{\leftarrow}(\nu)(f \cdot g, x))} (\Phi \odot \Psi)(\xi) \wedge \mathcal{F}(\rho_2)$
$= \bigvee_{\xi \times \rho_2 \leq \varphi^{\leftarrow}(\nu)} (\Phi \odot \Psi)(\xi) \wedge \mathcal{F}(\rho_2)$
$= ev^{\Rightarrow}(\Phi \odot \Psi \times \mathcal{F})(\nu)$, i.e., $ev^{\Rightarrow}(\Phi \times ev^{\Rightarrow}(\Psi \times \mathcal{F})) \leq ev^{\Rightarrow}(\Phi \odot \Psi \times \mathcal{F})$.
Thus, we have
$c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \to \lim ev^{\Rightarrow}((\Phi \odot \Psi) \times \mathcal{F}))(f \cdot g(x))$, i.e.,
$c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq c - \lim \Phi \odot \Psi(f \cdot g)$.
Similarly, one can show that $c - \lim^{(-1)} \Phi(f) \wedge c - \lim^{(-1)} \Psi(g) \leq c - \lim^{(-1)} \Phi \odot \Psi(f \cdot g)$.
(CGI) Let $\Phi \in \mathcal{F}_L^s(\mathcal{H}(X)$ and $f \in \mathcal{H}(X)$. Then we have from Definition 3.13 (3.1),
$c - \lim \Phi(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \to \lim ev^{\Rightarrow}(\Phi \times \mathcal{F})(f(x))]$ (3.3)
We apply the continuities of $\jmath : x \longrightarrow x^{-1}$ and the composition $\jmath \circ ev : \mathcal{H}(X) \times X \longrightarrow X$
together with the fact that $(ev^{\Rightarrow}(\Phi \times \mathcal{F}))^{-1} = ev^{\Rightarrow}(\Phi^{-1} \times \mathcal{F})$. In fact,
for any $\nu \in L^{\mathcal{H}(X) \times X}$, $(ev^{\Rightarrow}(\Phi \times \mathcal{F}))^{-1}(\nu) = ev^{\Rightarrow}(\Phi \times \mathcal{F})(\nu^{-1}) = \Phi \times \mathcal{F}\left(ev^{\leftarrow}(\nu^{-1})\right)$
$= \bigvee_{\nu_1 \times \nu_2 \leq ev^{\leftarrow}(\nu^{-1}), \nu_1 \in L^{\mathcal{H}(X)}, \nu_2 \in L^X} \Phi(\nu_1) \wedge \mathcal{F}(\nu_2)$ (3.4).
Now for any $f \in \mathcal{H}(x)$ and $x \in X$, we have
$\nu_1(f) \wedge \nu_2(x) \leq \nu^{-1}(ev(f, x)) = \nu^{-1}(f(x)) = \nu((f(x))^{-1})$
$= \nu(f^{-1}(x)) = (f^{-1})^{\leftarrow}(\nu)(x) \implies \nu_1^{-1}(f^{-1}) \wedge \nu_2(x) \leq (f^{-1})^{\leftarrow}(\nu)(x)$
$\implies (\nu_1^{-1} \times \nu_2)(f^{-1}, x) \leq ev^{\leftarrow}(\nu)(f^{-1}, x)$, i.e., $\nu_1^{-1} \times \nu_2 \leq ev^{\leftarrow}(\nu)$.
Then continuing after (3.4), we have
$(ev^{\Rightarrow}(\Phi \times \mathcal{F}))^{-1}(\nu) = \bigvee_{\nu_1 \times \nu_2 \leq ev^{\leftarrow}(\nu^{-1}), \nu_1 \in L^{\mathcal{H}(X)}, \nu_2 \in L^X} \Phi(\nu_1) \wedge \mathcal{F}(\nu_2)$
$\leq \bigvee_{\nu_1^{-1} \times \nu_2 \leq ev^{\leftarrow}(\nu)} \left(\Phi^{-1}(\nu_1^{-1}) \wedge \mathcal{F}(\nu_2)\right) = ev^{\Rightarrow}\left(\Phi^{-1} \times \mathcal{F}\right)(\nu)$, i.e.,
$(ev^{\Rightarrow}(\Phi \times \mathcal{F}))^{-1} \leq ev^{\Rightarrow}(\Phi^{-1} \times \mathcal{F})$. Therefore, it follows from (3.3) that
$c - \lim \Phi(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \to \lim ev^{\Rightarrow}(\Phi \times \mathcal{F})(f(x))]$
$\leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \to \lim ev^{\Rightarrow}(\Phi^{-1} \times \mathcal{F})(f^{-1}(x))]$
$= c - \lim \Phi^{-1}(f^{-1})$. Similarly, one can show that $c - \lim^{(-1)} \Phi(f) \leq c - \lim^{(-1)} \Phi^{-1}(f^{-1})$
which after combing together imply that $\mathfrak{c} - \lim \Phi(f) \leq \mathfrak{c} - \lim \Phi^{-1}(f^{-1})$. □

Lemma 3.16. *Let $(L, \leq, \wedge)$ be a frame. Then the stratified strong L-convergence group $(\mathcal{H}(X), \cdot, \mathfrak{c} - \lim)$ has the property that for any stratified strong L-convergence space $(Y, \cdot, \lim_Y)$, the continuity of the mapping $\zeta : (Y, \lim_Y) \longrightarrow (\mathcal{H}(X), \mathfrak{c} - \lim)$ implies the continuity of $\overline{\zeta} : Y \times X \longrightarrow X$, where $\overline{\zeta}(y, x) = \zeta(y)(x)$, ($\zeta(y) \in \mathcal{H}(X)$, and $x \in X$)*

Proof. Since ζ and the identity mappings id_X are continuous, $\zeta \times id_X$ is continuous, it follows then in conjunction with the continuity of ev, the composition $ev \circ \zeta \times id_X$ is continuous as shown in the commutative diagram below. Hence $\overline{\zeta}$ is continuous.

$$\begin{array}{ccc} Y \times X & \xrightarrow{\zeta \times id_X} & \mathcal{H}(X) \times X \\ & \overline{\zeta} \searrow & ev \downarrow \\ & & X \end{array}$$

□

Now we present a characterization theorem.

Theorem 3.17. *Let $(L, \leq, \wedge)$ be a frame. Then the triple $((G, \cdot, \lim_G), (X, \lim_X), \varphi)$ is a stratified strong L-convergence transformation group on a stratified strong L-convergence space $(X, \lim_X)$ with respect to the mapping $\varphi\colon G \times X \longrightarrow X, (g, x) \longmapsto \varphi(g, x)$, if and only if $\chi\colon (G, \lim_G) \longrightarrow (\mathcal{H}(X), \mathfrak{c} - \lim)$ is a continuous homomorphism, where χ is defined by $\chi(g)(x) = \varphi(g, x)$, $g \in G$, and $x \in X$.*

Proof. Let $((G, \cdot, \lim_G), (X, \lim_X), \varphi) \in |\mathbf{SS}L\textbf{-CONVTGRP}|$. Define $\chi(g)(x) = \varphi(g, x)$ for $g \in G$ and $x \in X$. Then applying Definition 3.5(CTG3), we have for any $g' \in G$, $\chi_{gg'}(x') = \varphi(gg', x) = \varphi(g, \varphi(g', x)) = \varphi\left(g, \chi_{g'}(x)\right) = \chi_g(\chi_{g'}(x)) = \left(\chi_g \circ \chi_{g'}\right)(x)$ implying $\chi_{gg'} = \chi_g \circ \chi_{g'}$. So, $\chi\colon G \longrightarrow \mathcal{H}(X)$ is a homomorphism. To show that the mapping $\chi\colon (G, \lim_G) \longrightarrow (\mathcal{H}(X), \mathfrak{c} - \lim)$ is continuous, we let $\mathcal{F} \in \mathcal{F}^s_L(G)$ and $g \in G$, and show that $\lim_G \mathcal{F}(g) \leq \mathfrak{c} - \lim \chi(\mathcal{F})(\chi_g)$. Now let $x \in X$ and $\mathcal{G} \in \mathcal{F}^s_L(X)$. Put $\chi^{\Rightarrow}(\mathcal{F}) = \Phi \in \mathcal{F}^s_L(\mathcal{H}(X))$. Since φ is continuous, we have
$\lim_G \mathcal{F}(g) \wedge \lim_X \mathcal{G}(x) \leq \lim_X \varphi^{\Rightarrow}(\Phi \times \mathcal{G})(\chi_g(x))$, whence $\chi_g(x) = \varphi(g, x)$.
Then one can write $\lim_G \mathcal{F}(g) \leq \lim_X \mathcal{G}(x) \to \lim_X ev^{\Rightarrow}(\Phi \times \mathcal{G})((\chi_g(x))$. This is true for any $\mathcal{G} \in \mathcal{F}^s_L(X)$ and for any $x \in X$ that
$\lim_G \mathcal{F}(g) \leq \bigwedge_{\mathcal{G} \in \mathcal{F}^s_L(X)} \bigwedge_{x \in X}[\lim_X \mathcal{G}(x) \to \lim_X ev^{\Rightarrow}(\Phi \times \mathcal{G})(\chi_g(x))]$
$= c - \lim \Phi(\chi_g)$, i.e., $\lim_G(\mathcal{F})(g) \leq c - \lim \Phi(\chi_g)$ (3.5).
Next, as $(G, \cdot, \lim_G) \in |\mathbf{SS}L\textbf{-CONVGRP}|$, for any $\mathcal{F} \in \mathcal{F}^s_L(G)$, $\lim_G \mathcal{F}(g) \leq \lim_G \mathcal{F}^{-1}(g^{-1})$, and since $\chi\colon G \longrightarrow \mathcal{H}(X)$ is a homomorphism $\chi_{g^{-1}} = (\chi_g)^{-1}$, we have
$\lim_G \mathcal{F}(g) \leq \lim_X \mathcal{G}(x) \to \lim_X ev^{\Rightarrow}\left(\Phi^{-1} \times \mathcal{G}\right)(\chi^{-1}_g(x))$
implying that $\lim_G \mathcal{F}(g) \leq \bigwedge_{\mathcal{G} \in \mathcal{F}^s_L(X)} \bigwedge_{x \in X}[\lim_X \mathcal{G}(x) \to \lim ev^{\Rightarrow}\left(\Phi^{-1} \times \mathcal{G}\right)(\chi^{-1}_g(x))]$.
This mens that $\lim_G \mathcal{F}(g) \leq c - \lim^{(-1)} \Phi(\chi_g)$ (3.6) .
Hence from (3.5) and (3.6), we have $\lim_G \mathcal{F}(g) \leq c - \lim \Phi(\chi_g) \wedge c - \lim \Phi^{-1}(\chi^{-1}_g)$.
To show the converse, let $\chi\colon G \longrightarrow \mathcal{H}(X)$ be continuous homomorphism, $g, h \in G$ and $x \in X$. Then $\varphi(gh, x) = \chi_{gh}(x) = \chi_g(\chi_h(x)) = \chi_g(\varphi(h, x)) = \varphi(g, \varphi(h, x))$ while the other condition is trivially true. The continuity of φ follows from the Lemma 3.16 with $Y = G$ and $\overline{\zeta} = \varphi$ (see also the diagram above).
Hence, the triple $((G, \cdot, \lim_G), (X, \lim_X), \varphi) \in |\mathbf{SS}L\textbf{-CONVTGRP}|$ over $(X, \lim_X) \in |\mathbf{SS}L\textbf{-CONV}|$ with respect to φ. □

Theorem 3.18. *Let $(L, \leq, \wedge)$ be a frame, $(X, \cdot, \lim_X) \in |\mathbf{SS}L\textbf{-CONVGRP}|$ and $(X, \lim_X) \in |\mathbf{SS}L\textbf{-CONV}|$. If $\mathcal{H}(X)$ is a group of homeomorphisms of $(X, \lim_X)$, then X is isomorphically embedded in $(\mathcal{H}(X), \mathfrak{c} - \lim)$.*

Proof. Let $\chi\colon X \longrightarrow \mathcal{H}(X)$, $x \longmapsto \chi_x$ given by $\chi_x(x') = xx'$ for any $x' \in X$. The mapping χ is an injective-homomorphism, we show that the mapping χ is continuous, for, let $\mathcal{F} \in \mathcal{F}^s_L(X)$, we show that $\lim \mathcal{F}(x) \leq \mathfrak{c} - \lim \chi^{\Rightarrow}(\mathcal{F})(\chi_x)$.
Since $(X, \cdot, \lim_X)$ is a stratified strong L-convergence group, for any $x' \in X$ and $\mathcal{G} \in \mathcal{F}^s_L(X)$, we have: $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(x') \leq \lim (\mathcal{F} \odot \mathcal{G})(xx') = \lim (\mathcal{F} \odot \lim \mathcal{G})(\chi_x(x'))$.
But it follows that $\mathcal{F} \odot \mathcal{G} \leq \chi^{\Rightarrow}(\mathcal{F})\mathcal{G}$, so
$\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(x') \leq \lim ev^{\Rightarrow}(\chi^{\Rightarrow}(\mathcal{F}) \times \mathcal{G})(\chi_x(x'))$
$\Rightarrow \lim \mathcal{F}(x) \leq \lim \mathcal{G}(x') \longrightarrow \lim ev^{\Rightarrow}(\chi^{\Rightarrow}(\mathcal{F}) \times \mathcal{G})(\chi_x(x'))$
This is true for any $\mathcal{G} \in \mathcal{F}^s_L(X)$ and any $x' \in X$. This means that
$\lim \mathcal{F}(x) \leq \bigwedge_{\mathcal{G} \in \mathcal{F}^s_L(X)} \bigwedge_{x' \in X}[\lim \mathcal{G}(x') \longrightarrow \lim ev^{\Rightarrow}(\chi^{\Rightarrow}(\mathcal{F}) \times \mathcal{G})(\chi_x(x'))]$
$= c - \lim \chi^{\Rightarrow}(\mathcal{F})(\chi_x)$. Similarly, $\lim \mathcal{F}(x) \leq c - \lim^{(-1)} \chi^{\Rightarrow}(\mathcal{F})(\chi_x)$
Hence $\lim \mathcal{F}(x) \leq \mathfrak{c} - \lim \chi^{\Rightarrow}(\mathcal{F})(\chi_x)$.
Now let us show that the mapping $\chi^{-1}_{|\chi(X)}\colon \chi(X) \longrightarrow X$ is continuous. Pick $\chi_x \in \chi(X)$ and suppose $\Phi \in \mathcal{F}^s_L(\chi(X))$; we show that $\mathfrak{c} - \lim_{|\chi(X)} \Phi(\chi_x) \leq \lim\ \chi^{-1}(\Phi)\left(\chi^{-1}(\chi_x)\right)$.
Since $\chi^{-1}_{|\chi(X)}(\chi_x) = x = xe = \chi_x(e) = ev(\chi_x, e)$, upon using the continuity of $ev\colon \mathcal{H}(X) \times X \longrightarrow X$,
we have $c - \lim_{|\chi(X)} \Phi(\chi_x) = \bigwedge_{\mathcal{G} \in \mathcal{F}^s_L(X)} \bigwedge_{x' \in X}[\lim \mathcal{G}(x') \longrightarrow \lim ev^{\Rightarrow}(\Phi \times \mathcal{G})(\chi_x(x'))]$
$\leq \lim[e](e) \longrightarrow \lim ev^{\Rightarrow}(\Phi \times [e])(ev(\chi_x, e)) = \lim ev^{\Rightarrow}(\Phi \times [e])(ev(\chi_x, e))$
This implies that $c - \lim_{|\chi(X)} \Phi(\chi_x) \leq \lim ev^{\Rightarrow}(\Phi \times [e])(ev(\chi_x, e))$

$= \lim(\chi^{-1})^{\Rightarrow}(\Phi)\left(\chi^{-1}(\chi_x)\right)$ (3.7).
Hence $\mathfrak{c}-\lim_{|\chi(X)}\Phi(\chi_x) \le \lim\,(\chi^{-1})^{\Rightarrow}(\Phi)\left(\chi^{-1}(\chi_x)\right)$, showing that
$\chi^{-1}_{|\chi(X)}: \chi(X) \longrightarrow X, \chi_x \longmapsto \chi^{-1}(\chi_x)$ is continuous. In fact, for any $\nu \in L^X$,
we get $ev^{\Rightarrow}\,(\Phi\times[e])\,(\nu) = \bigvee_{\nu_1\times\nu_2\le ev^{\leftarrow}(\nu),\ \nu_1\in L^{\chi(X)},\ \nu_2\in L^X}\ \Phi(\nu_1)\wedge[e](\nu_2)$
$\le \bigvee_{\nu_1\times\nu_2\le ev^{\leftarrow}(\nu),\ \nu_1\in L^{\chi(X)},\ \nu_2\in L^X}\Phi\,(\nu_1\wedge\nu_2(e))$ (3.8).
Now since $\nu_1(\chi_x)\wedge\nu_2(e) \le \nu(ev(\chi_x,e)) = \nu(\chi_x(e)) = \nu(x)$
$= \nu(\chi^{-1}(\chi_x)) = \left((\chi^{-1})^{\leftarrow}(\nu)\right)(\chi_x)$, we have from (3.8) that
(3.8) $\le \Phi\left((\chi^{-1})^{\leftarrow}(\nu)\right) = (\chi^{-1})^{\Rightarrow}(\Phi)(\nu)$, i.e., one obtains:
$ev^{\Rightarrow}\,(\Phi\times[e]) \le (\chi^{-1})^{\Rightarrow}(\Phi)$, which is used in (3.7). This ends the proof. □

4. Construction of a stratified enriched lattice-valued convergence transformation group

Definition 4.1. *Let $L = (L,\le,\wedge)$ be a frame, $(X,\lim) \in |SSL\text{-}\mathbf{CONV}|$, $(G,\cdot) \in |\mathbf{GRP}|$, and $\varphi: G\times X \longrightarrow X$ be a mapping satisfying the following:*
(1) $\varphi(e,x) = x$ for all $x\in X$;
(2) $\varphi(g\cdot h,x) = \varphi(g,\varphi(h,x))$, for all $g,h\in G$ and $x\in X$;
(3) $\varphi(g,.): X\longrightarrow X, x\longmapsto \varphi(g,x) = \varphi_g(x)$ is continuous , for each $g\in G$.
Define $\mathfrak{c}-\lim$ on G for all $g\in G$ and for all $\Phi\in\mathcal{F}^s_L(G)$ by
$\mathfrak{c}-\lim = c-\lim\ \wedge c-\lim^{(-1)}$ *such that*
$c-\lim\Phi(g) = \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(X)}\bigwedge_{x\in X}[\lim\mathcal{F}(x)\to\lim\varphi^{\Rightarrow}\,(\Phi\times\mathcal{F})\,(\varphi(g,x))]$ (4.1)
$c-\lim^{(-1)}\Phi(g) = \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(X)}\bigwedge_{x\in X}[\lim\mathcal{F}(x)\to\lim\varphi^{\Rightarrow}\left(\Phi^{-1}\times\mathcal{F}\right)(\varphi(g^{-1},x))$ (4.2)

Theorem 4.2. *Let $L = (L,\le,\wedge)$ be a frame. Let $(X,\lim)\in|SSL\text{-}\mathbf{CONV}|$, and $(G,\cdot)\in|\mathbf{GRP}|$. Then $\mathfrak{c}-\lim$ is a stratified strong L-convergence structure on G such that the triple $((G,\cdot,\mathfrak{c}-\lim),(X,\lim),\varphi)\in|SSL\text{-}\mathbf{CONVTGRP}|$ on $(X,\lim)\in|SSL\text{-}\mathbf{CONV}|$ with respect to φ.*

Proof. We need to prove the following three items:
(i) $(G,\cdot,\mathfrak{c}-\lim)\in|\mathbf{SS}L\text{-}\mathbf{CONVGRP}|$;
(ii) $\varphi: G\times X\longrightarrow X$ is continuous;
(iii) φ satisfies (1) and (2) above.
(iii) follows from the Definition 4.1(3). We prove (i), note that we use f, g the elements of G, and e as its identity element:
(C1) (a) Let $f\in G$ with $[f]\in\mathcal{F}^s_L(G)$, and $x\in X$. In view of (4.1), we have for all $\mathcal{F}\in\mathcal{F}^s_L(X)$ and for all $x\in X$:
$c-\lim\Phi(f)\le[\lim\mathcal{F}(x)\to\lim\varphi^{\Rightarrow}\,(\Phi\times\mathcal{F})\,(\varphi(f,x))]$.
Now due to continuity of $\varphi(g,.): X\longrightarrow X$, we have in particular, for any $f\in G$: $\lim[x](x)\le\lim\varphi^{\Rightarrow}([f]\times[x])(\varphi(f,x))$. But then
$\top = \lim[x](x)\to\lim\varphi^{\Rightarrow}([f]\times[x])(\varphi(f,x))\le c-\lim[f](f)$, i.e., $c-\lim[f](f) = \top$.
(b) Let $f\in G$, and since G is a group, we have $f^{-1}\in G$ and hence by continuity of $\varphi(f^{-1},.)$, in particular, one obtains:
$\lim[x](x)\le\lim\varphi^{\Rightarrow}([f^{-1}]\times[x])(\varphi(f^{-1},x))$, and then we have $\top = \lim[x](x)\to\lim\varphi^{\Rightarrow}([f^{-1}]\times[x])(\varphi(f^{-1},x))\le c-\lim^{(-1)}[f](f)$, i.e., $c-\lim^{(-1)}[f](f) = \top$. Hence combining (a) and (b), we get $\mathfrak{c}-\lim[f](f) = \top$.
(C2) (a) Let $\Phi,\Psi\in\mathcal{F}^s_L(G)$ with $\Psi\le\Phi$. Then $\varphi^{\Rightarrow}(\Psi\times\mathcal{F})\le\varphi^{\Rightarrow}(\Phi\times\mathcal{F})$
which implies that $\lim\varphi^{\Rightarrow}(\Psi\times\mathcal{F})\le\lim\varphi^{\Rightarrow}(\Phi\times\mathcal{F})$. Then by using (4.1),
we have for any $f\in G$, $c-\lim\Psi(f) = \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(G)}\bigwedge_{x\in X}[\lim\mathcal{F}(x)\longrightarrow\lim\varphi^{\Rightarrow}\,(\Psi\times\mathcal{F})\,(\varphi(f,x))]$
$\le\bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(G)}\bigwedge_{x\in X}[\lim\mathcal{F}(x)\longrightarrow\lim\varphi^{\Rightarrow}\,(\Phi\times\mathcal{F})\,(\varphi(f,x))] = c-\lim\Phi(f)$.
(b) Let $\Phi,\Psi\in\mathcal{F}^s_L(G)$ with $\Psi\le\Phi$. Then $\varphi^{\Rightarrow}(\Psi^{-1}\times\mathcal{F})\le\varphi^{\Rightarrow}(\Phi^{-1}\times\mathcal{F})$
which implies that $\lim\varphi^{\Rightarrow}(\Psi^{-1}\times\mathcal{F})\le\lim\varphi^{\Rightarrow}(\Phi^{-1}\times\mathcal{F})$. Then using (4.2),
we get for any $f\in G$
$c-\lim^{(-1)}\Psi(f) = \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(G)}\bigwedge_{x\in X}[\lim\mathcal{F}(x)\longrightarrow\lim\varphi^{\Rightarrow}\left(\Psi^{-1}\times\mathcal{F}\right)(\varphi(f^{-1},x))]$

$\leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(G)} \bigwedge_{x\in X}[\lim \mathcal{F}(x) \longrightarrow \lim \varphi^{\Rightarrow}\left(\Phi^{-1}\times\mathcal{F}\right)(\varphi(f^{-1},x))] = c - \lim^{(-1)}\Phi(f)$.
Hence, combining (a) and (b), we have
$\mathfrak{c} - \lim \Psi(f) = c - \lim \Psi(f) \wedge c - \lim^{(-1)}\Psi(f) \leq c - \lim \Phi(f) \wedge c - \lim^{(-1)}\Phi(f)$
$= \mathfrak{c} - \lim \Phi(f)$, i.e., $\mathfrak{c} - \lim \Psi(f) \leq \mathfrak{c} - \lim \Phi(f)$.
(C3s) Let $\Phi, \Psi \in \mathcal{F}_L^s(G)$ and $f \in G$. Then we have
(a) $c - \lim \Phi(f) \wedge c - \lim \Psi(f) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(G)} \bigwedge_{x\in X}[\lim \mathcal{F}(x) \longrightarrow \lim \varphi^{\Rightarrow}(\Phi\times\mathcal{F})(\varphi(f,x))] \wedge$
$\bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(G)} \bigwedge_{x\in X}[\lim \mathcal{F}(x) \longrightarrow \lim \varphi^{\Rightarrow}(\Psi\times\mathcal{F})(\varphi(f,x))]$
$\leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(G)} \bigwedge_{x\in X}[\lim \mathcal{F}(x) \longrightarrow \lim[\varphi^{\Rightarrow}(\Phi\times\mathcal{F}) \wedge \varphi^{\Rightarrow}(\Psi\times\mathcal{F})](\varphi(f,x))]$
$= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(G)} \bigwedge_{x\in X}[\lim \mathcal{F}(x) \longrightarrow \lim \varphi^{\Rightarrow}((\Phi\wedge\Psi)\times\mathcal{F}))(\varphi(f,x))] = c - \lim(\Phi\wedge\Psi)(f)$,
(b) This follows in an exact fashion as in (a) showing that
$c - \lim^{(-1)}\Phi(f) \wedge c - \lim^{(-1)}\Psi(f) \leq c - \lim^{(-1)}(\Phi\wedge\Psi)(f)$.
Hence, combining (a) and (b), we get $\mathfrak{c} - \lim \Phi(f) \wedge \mathfrak{c} - \lim \Psi(f) \leq \mathfrak{c} - \lim(\Phi\wedge\Psi)(f)$.
(CGM) Let $\Phi, \Psi \in \mathcal{F}_L^s(G)$. Then we prove that for any $f, g \in G$:
$\mathfrak{c} - \lim \Phi(f) \wedge \mathfrak{c} - \lim \Psi(g) \leq \mathfrak{c} - \lim(\Phi\odot\Psi)(f\cdot g)$. For that we proceed as follows:
(a) $c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq c - \lim(\Phi\odot\Psi)(f\cdot g)$
Now in view of (4.1), $c - \lim \Phi(f) \wedge (c - \lim \Psi(g) \wedge \lim \mathcal{F}(x))$
$\leq c - \lim \Phi(f) \wedge \lim \varphi^{\Rightarrow}(\Psi\times\mathcal{F})(\varphi(g,x))$
$\leq \lim \varphi^{\Rightarrow}(\Phi\times\varphi^{\Rightarrow}(\Psi\times\mathcal{F}))(\varphi(f,\varphi(g,x)))$, i.e.,
$c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq \lim \mathcal{F}(x) \to \lim \varphi^{\Rightarrow}(\Phi\times\varphi^{\Rightarrow}(\Psi\times\mathcal{F}))(\varphi(f,\varphi(g,x)))$;
this is true for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and for all $x \in X$, and thus,
$c - \lim \Phi(f) \wedge c - \Psi(g) \leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(X)} \bigwedge_{x\in X}[\lim \mathcal{F}(x) \to \lim \varphi^{\Rightarrow}(\Phi\times\varphi^{\Rightarrow}(\Psi\times\mathcal{F}))(\varphi(f,\varphi(g,x)))]$.
Applying Proposition 2.6 along with the fact that
$\varphi^{\Rightarrow}(\Phi\times\varphi^{\Rightarrow}(\Psi\times\mathcal{F}))(\varphi(f,\varphi(g,x)) \leq \varphi^{\Rightarrow}(\Phi\odot\Psi\times\mathcal{F})(\varphi(f\cdot g,x))$,
we get $c - \lim \Phi(f) \wedge c - \Psi(g) \leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X)} \bigwedge_{x\in X}[\lim \mathcal{F}(x) \to \lim \varphi^{\Rightarrow}(\Phi\odot\Psi\times\mathcal{F})(\varphi(f\cdot g,x))]$
$= c - \lim(\Phi\odot\Psi)(f\cdot g)$.
In fact, for $\nu \in L^{G\times(G\times X)}$, upon manipulating the condition (2) of the Definition 4.1, i.e., $\varphi(f,\varphi(g,x)) = \varphi(f\cdot g,x)$, we have
$\varphi^{\Rightarrow}(\Phi\times\varphi^{\Rightarrow}(\Psi\times\mathcal{F}))(\nu) = [\Phi\times\varphi^{\Rightarrow}(\Psi\times\mathcal{F})](\varphi^{\leftarrow}(\nu))$
$= \bigvee_{\nu_1\times\nu_2\leq\varphi^{\leftarrow}(\nu),\nu_1\in L^G,\nu_2\in L^{G\times X}}[\Phi(\nu_1) \wedge \left(\bigvee_{\rho_1\times\rho_2\leq\varphi^{\leftarrow}(\nu_2),\rho_1\in L^G,\rho_2\in L^X}\Psi(\rho_1)\wedge\mathcal{F}(\rho_2)\right)]$
$= \bigvee_{\nu_1\times\nu_2\leq\varphi^{\leftarrow}(\nu);\rho_1\times\rho_2\leq\varphi^{\leftarrow}(\nu_2)}(\Phi(\nu_1)\wedge\Psi(\rho_1))\wedge\mathcal{F}(\rho_2)$
$\leq \bigvee_{\nu_1(f)\wedge\nu_2(\varphi(g,x))\leq\nu(\varphi(f,\varphi(g,x)));\rho_1(g)\wedge\rho_2(x)\leq\nu_2(\varphi(g,x))}(\Phi(\nu_1)\wedge\Psi(\rho_1))\wedge\mathcal{F}(\rho_2)$
$\leq \bigvee_{\nu_1(f)\wedge\rho_1(g)\wedge\rho_2(x)\leq\varphi^{\leftarrow}(\nu)(f\cdot g,x)}(\Phi(\nu_1)\wedge\Psi(\rho_1))\wedge\mathcal{F}(\rho_2)$
$\leq \bigvee_{\nu_1(f)\wedge\rho_1(g)\wedge\rho_2(x)\leq\varphi^{\leftarrow}(\nu)(f\cdot g,x)}\left(\bigvee_{\nu_1\odot\rho_1\leq\xi}(\Phi\odot\Psi)(\xi)\right)\wedge\mathcal{F}(\rho_2)$
$= \bigvee_{\nu_1(f)\wedge\rho_1(g)\wedge\rho_2(x)\leq\varphi^{\leftarrow}(\nu)(f\cdot g,x);\nu_1(f)\wedge\rho_1(g)\leq\xi(f\cdot g)}(\Phi\odot\Psi)(\xi)\wedge\mathcal{F}(\rho_2)$
$\leq \bigvee_{\xi(f\cdot g)\wedge\rho_2(x)\leq\varphi^{\leftarrow}(\nu)(f\cdot g,x))}(\Phi\odot\Psi)(\xi)\wedge\mathcal{F}(\rho_2)$
$= \bigvee_{\xi\times\rho_2\leq\varphi^{\leftarrow}(\nu)}(\Phi\odot\Psi)(\xi)\wedge\mathcal{F}(\rho_2)$
$= \varphi^{\Rightarrow}(\Phi\odot\Psi\times\mathcal{F})(\nu)$, i.e., $\varphi^{\Rightarrow}(\Phi\times\varphi^{\Rightarrow}(\Psi\times\mathcal{F})) \leq \varphi^{\Rightarrow}(\Phi\odot\Psi\times\mathcal{F})$.
(b) Following almost in the similar fashion as in (a), we can arrive at :
$c - \lim^{(-1)}\Phi(f) \wedge c - \lim^{(-1)}\Psi(g) \leq c - \lim^{(-1)}(\Phi\odot\Psi)(f\cdot g)$
Hence combining (a) and (b), one obtains: $\mathfrak{c} - \lim \Phi(f) \wedge \mathfrak{c} - \lim \Psi(g) \leq \mathfrak{c} - \lim \Phi\odot\Psi(f\cdot g)$.
(CGI) This follows at once from the Definition 4.1 (4.2) while for the other part
we just use the inverse of the inverse of f, and the fact that if $\Phi \in \mathcal{F}_L^s(G)$,
then $\Phi^{-1} \in \mathcal{F}_L^s(G)$.
(ii) We show that the mapping $\varphi\colon (G\times X, c - \lim_G \times \lim_X) \longrightarrow (X, \lim_X)$,
$(f,x) \longmapsto \varphi(f,x)$ is continuous. For, let $\Upsilon \in \mathcal{F}_L^s(G\times X)$ and $(f,x) \in G\times X$.
Then we have $(c - \lim_G \times \lim_X)\Upsilon(f,x) = c - \lim_G pr_G^{\Rightarrow}(\Upsilon)(f) \wedge \lim_X pr_X^{\Rightarrow}(\Upsilon)(x)$
$\leq \lim_X pr_X^{\Rightarrow}(\Upsilon)(x) \to \lim_X \varphi^{\Rightarrow}\left(pr_G^{\Rightarrow}(\Upsilon)\times pr_X^{\Rightarrow}(\Upsilon)\right)(\varphi(f,x)) \wedge \lim_X pr^{\Rightarrow}(\Upsilon)(x)$.
But as it follows from Proposition 3.6 [30] that $pr_G^{\Rightarrow}(\Upsilon)\times pr_X^{\Rightarrow}(\Upsilon) \leq \Upsilon$,
we get $(c - \lim_G \times \lim_X)\Upsilon(f,x) = c - \lim_G pr_G^{\Rightarrow}(\Upsilon)(f) \wedge \lim_X pr_X^{\Rightarrow}(\Upsilon)(x)$

$\leq \lim_X pr_X^{\Rightarrow}(\Upsilon)(x) \to \lim_X \varphi^{\Rightarrow}(\Upsilon)(\varphi(f,x)) \ \wedge \ \lim_X pr_X^{\Rightarrow}(\Upsilon)(x)$
$\leq \lim_X \varphi^{\Rightarrow}(\Upsilon)(\varphi(f,x))$, i.e., $(c - \lim_G \times \lim_X)\, \Upsilon(f,x) \leq \lim_X \varphi^{\Rightarrow}(\Upsilon)(\varphi(f,x))$.
Similarly, one can show (4.2). Hence, $\mathfrak{c} - (\lim_G \times \lim_X)\, \Upsilon(f,x) \leq \lim_X \varphi^{\Rightarrow}(\Upsilon)\,(\varphi(f,x))$.
This ends the proof. $\square$

5. Examples: Continuous action of convergence approach groups on convergence approach spaces

Definition 5.1. *[36, 37] A pair (X,λ) is called a* convergence approach space, *where the mapping $\lambda\colon \mathbb{F}(X) \longrightarrow [0,\infty]^X$ is called a* convergence approach structure *on X provided:*
(CAL1) $\lambda(\dot{x})(x) = 0$, where $\dot{x} = \{A \subseteq X\colon x \in A\}$;
(CAL2) if $\mathbb{F},\mathbb{G} \in \mathbb{F}(X)$ with $\mathbb{F} \leq \mathbb{G}$, then $\lambda(\mathbb{G}) \leq \lambda(\mathbb{F})$;
(CAL3) $\forall \mathbb{F},\mathbb{G} \in \mathbb{F}(X)$, $\lambda\,(\mathbb{F} \wedge \mathbb{G}) = \lambda(\mathbb{F}) \vee \lambda(\mathbb{G})$.
A mapping $f\colon\ (X,\lambda) \longrightarrow (Y,\sigma)$ between convergence approach spaces is said to a contraction *if for each $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$, $\sigma\,(f(\mathbb{F}))\,(f(x)) \leq \lambda(\mathbb{F})(x)$.*
*Let **CAP** denote the category whose objects are all convergence approach spaces and morphisms are all contraction mappings.*

Theorem 5.2. *[37] The category **CAP** is a topological construct. In particular, given a source $f_j\colon X \longrightarrow \left(Y_j,\sigma_j\right)$, $j \in J$, the initial approach convergence structure λ on X is given for all $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$ by $\lambda(\mathbb{F})(x) = \bigvee_{j\in J} \sigma_j \left(f_j^{\rightarrow}(\mathbb{F}\right)(f_j(x))$.*

Definition 5.3. *[5, 38] The triple $(G,\cdot,\lambda)$ is called a* convergence approach group *if the following are fulfilled:*
(CAG1) $(G,\cdot)$ is a group;
(CAG2) $(G,\lambda) \in |\boldsymbol{CAP}|$;
(CAGM) $\forall \mathbb{F},\mathbb{G} \in \mathbb{F}(G)$, $x,y \in G$: $\lambda\,(\mathbb{F} \odot \mathbb{G})\,(xy) \leq \lambda(\mathbb{F})(x) \vee \lambda(\mathbb{G})(y)$;
(CAGI) $\forall \mathbb{F} \in \mathbb{F}(G)$, $x \in G$: $\lambda(\mathbb{F}^{-1})(x^{-1}) \leq \lambda(\mathbb{F})(x)$.
*The category of all convergence approach groups and group homomorphisms which are contraction mappings is denoted by **CAPGRP**.*

Example 5.4. *[5] Let $(X,\lambda_X) \in |\boldsymbol{CAP}|$, and λ_Y be a convergence approach structure on a group Y. If $C(X,Y) = \{h\colon X \longrightarrow Y\colon f \text{ is a contraction}\}$, the set of all contraction mappings from (X,λ_X) into (Y,λ_Y). If we define pointwise that $(fg)(x) = f(x)g(x)$ and $f^{-1}(x) = (f(x))^{-1}$ for all $f,g \in C(X,Y)$ and $x \in X$, then $(C(X,Y),\cdot)$ is a group and the triple $(C(X,Y),\cdot,\lambda_c)$ is a convergence approach group, where $\lambda_c\colon \mathbb{F}(C(X,Y)) \longrightarrow [0,\infty]^{C(X,Y)}$ is defined by $\lambda_c(\Phi)(f) = \bigwedge \mathcal{L}(\Phi,f)$ and*
$\mathcal{L}(\Phi,f) = \{\alpha \in [0,\infty]\colon \forall\ \mathbb{F} \in \mathbb{F}(X), x \in X, \lambda_Y\,(ev(\Phi \times \mathbb{F}))\,(f(x)) \leq \lambda_X(\mathbb{F})(x) \vee \alpha\}$, for all $\Phi \in \mathbb{F}\,(C(X,Y))$, $f \in C(X,Y)$ and $ev\colon C(X,Y) \times X \longrightarrow X$, $(f,x) \longmapsto f(x)$, the evaluation mapping.

Example 5.5. *[17] Let $(X,\lambda) \in |\boldsymbol{CAP}|$ and consider $\mathcal{H}(X) = \{f\colon X \longrightarrow X\colon\ f$ is homeomorphism, i.e., f is bijective, f and f^{-1} are contractions$\}$.*
Define a convergence approach structure $\lambda_{\mathfrak{c}}$ on $\mathcal{H}(X)$, the group of homeomorphism under composition: for $\Phi \in \mathcal{F}(\mathcal{H}(X))$ and $f \in \mathcal{H}(X)$ we define

$$\mathcal{L}(\Phi,f) = \{\alpha \in [0,\infty]\colon\ \forall \mathbb{F} \in \mathbb{F}(X),\ x \in X.\ \lambda\,(ev(\Phi \times \mathbb{F}))\,(f(x)) \vee \lambda\left(ev(\Phi^{-1} \times \mathcal{F})\right)(f^{-1}(x)) \leq \lambda(\mathbb{F})(x) \vee \alpha\},$$

whence $\lambda_{\mathfrak{c}}\colon \mathbb{F}(\mathcal{H}(X)) \longrightarrow [0,\infty]^{\mathcal{H}(X)}$ is defined by $\lambda_{\mathfrak{c}}(\Phi)(f) = \bigwedge \mathcal{L}(\Phi,f)$. Then $(\mathcal{H}(X),\cdot,\lambda_c)$ is a convergence approach group.

Let $S\colon [0,1] \to [0,\infty]$ be a strictly decreasing surjective mapping such that $S(1) = 0$, which is also order reversing and satisfies that $S\left(\bigwedge_{j\in J} \alpha_j\right) = \bigvee_{j\in J} S(\alpha_j)$ and $S\left(\bigvee_{j\in J} \alpha_j\right) = \bigwedge_{j\in J} S(\alpha_j)$. For this map S, there exists inverse $S^{-1}\colon [0,\infty] \to [0,1]$ which is strictly decreasing and surjective, and therefore share the properties of $\mathcal{S}$. We recall from [30](see also, [41]) that if $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$, then $\Phi_{\mathcal{F}} \in \mathbb{F}(X)$ where

$$\Phi_{\mathcal{F}} = \{A \subseteq X : \mathcal{F}(1_A) = 1\}.$$

Also, recall that if $\mathbb{F} \in \mathbb{F}(X)$, then we define the $[0,1]$-filter $\mathcal{F}^{\mathbb{F}}$ for any $\nu \in [0,1]^X$ by

$$\mathcal{F}^{\mathbb{F}}(\nu) = \bigvee\{\alpha \in [0,1] : \ \nu^{\alpha} \in \mathbb{F}\},$$

where $\nu^{\alpha} = \{x \in X : \ \nu(x) \geq \alpha\}$.

Proposition 5.6. *[31] If $(X, \lambda) \in |\boldsymbol{CAP}|$, then $(X, \lim_{\lambda}) \in |\boldsymbol{SS}[0,1]\text{-}\boldsymbol{CONV}|$, where $\lim_{\lambda}(\mathcal{F})(x) = S^{-1}(\lambda(\Phi_{\mathcal{F}})(x))$, for all $\mathcal{F} \in \mathcal{F}^{s}_{[0,1]}(X)$.*

Lemma 5.7. *[31] If $f : (X, \lambda) \longrightarrow (X', \lambda')$ is a contraction, then $f : (X, \lim_{\lambda}) \longrightarrow (X', \lim_{\lambda'})$ is continuous.*

Proposition 5.8. *[2] If $(X, \cdot, \lambda) \in |\boldsymbol{CAPGRP}|$, then $(X, \cdot, \lim_{\lambda}) \in |\boldsymbol{SS}[0,1]\text{-}\boldsymbol{CONVGRP}|$*

Lemma 5.9. *Let $(G, \cdot, \lambda), (G', \cdot, \lambda') \in |\boldsymbol{CAPGRP}|$ and $f : (G, \cdot, \lambda) \longrightarrow (G', \cdot, \lambda')$ be a group homomorphism and contraction mapping. Then the mapping $f : (G, \cdot, \lim_{\lambda}) \longrightarrow (G', \cdot, \lim_{\lambda'})$ is a continuous group homomorphism.*

Proof. This follows at once from the Lemma 5.7 and Proposition 5.8. □

Corollary 5.10.

$$\mathfrak{F} : \begin{cases} \boldsymbol{CAPGRP} & \longrightarrow & \boldsymbol{SS[0,1]\text{–}CONVGRP} \\ (X, \cdot, \lambda) & \longmapsto & (X, \cdot, \lim_{\lambda}) \\ f & \longmapsto & f \end{cases},$$

is a functor.

Proposition 5.11. *[31] Let $(X, \lim) \in |\boldsymbol{SS}[0,1]\text{-}\boldsymbol{CONV}|$. Then $(X, \lambda_{\lim}) \in |\boldsymbol{CAP}|$, where $\lambda_{\lim}(\mathbb{F})(x) = S\left(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \leq \mathbb{F}\}\right)$.*

Lemma 5.12. *[31] If $f : (X, \lim) \longrightarrow (X', \lim')$ is a continuous mapping, then $f : (X, \lambda_{\lim}) \longrightarrow (X', \lambda_{\lim'})$ is a contraction mapping.*

Proposition 5.13. *Let $(X, \cdot, \lim) \in |\boldsymbol{SS}[0,1]\text{-}\boldsymbol{CONVGRP}|$. Then $(X, \cdot, \lambda_{\lim}) \in |\boldsymbol{CAPGRP}|$, where for any $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$ $\lambda_{\lim}(\mathbb{F})(x) = S\left(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \leq \mathbb{F}\}\right)$.*

Proof. We only need to check the condition(CAGM): For this, let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ and $x, y \in X$, then
$\lambda_{\lim}(\mathbb{F})(x) \vee \lambda_{\lim}(\mathbb{G})(y)$
$= S\left(\bigvee\{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \leq \mathbb{F}\}\right) \vee S\left(\bigvee\{\lim \mathcal{G}(y) : \Phi_{\mathcal{G}} \leq \mathbb{G}\}\right)$
$= S\left(\bigvee\{\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y) : \Phi_{\mathcal{F}} \leq \mathbb{F}, \Phi_{\mathcal{G}} \leq \mathbb{G}\}\right)$
$\geq S\left(\bigvee\{\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y) : \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}} \leq \mathbb{F} \odot \mathbb{G}\}\right) \geq S\left(\bigvee\{\lim (\mathcal{F} \odot \mathcal{G})(xy) : \Phi_{\mathcal{F} \odot \mathcal{G}} \leq \mathbb{F} \odot \mathbb{G}\}\right) = \lambda_{\lim}(\mathbb{F} \odot \mathbb{G})(xy)$,
i.e., $\lambda_{\lim}(\mathbb{F} \odot \mathbb{G})(xy) \leq \lambda_{\lim}(\mathbb{F})(x) \vee \lambda_{\lim}(\mathbb{G})(y)$.
In fact, $A \in \Phi_{\mathcal{F} \odot \mathcal{G}} \iff \mathcal{F} \odot \mathcal{G}(1_A) = \top$ if and only if there are $\nu_1 := 1_F$ and $\nu_2 := 1_G$ with $1_F \cdot 1_G \leq 1_A$ such that $\mathcal{F}(1_F) = \top$ and $\mathcal{G}(1_G) = \top$, whence $F \in \Phi_{\mathcal{F}}$ and $G \in \Phi_{\mathcal{G}}$ but as it follows that $F \cdot G \subseteq A$ and as $F \cdot G \in \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}$ and since $\Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}$ is a filter yields $A \in \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}$. Hence $\Phi_{\mathcal{F} \odot \mathcal{G}} \leq \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}$. □

Lemma 5.14. *If $(X, \cdot, \lim), (X', \cdot, \lim') \in |\boldsymbol{SS}[0,1]\text{-}\boldsymbol{CONVGRP}|$ and $f : (X, \cdot, \lim) \longrightarrow (X', \cdot, \lim')$ is a continuous group homomorphism, then $f : (X, \cdot, \lambda_{\lim}) \longrightarrow (X', \cdot, \lambda_{\lim'})$ is a group homomorphism and a contraction mapping.*

Proof. In view of the Lemma 5.12 and Proposition 5.13, we are done. □

Corollary 5.15.

$$\mathfrak{G} : \begin{cases} \boldsymbol{SS[0,1]\text{–}CONVGRP} & \longrightarrow & \boldsymbol{CAPGRP} \\ (X, \cdot, \lim) & \longmapsto & (X, \cdot, \lambda_{\lim}) \\ f & \longmapsto & f \end{cases},$$

is a functor.

Theorem 5.16. *$\boldsymbol{CAPGRP}$ a reflective subcategory of $\boldsymbol{SS}[0,1]$-$\boldsymbol{CONVGRP}$*

Proof. From Corollary 5.10 and Corollary 5.15 in conjunction with the Proposition 5.5 [31](see also, Propositions 5.3 and 5.4 [31]), we obtain the embedding $\mathbf{SS}[0,1]-\mathbf{CONVGRP} \xrightarrow{\mathfrak{G}} \mathbf{CAPGRP}$ and $\mathbf{CAPGRP} \xrightarrow{\mathfrak{F}} \mathbf{SS}L-\mathbf{CONVGRP}$ such that $\mathfrak{F}\circ\mathfrak{G} \geq id_{\mathbf{SS}[0,1]-\mathbf{CONVGRP}}$ and $\mathfrak{G}\circ\mathfrak{F} = id_{CAPGRP}$. □

Lemma 5.17. *Let $(X,\cdot,\lim) \in |\boldsymbol{SS}[0,1]$-$\boldsymbol{CONVGRP}|$. Then $\left(X,\cdot,\lambda^{\lim}\right) \in |\boldsymbol{CAPGRP}|$, where $\lambda^{\lim}(\mathbb{F})(x) = S\left(\lim \mathcal{F}^{\mathbb{F}}(x)\right)$.*

Proof. Let $\mathbb{F},\mathbb{G} \in \mathbb{F}(X)$ and $x,y \in X$
$\lambda^{\lim}(\mathbb{F})(x) \vee \lambda^{\lim}(\mathbb{G})(y)$
$= S\left(\lim \mathcal{F}^{\mathbb{F}}(x)\right) \vee S\left(\lim \mathcal{G}^{\mathbb{G}}(y)\right)$
$= S\left(\lim \mathcal{F}^{\mathbb{F}}(x) \wedge \lim \mathcal{G}^{\mathbb{G}}(y)\right)$
$\geq S\left(\lim(\mathcal{F}^{\mathbb{F}} \odot \mathcal{G}^{\mathbb{G}})(xy)\right)$
$\geq S\left(\lim\left(\mathcal{F}\odot\mathcal{G}\right)^{\mathbb{F}\odot\mathbb{G}}(xy)\right) = \lambda^{\lim}\left(\mathbb{F}\odot\mathbb{G}\right)(xy)$.
In fact, $\mathcal{F}^{\mathbb{F}} \odot \mathcal{G}^{\mathbb{G}}(\nu) = \bigvee_{\nu_1\cdot\nu_2\leq\nu} \mathcal{F}^{\mathbb{F}}(\nu_1) \wedge \mathcal{G}^{\mathbb{G}}(\nu_2)$
$= \bigvee_{\nu_1\cdot\nu_2\leq\nu}\left(\bigvee\{\alpha\in[0,1] : \nu_1^{\alpha}\in\mathbb{F}\} \wedge \bigvee\{\beta\in[0,1] : \nu_2^{\beta}\in\mathbb{G}\}\right)$
$\leq \bigvee_{\nu_1\cdot\nu_2\leq\nu}\left(\bigvee\{\alpha\wedge\beta\in[0,1] : \nu_1^{\alpha}\cdot\nu_2^{\beta}\in\mathbb{F}\odot\mathbb{G}\}\right)$
$\leq \bigvee\{\gamma\in[0,1] : \nu^{\gamma}\in\mathbb{F}\odot\mathbb{G}\}$
$= (\mathcal{F}\odot\mathcal{G})^{\mathbb{F}\odot\mathbb{G}}$; this is true, since $z \in \nu_1^{\alpha}\cdot\nu_2^{\beta}$ if and only if there are $x \in \nu^{\alpha}$ and $y \in \nu_2^{\beta}$ such that $z = xy$. Now $\nu(z) \geq \nu_1\cdot\nu_2(z) \geq \nu_1(x)\wedge\nu_2(y) \geq \alpha\wedge\beta$ which implies $z \in \nu^{\alpha\wedge\beta}$, but as $\mathbb{F}\odot\mathbb{G}$ is a filter, we have $\nu^{\alpha\wedge\beta}\in\mathbb{F}\odot\mathbb{G}$; taking $\alpha\wedge\beta=\gamma$, we have $\nu^{\gamma}\in\mathbb{F}\odot\mathbb{G}$. □

Definition 5.18. *Let $(X,\lambda_X) \in |\boldsymbol{CAP}|$ and $(G,\cdot,\lambda_G) \in |\boldsymbol{CAPGRP}|$. Then the triple $\left((G,\cdot,\lambda_G),(X,\lambda_X),\varphi\right)$ or in short (G,X,φ) is called a* convergence approach transformation group *on convergence approach space (X,λ_X) with respect to the mapping $\varphi\colon G\times X \longrightarrow X$, $(g,x)\longmapsto\varphi(g,x)$, if φ satisfies the conditions:*
(CATG1) $\forall\mathbb{K}\in\mathbb{F}(G)$, $\forall\mathbb{F}\in\mathbb{F}(X)$, and $\forall g\in G$, $\forall x\in X$: $\lambda_X\left(\varphi^{\rightarrow}\left(\mathbb{K}\times\mathbb{F}\right)\right)(\varphi(g,x)) \leq \lambda_G(\mathbb{K})(g)\vee\lambda_X(\mathbb{F})(x)$;
(CATG2) $\varphi(e,x) = x$, $\forall x\in X$;
(CATG3) $\varphi(g\cdot h,x) = \varphi(g,\varphi(h,x))$, $\forall g,h\in G$, and $\forall x\in X$.
φ is called continuous action, *G is called* phase group *while X is called* phase space*.*
The category of convergence approach transformation groups denoted by $\boldsymbol{CAPTGRP}$ consists of all convergence approach transformation groups as objects, and all pairs $(k,f)\colon (G,X,\varphi)\longrightarrow(G',X',\varphi')$ as morphism, where
(TG1) $G\xrightarrow{k}G'$ is a $\boldsymbol{CAPGRP}$-morphism, i.e., a group homomorphism and contraction mapping;
(TG2) $X\xrightarrow{f}X'$ is a $\boldsymbol{CAP}$-morphism, i.e., a contraction mapping such that $\varphi'\circ(k\times f) = f\circ\varphi$.

Remark 5.19. *If $(k,f) : (G,X,\varphi)\longrightarrow(G',X',\psi)$ and $(k',h) : (G',X',\psi)\longrightarrow\left(G'',X'',\chi\right)$ are morphisms of stratified enriched lattice-valued convergence transformation groups, then the composition $(k'k,hf) : (G,X,\varphi)\longrightarrow\left(G'',X'',\chi\right)$ is again a morphism of a stratified enriched lattice-valued convergence transformation group, where the composition is defined by $(k',h)\circ(k,f) = (k'k,hf)$. Moreover, if $G=G'=G''$ and $k=k'=id_G$, then clearly $k'k = id_G$. Similarly, one can describe the composition of morphisms for the case of convergence approach transformation groups as well as for probabilistic convergence transformation groups under t-norm $*$ as given below in Section 6.*

Example 5.20. *Any convergence approach group $(G,\cdot,\lambda)$ can be made into a convergence approach transformation group on itself in the following way: $((G,\cdot,\lambda),(G,\lambda),\varphi)$ with respect to the mapping $\varphi\colon G\times X\longrightarrow X$, defined by $\varphi(g,x) = gx$, where the condition (CATG1) stand as follows:*

$$\forall\mathbb{K},\mathbb{F}\in\mathbb{F}(G) \text{ and } \forall g,x\in G\text{: } \lambda\left(\mathbb{K}\odot\mathbb{F}\right)(gx) \leq \lambda(\mathbb{K})(g)\vee\lambda(\mathbb{F})(x).$$

Theorem 5.21. *If $\left((G,\cdot,\lambda_G),(X,\lambda_X),\varphi\right) \in |\boldsymbol{CAPTGRP}|$, then*
$\left((G,\cdot,\lim_{\lambda_G}),(X,\lim_{\lambda_X}),\varphi\right) \in |\boldsymbol{SS}[0,1]$-$\boldsymbol{CONVTGRP}|$, where $\lim_{\lambda}\mathcal{F}(x) = \mathcal{S}^{-1}\left(\lambda\left(\Phi_{\mathcal{F}}\right)(x)\right)$.

Proof. We only prove the condition(CTG1): For this, we let $\mathcal{K} \in \mathcal{F}_L^s(G)$, $\mathcal{F} \in \mathcal{F}_L^s(X)$, $g \in G$ and $x \in X$. Then we have:
$\lim_{\lambda_G}(\mathcal{K})(g) \wedge \lim_{\lambda_X}(\mathcal{F})(x)$
$= \mathcal{S}^{-1}(\lambda_G(\Phi_{\mathcal{K}})(g)) \wedge \mathcal{S}^{-1}(\lambda_X(\Phi_{\mathcal{F}})(x))$
$= \mathcal{S}^{-1}(\lambda_G(\Phi_{\mathcal{K}})(g) \vee \lambda_X(\Phi_{\mathcal{F}})(x))$
$\leq \mathcal{S}^{-1}(\lambda_X(\Phi_{\mathcal{K}} \times \Phi_{\mathcal{F}})(\varphi(g,x)))$
$\leq \mathcal{S}^{-1}(\lambda_X(\Phi_{\mathcal{K}\times\mathcal{F}})(\varphi(g,x)))$ (since both λ_X and $\mathcal{S}^{-1}$ are order reversing)
$= \lim_{\lambda_X}(\mathcal{K} \times \mathcal{F})(\varphi(g,x))$, i.e.,
$\lim_{\lambda_G}(\mathcal{K})(g) \wedge \lim_{\lambda_X}(\mathcal{F})(x) \leq \lim_{\lambda_X}(\mathcal{K} \times \mathcal{F})(\varphi(g,x))$.
In fact, $\Phi_{\mathcal{K}} \times \Phi_{\mathcal{F}} = \{K \times F \colon K \in \Phi_{\mathcal{K}}, F \in \Phi_{\mathcal{F}}\}$
$= \{K \times F \colon \mathcal{K}(1_K) = 1, \mathcal{F}(1_F) = 1\}$
$\leq \{K \times F \colon \mathcal{K} \times \mathcal{F}(1_{K\times F}) = 1\}$
$\leq \{H \colon \mathcal{K} \times \mathcal{F}(1_H) = 1\}$
$= \Phi_{\mathcal{K}\times\mathcal{F}}$. □

Corollary 5.22.

$$\mathfrak{N} : \begin{cases} \textbf{\textit{CAPTGRP}} & \longrightarrow & \textbf{\textit{SS[0,1]-CONVTGRP}} \\ ((G,\cdot,\lambda_G),(X,\lambda_X),\varphi) & \longmapsto & ((G,\cdot,\lim_{\lambda_G}),(X,\lim_{\lambda_X}),\varphi) \\ (k,f) & \longmapsto & (k,f) \end{cases},$$

is a functor.

Theorem 5.23. *If* $((G,\cdot,\lim_G),(X,\lim_X),\varphi) \in |\textbf{\textit{SS}}[0,1]\text{-}\textbf{\textit{CONVTGRP}}|$, *then* $((G,\cdot,\lambda_{\lim_G}),(X,\lambda_{\lim_X}),\varphi) \in |\textbf{\textit{CAPTGRP}}|$, *where* $\lambda_{\lim_G}(\mathbb{K}) = S(\bigvee \lim_G \mathcal{K}(g) \colon \Phi_{\mathcal{K}} \leq \mathbb{K})$.

Proof. We only show that the condition (CATG1) is satisfied. To do so, let $\mathbb{K} \in \mathbb{F}(G)$, $\mathbb{F} \in \mathbb{F}(X)$, $g \in G$ and $x \in X$. Then
$\lambda_{\lim_G}(\mathbb{K})(g) \vee \lambda_{\lim_X}(\mathbb{F})(x)$
$= \mathcal{S}(\bigvee\{\lim_G \mathcal{K}(g) \colon \Phi_{\mathcal{K}} \leq \mathbb{K}\}) \vee \mathcal{S}(\bigvee\{\lim_X \mathcal{F}(x) \colon \Phi_{\mathcal{F}} \leq \mathbb{F}\})$
$= \mathcal{S}(\bigvee\{\lim_G \mathcal{K}(g) \wedge \lim_X \mathcal{F}(x) \colon \Phi_{\mathcal{K}} \leq \mathbb{K}, \Phi_{\mathcal{F}} \leq \mathbb{F}\})$
$\geq \mathcal{S}(\bigvee\{\lim_X \varphi^{\Rightarrow}(\mathcal{K} \times \mathcal{F})(\varphi(g,x)) \colon \Phi_{\mathcal{K}\times\mathcal{F}} \leq \mathbb{K} \times \mathbb{F}\})$
$= \lambda_{\lim_X}(\varphi^{\rightarrow}(\mathbb{K} \times \mathbb{F}))(\varphi(g,x))$,
i.e., $\lambda_{\lim_X}(\varphi^{\rightarrow}(\mathbb{K} \times \mathbb{F}))(\varphi(g,x)) \leq \lambda_{\lim_G}(\mathbb{K})(g) \vee \lambda_{\lim_X}(\mathbb{F})(x)$. □

In view of the previous results, and due to the references [31] and [41] we have the following:

Corollary 5.24.

$$\mathfrak{M} : \begin{cases} \textbf{\textit{SS[0,1]-CONVTGRP}} & \longrightarrow & \textbf{\textit{CAPTGRP}} \\ ((G,\cdot,\lim_G),(X,\lim_X),\varphi) & \longmapsto & ((G,\cdot,\lambda_{\lim_G}),(X,\lambda_{\lim_X}),\varphi) \\ (k,f) & \longmapsto & (k,f) \end{cases},$$

is a functor.

Theorem 5.25. *The category* ***CAPTGRP*** *is isomorphic to a reflective subcategory of the category* ***SS***$[0,1]$-***CONVTGRP***.

Proof. This proof is almost similar to the proofs given in [31] except some algebraic parts; however, for the sake of completeness we give here an outline of the proof, and refer to [31] for the details. Remark that it follows from Theorem 5.16 that **CAPGRP** is a reflective subcategory of **SS**$[0,1]$-**CONVGRP**. In view of Corollary 5.22 and Corollary 5.24, and, among others, we need to see that the functor $\mathfrak{N}\colon$ **CAPTGRP**$\longrightarrow$ **SS**$[0,1]$-**CONVTGRP** is injective on objects, and full.

1. To show $\mathfrak{N}$ is injective on objects, we proceed as follows:
   Let $((G,\cdot,\lambda_G),(X,\lambda_X),\varphi) \neq \left((G,\cdot,\lambda'_G),(X,\lambda'_X),\psi)\right)$, we claim that
   $((G,\cdot,\lim_{\lambda_G}),(X,\lim_{\lambda_X}),\varphi) \neq \left((G,\cdot,\lim_{\lambda'_G}),(X,\lim_{\lambda'_X}),\psi\right)$. Then first, if $\lambda_G \neq \lambda'_G$ (resp. $\lambda_X \neq \lambda'_X$), then it follows at once from Proposition 4.5 [31] that $\lim_{\lambda_G} \neq \lim_{\lambda'_G}$ (resp. $\lim_{\lambda_X} \neq \lim_{\lambda'_X}$). Secondly, if $\varphi \neq \psi$, then we immediately get that $((G,\cdot,\lim_{\lambda_G}),(X,\lim_{\lambda_X}),\varphi) \neq \left((G,\cdot,\lim_{\lambda'_G}),(X,\lim_{\lambda'_X}),\psi\right)$.
2. To prove that $\mathfrak{N}$ is full, we make use of the morphisms of these categories. For, let $((G,\cdot,\lambda_G),(X,\lambda_X),\varphi)$, $((H,\cdot,\lambda_H),(Y,\lambda_Y),\psi) \in |$ **CAPTGRP**$|$, and assume that
   $(k,f)\colon ((G,\cdot,\lim_{\lambda_G}),(X,\lim_{\lambda_X}),\varphi) \longrightarrow ((H,\cdot,\lim_{\lambda_H}),(Y,\lim_{\lambda_Y}),\psi)$ is a morphism in **SS**$[0,1]$-**CONVTGRP**, i.e., $k\colon G \longrightarrow H$ is a **SS**$[0,1]$-**CONVGRP**-morphism, meaning k is continuous group homomorphism while $f\colon X \longrightarrow Y$ is a **SS**$[0,1]$-**CONV**-morphism, meaning f is a continuous mapping such that $\psi \circ (k \times f) = f \circ \varphi$. Now due to the preceding results both of these morphisms imply that (k,f) is a morphism in **CAPTGRP**, whence the group homomorphism for the case k remains the same; moreover, one can observe that $\psi \circ (k \times f) = f \circ \varphi$ holds good, where ψ, φ, k and f are all respectively contraction mappings.

Upon using Proposition 5.5 [31] in conjunction with the Theorem 5.23 and Lemma 5.14, we conclude that **CAPTGRP** is isomorphic to a reflective subcategory of **SS**$[0,1]$-**CONVTGRP**. This ends the proof. □

We consider below another type of examples where probabilistic convergence structure is given in [25] and probabilistic convergence group in question is introduced in [3]. In what follows we consider $L = ([0,1],\leq,*,\wedge)$ which is an enriched *cl*-premonoid.

6. Examples: Continuous action of probabilistic convergence groups on probabilistic convergence spaces under triangular norms

Definition 6.1. *[25] A pair $(X,\mathbb{C})$ with $\mathbb{C} = (\mathfrak{c}_x)_{x\in X}$, where $\mathfrak{c}_x\colon \mathbb{F}(X) \longrightarrow [0,1]$ is called a probabilistic convergence space under t-norm $*$ if and only if the following conditions are fulfilled:*
(PC1) $\forall x \in X$: $\mathfrak{c}_x(\dot{x}) = 1$;
(PC2) $\forall x \in X$, $\forall \mathbb{F},\mathbb{G} \in \mathbb{F}(X)$ with $\mathbb{F} \leq \mathbb{G}$ implies $\mathfrak{c}_x(\mathbb{F}) \leq \mathfrak{c}_x(\mathbb{G})$;
(PC3) $\forall x \in X$, $\forall \mathbb{F},\mathbb{G} \in \mathbb{F}(X)$, $\mathfrak{c}_x(\mathbb{F}) * \mathfrak{c}_x(\mathbb{G}) \leq \mathfrak{c}_x(\mathbb{F} \wedge \mathbb{G})$.
A mapping $f\colon (X,\mathbb{C}) \longrightarrow (X',\mathbb{C}')$ between probabilistic convergence spaces $(X,\mathbb{C})$ and $(X',\mathbb{C}')$ is called continuous if and only if for all $x \in X$ and for all $\mathbb{F} \in \mathbb{F}(X)$, $\mathfrak{c}_x(\mathbb{F}) \leq \mathfrak{c}_{f(x)}(f(\mathbb{F}))$.
*The category of probabilistic convergence spaces under t-norm $*$, and continuous mappings between them is denoted by **PCONV***.*

Definition 6.2. *[3] A triple $(X,\cdot,\mathbb{C})$ is called a* probabilistic convergence group *under t-norm $*$ if and only if the following are true:*
*(PCG1) $(X,\cdot) \in |$**GRP**$|$;*
*(PCG2) $(X,\mathbb{C}) \in |$**PCONV**$^*|$;*
(PCGM) $\forall \mathbb{F},\mathbb{G} \in \mathbb{F}(X)$ $\forall x,y \in X$: $\mathfrak{c}_x(\mathbb{F}) * \mathfrak{c}_y(\mathbb{G}) \leq \mathfrak{c}_{xy}(\mathbb{F} \odot \mathbb{G})$;
(PCGI) $\forall x \in X$, $\forall \mathbb{F} \in \mathbb{F}(X)$: $\mathfrak{c}_x(\mathbb{F}) \leq \mathfrak{c}_{x^{-1}}(\mathbb{F}^{-1})$.
*The category of all probabilistic convergence groups under t-norm $*$ and continuous group homomorphisms is denoted by **PCONVGRP***.*

Lemma 6.3. *[41] **PCONV*** is a isomorphic to a full subcategory of **S**$[0,1]$-**CONV**.*

Proposition 6.4. *[3] If $(X,\cdot,\mathbb{C})$ is a probabilistic convergence group under t-norm $*$, then $(X,\cdot,\lim_{\mathbb{C}})$ is a stratified $[0,1]$-convergence group, where $\lim_{\mathbb{C}}(\mathcal{F})(x) = \mathfrak{c}_x(\Phi_{\mathcal{F}})$, for any $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$ and $x \in X$.*

Lemma 6.5. *[41] **PCONV*** is a isomorphic to a reflective subcategory of **S**$[0,1]$-**CONV**.*

Proposition 6.6. *If $(X,\cdot,\lim)$ is a stratified $[0,1]$-valued convergence group, then $(X,\cdot,\mathbb{C}_{\lim})$ is a probabilistic convergence group under t-norm $*$, where $\mathfrak{c}_x(\mathbb{F}) = \bigvee\{\lim \mathcal{F}(x)\colon \Phi_{\mathcal{F}} \leq \mathbb{F}\}$.*

Proof. In view of Lemma 6.5, we only need to prove (PCGM).
For, let $x, y \in X$ and $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$. Then we have
$\mathfrak{c}_x(\mathbb{F}) * \mathfrak{c}_y(\mathbb{G})$
$= \bigvee_{\Phi_{\mathcal{F}} \leq \mathbb{F}} \lim \mathcal{F}(x) * \bigvee_{\Phi_{\mathcal{G}} \leq \mathbb{G}} \lim \mathcal{G}(y)$
$= \bigvee_{\Phi_{\mathcal{F}} \leq \mathbb{F}, \Phi_{\mathcal{G}} \leq \mathbb{G}} \lim \mathcal{F}(x) * \lim \mathcal{G}(y)$
$\leq \bigvee_{\Phi_{\mathcal{F} \odot \mathcal{G}} \leq \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}} \leq \mathbb{F} \odot \mathbb{G}} \lim \mathcal{F} \odot \mathcal{G}(xy)$ (by Definition 3.3(CGM))
$= \bigvee_{\Phi_{\mathcal{F} \odot \mathcal{G}} \leq \mathbb{F} \odot \mathbb{G}} \lim \mathcal{F} \odot \mathcal{G}(xy)$
$= \mathfrak{c}_{xy}(\mathbb{F} \odot \mathbb{G})$. That is, $\mathfrak{c}_x(\mathbb{F}) * \mathfrak{c}_y(\mathbb{G}) \leq \mathfrak{c}_{xy}(\mathbb{F} \odot \mathbb{G})$. Condition (PCGI) follows immediately from the continuity condition in Lemma 6.5. □

Due to preceding results we have the following:

Corollary 6.7.

$$\mathfrak{P} : \begin{cases} \textbf{\textit{S[0,1]}}-\textbf{\textit{CONVGRP}} & \longrightarrow & \textbf{\textit{PCONVGRP}}^* \\ (X, \cdot, \lim) & \longmapsto & (X, \cdot, \mathbb{C}_{\lim}) \\ f & \longmapsto & f \end{cases},$$

and

$$\mathfrak{R} : \begin{cases} \textbf{\textit{PCONVGRP}}^* & \longrightarrow & \textbf{\textit{S[0,1]}}-\textbf{\textit{CONVGRP}} \\ (X, \cdot, \mathbb{C}) & \longmapsto & (X, \cdot, \lim_{\mathbb{C}}) \\ f & \longmapsto & f \end{cases},$$

are functors.

Proof. Objects wise correspondence can be seen from Proposition 6.4 and Proposition 6.6 in conjunction with Lemma 6.5 while morphisms are clearly true since in Lemma 6.3 and 6.5 these are already proved, whence the case for group homomorphisms are also true. □

Theorem 6.8. ***PCONVGRP**** *is isomorphic to a reflective subcategory of* ***S***[0, 1]-***CONVGRP***.

Proof. In view of Lemma 6.3, the functor $\mathfrak{R}$: **PCONVGRP*** $\longrightarrow$ **S**[0, 1]-**CONVGRP** is injective on objects and full. Then it follows upon using Lemma 6.3 that the category **PCONVGRP*** is isomorphic to a full subcategory of **S**[0, 1]-**CONVGRP**. To prove the reflectivity remark that due to Lemma 6.9 [41] in conjunction with the Proposition 6.6, one can show that the functors in Corollary 6.7:

$$\mathbf{S}[0,1]\text{-}\mathbf{CONVGRP} \xrightarrow{\mathfrak{P}} \mathbf{PCONVGRP}^* \text{ and } \mathbf{PCONVGRP}^* \xrightarrow{\mathfrak{R}} \mathbf{S}[0,1]\text{-}\mathbf{CONVGRP}$$

yield that $\mathfrak{R} \circ \mathfrak{P} \geq id_{\mathbf{S}[0,1]-\mathbf{CONVGRP}}$, and $\mathfrak{P} \circ \mathfrak{R} = id_{\mathbf{PCONVGRP}^*}$.
Hence we infer that **PCONVGRP*** is isomorphic to a reflective subcategory of **S**[0, 1]-**CONVGRP**. □

Definition 6.9. *Let* $(X, \mathbb{C}) \in |$***PCONV****$|$ *and* $(G, \cdot, \mathbb{C}) \in |$***PCONVGRP****$|$. *Then the triple* $((G, \cdot, \mathbb{C}), (X, \mathbb{C}), \curlywedge)$ *or in short* $(G, X, \curlywedge)$ *is called a* probabilistic convergence transformation group under t-norm $*$ *on probabilistic convergence space* $(X, \mathbb{C})$ *under t-norm* $*$ *with respect to the mapping* $\curlywedge: G \times X \longrightarrow X$, $(g, x) \longmapsto \curlywedge(g, x)$, *if* $\curlywedge$ *satisfies the conditions:*
(PCTG1) $\forall \mathbb{K} \in \mathbb{F}(G)$, $\forall \mathbb{T} \in \mathbb{F}(X)$, *and* $\forall g \in G$, $\forall x \in X$: $\mathfrak{c}_g(\mathbb{K}) * \mathfrak{c}_x(\mathbb{T}) \leq \mathfrak{c}_{\curlywedge(g,x)}(\curlywedge^{\rightarrow}(\mathbb{K} \times \mathbb{T}))$;
(PCTG2) $\curlywedge(e, x) = x$, $\forall x \in X$;
(CATG3) $\curlywedge(g \cdot h, x) = \curlywedge(g, \curlywedge(h, x))$, $\forall g, h \in G$, *and* $\forall x \in X$.
$\curlywedge$ *is called* continuous action, *G is called* phase group *while X is called* phase space.
The category of probabilistic convergence transformation groups under t-norm $*$ *is denoted by* ***PCONVTGRP**** *consists of all probabilistic convergence transformation groups under t-norm* $*$ *as objects, and all pairs* (k, f): $(G, X, \curlywedge) \longrightarrow (G', X', \curlywedge')$ *as morphism, where*
(PCTG1) $G \xrightarrow{k} G'$ *is a* ***PCONVGRP****-*morphism, i.e., a group homomorphism and continuous mapping;*
(PCTG2) $X \xrightarrow{f} X'$ *is a* ***PCONV****-*morphism, i.e., a continuous mapping such that* $\curlywedge' \circ (k \times f) = f \circ \curlywedge$.

Theorem 6.10. *If* $((G,\cdot,\mathbb{C}_G),(X,\mathbb{C}_X),\curlywedge) \in |$***PCONVTGRP****$^*|$, *then* $((G,\cdot,\lim_{\mathbb{C}_G}),(X,\lim_{\mathbb{C}_X}),\curlywedge) \in |$***S***$[0,1]$-***CONVTGRP***$|$.

Proof. It suffices to prove the condition (CTG1). For, let $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(G)$ and $\mathcal{G} \in \mathcal{F}^s_{[0,1]}(X)$, and $(g,x) \in G \times X$. Then we have
$\lim_{\mathbb{C}_G} \mathcal{F}(g) * \lim_{\mathbb{C}_X} \mathcal{G}(x)$
$= \mathfrak{c}_g(\Phi_{\mathcal{F}}) * \mathfrak{c}_x(\Phi_{\mathcal{G}})$
$\le \mathfrak{c}_{\curlywedge(g,x)}(\curlywedge^{\rightarrow}(\Phi_{\mathcal{F}} \times \Phi_{\mathcal{G}}))$
$\le \mathfrak{c}_{\curlywedge(g,x)}\left(\Phi_{\curlywedge^{\rightarrow}(\mathcal{F}\times\mathcal{G})}\right)$
$= \lim_{\mathbb{C}_X}(\curlywedge^{\rightarrow}(\mathcal{F}\times\mathcal{G}))(\curlywedge(g,x))$. That is, we have
$\lim_{\mathbb{C}_G} \mathcal{F}(g) * \lim_{\mathbb{C}_X} \mathcal{G}(x) \le \lim_{\mathbb{C}_X}(\curlywedge^{\rightarrow}(\mathcal{F}\times\mathcal{G}))(\curlywedge(g,x))$. □

Corollary 6.11.

$$\mathfrak{Y}: \begin{cases} \textbf{\textit{PCONVTGRP}}^* & \longrightarrow & \textbf{\textit{S[0,1]}}-\textbf{\textit{CONVTGRP}} \\ ((G,\cdot,\mathbb{C}_G),(X,\mathbb{C}_X),\curlywedge) & \longmapsto & ((G,\cdot,\lim_{\mathbb{C}_G}),(X,\lim_{\mathbb{C}_X}),\curlywedge) \\ (k,f) & \longmapsto & (k,f) \end{cases},$$

is a functor.

Theorem 6.12. *If* $((G,\cdot,\lim_G),(X,\lim_X),\curlywedge) \in |$***S***$[0,1]$-***CONVTGRP***$|$, *then*
$((G,\cdot,\mathbb{C}_{\lim_G}),(X,\mathbb{C}_{\lim_X}),\curlywedge) \in |$***PCONVTGRP****$^*|$.

Proof. We only need to prove the condition (PCTG1). In order to do so, let $\mathbb{K} \in \mathbb{F}(G), \mathbb{T} \in \mathbb{F}(X)$, $(g,x) \in G\times X$, then we have
$\mathfrak{c}_g(\mathbb{K}) * \mathfrak{c}_x(\mathbb{T})$
$= \bigvee_{\Phi_{\mathcal{K}}\le\mathbb{K}} \lim_G \mathcal{K}(g) * \bigvee_{\Phi_{\mathcal{G}}\le\mathbb{T}} \lim_X \mathcal{G}(x)$
$= \bigvee_{\Phi_{\mathcal{K}}\le\mathbb{K},\Phi_{\mathcal{G}}\le\mathbb{T}} \lim_G \mathcal{K}(g) * \lim_X \mathcal{G}(x)$
$\le \bigvee_{\Phi_{\mathcal{K}}\le\mathbb{K},\Phi_{\mathcal{G}}\le\mathbb{T}} \lim_X(\curlywedge^{\Rightarrow}(\mathcal{K}\times\mathcal{G}))(\curlywedge(g,x))$
$\le \bigvee_{\Phi_{\mathcal{K}\times\mathcal{G}}\le\Phi_{\mathcal{F}}\times\Phi_{\mathcal{G}}\le\mathbb{K}\times\mathbb{T}} \lim_X(\curlywedge^{\Rightarrow}(\mathcal{K}\times\mathcal{G}))(\curlywedge(g,x))$
$\le \bigvee_{\Phi_{\mathcal{K}\times\mathcal{G}}\le\mathbb{K}\times\mathbb{T}} \lim_X(\curlywedge^{\Rightarrow}(\mathcal{K}\times\mathcal{G}))(\curlywedge(g,x))$
$= \mathfrak{c}_{\curlywedge(g,x)}(\curlywedge^{\Rightarrow}(\mathbb{K}\times\mathbb{T}))$ □

Considering the morphisms as described in the Definition 6.9 and their composition in the light of the Remark 5.19, the Theorem 6.12 yields the following:

Corollary 6.13.

$$\mathfrak{Y}: \begin{cases} \textbf{\textit{S[0,1]}}-\textbf{\textit{CONVTGRP}} & \longrightarrow & \textbf{\textit{PCONVTGRP}}^* \\ ((G,\cdot,\lim_G),(X,\lim_X),\curlywedge) & \longmapsto & ((G,\cdot,\mathbb{C}_{\lim_G}),(X,\mathbb{C}_{\lim_X}),\curlywedge) \\ (k,f) & \longmapsto & (k,f) \end{cases},$$

is a functor.

Theorem 6.14. ***PCONVTGRP***** *is isomorphic to a reflective subcategory of* ***S***$[0,1]$-***CONVTGRP***.

Proof. It is pointed out in Theorem 6.8 that **PCONVGRP*** is isomorphic to a reflective subcategory of **S**$[0,1]$-**CONVGRP**; upon using Lemma 6.5 ([41]: Lemma 6.9), Theorem 6.12, and following almost similar proof as in Theorem 5.25, and noting that the morphism (k,f) in **S**$[0,1]$-**CONVTGRP** implies that (k,f) is a morphism in **PCONTVGRP*** such that $\curlywedge' \circ (k\times f) = f \circ \curlywedge$. Thus, we can conclude that **PCONTVGRP*** is isomorphic to a reflective subcategory of **S**$[0,1]$-**CONTVGRP**. □

7. Conclusion

In this article, we have discussed the action of enriched lattice-valued convergence groups providing the action of convergence approach groups as well as action of probabilistic convergence groups under t-norm $*$, as natural examples. In doing so, we have considered $L = (L, \leq, *, \wedge)$, an enriched cl-premonoid, and when dealing with the function space equipped with lattice-valued continuous convergence structure [30], we have taken $L = (L, \leq, \wedge)$, a frame [22, 49] or a complete Heyting algebra. However, in the present scenario, we have achieved a good number of interesting results, leaving the cases for $L = (L, \leq, *, \otimes)$, an arbitrary enriched cl-premonoid lattice, still an open problem. There are some interesting points worth mentioning that we intend to address in a separate paper: one, to look into these works from monoidal categorical perspective [26, 51], also this is one of the suggestions made by one of the referees; second, at present we considered the mapping $S\colon [0,1] \longrightarrow [0,\infty]$ in Sections 5 and 6 as a continuation of our previous work in [3] is originated from the papers [31] and [41] that we strictly followed hereof. But as mentioned by one of the referees that the mapping $S\colon [0,1] \longrightarrow [0,\infty]$ could be avoided following an alternative route, i.e., the examples of Section 5 can be captured by taking $L = \{0,1\}$ for filters and taking $M = ([0,\infty], \geq, +)$ (the extended half-line, opposite order, addition as quantale operation) for the other part; and the examples in Sections 6 by taking $L = \{0,1\}$ for the filters, and $M = ([0,1], \leq, *)$ with a continuous t-norm $*$ for other side; in this way, use of the mapping $S\colon [0,1] \longrightarrow [0,\infty]$ could be simplified. Thanks to the referee for raising this interesting point, of which, unfortunately, we were not aware of, will certainly be looked into while working in a paper devoted to monoidal categories as noted above. Furthermore, we will be interested in future to see if there is any relationship between probabilistic convergence transformation groups that we studied in [4], and the enriched lattice-valued convergence transformation groups that we considered herein this text; although, apparently these two approaches are different, given the fact that both the generalizations inherited their root from the work of Park [42]. However, we do not rule out the possibility of a connection between the examples given in Section 6 and in [4] (see also [6]), but these are beyond the scope of the present paper, and will be dealt with in one of our future articles.

Acknowledgement

We are thankful to the reviewers for generously giving their time to read our manuscript, and providing various pertinent advice which have greatly improved this article. We express our gratitude to the section editor Professor Dijana Mosic for her kind support.

Finally, We gratefully acknowledge the support given by the King Saud University, Deanship of Scientific Research, College of Science Research Center to carry out this work.

References

[1] J. Adámek, H. Herrlich, G. E. Streker, Abstract and Concrete Categories, New York, USA: John Wiley & Sons, 1990.

[2] T. M. G. Ahsanullah, J. Al-Mufarrij, Framed-valued stratified generalized convergence groups, Quaest. Math. **31**(3)(2008) 279–302.

[3] T. M. G. Ahsanullah, D. Gauld, J. Al-Mufarrij, F. Al-Thukair, Enriched lattice-valued convergence groups, Fuzzy Sets and Systems **238**(1)(2014) 71–88.

[4] T. M. G. Ahsanullah, G. Jäger, Probabilistic convergence trasnformation groups, Math. Slovaca **68**(6)(2018) 1447–1464.

[5] T. M. G. Ahsanullah, G. Jäger, On Approach limit groups and their uniformization. Internat. J. Contemp. Math. Sci. **9**(5)(2014) 195–213.

[6] T. M. G. Ahsanullah, F. Al-Thukair, Links between probabilistic convergence groups under triangular norms and enriched lattice-valued convergence groups, New Math. and Natural Computation **12**(2)(2016) 53–76.

[7] R. Arens, Topologies for homeomorphism groups, Amer. J. Math. **68**(4)(1946) 593–610.

[8] R. Beattie, H.-P. Butzmann, Convergence Structures and Applications to Functional Analysis, Kluwer Academic Publishers, Dordrecht, 2002.

[9] M. Baran, The notion of closedness in topological categories, Commentationes Mathematicae Universitatis Carolinae **34**(2)(1993) 383-395.

[10] M. Baran, B. Kula, A. Erciyes, $T0$ and $T1$ semiuniform convergence spaces, Filomat **27**(4)(2013) 534–546.

[11] M. Baran, S. Kula, T. M. Baran, M. Qasim, Closure operators in semiuniform convergence spaces, Filomat **31**(1)(2016) 131–140.

[12] M. Baran, M. Qasim, $T1$ approach spaces, Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68**(1)(2019) 784–800.

[13] G. E. Bredon, Introduction to Compact Transformation groups, New York, USA: Academic Press, 1972.

[14] N. Bourbaki, Elements of Mathematics: General Topology, Part I & II, Addison-Wesley Publishing, Reading, Massachusets, 1966.

[15] E. Čech, Topological Spaces, Interscience Publishers, John Wiley & Sons, London, 1966.
[16] A. Craig, G. Jäger, A common framework for lattice-valued uniform spaces and probabilistic uniform limit spaces, Fuzzy Sets and Systems **160**(9)(2009) 1177–1203.
[17] E. Colebunders, H. Boustique, P. Mikusiński, G. Richardson, Convergence approach spaces: Actions, Appl. Categor. Struct. **24**(2)(2016) 147–161.
[18] J. Dieudonné, On topological groups of homeomorphisms, Amer. J. Math. **70**(3)(1948) 659–680.
[19] Jinming Fang, Lattice-valued preuniform convergence spaces, Fuzzy Sets and Systems **251**(16)(2014) 52–70.
[20] P. V. Flores, R. N. Mohapatra, G. Richardson, Lattice-valued spaces: fuzzy convergence, Fuzzy Sets and Systems **157**(2006) 2947–2955.
[21] P. S. Gevorgyan, Groups of binary operations and binary G-spaces, Top. and its Appl. **201**(2016) 18–24.
[22] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, Continuous Lattices and Domains, Cambridge, UK: Cambridge University Press, 2003.
[23] W. H. Gottschalk, G. A. Hedlund, Topological Dynamics, American Mathematical Society, Coll. Pub., **36**, 2017.
[24] J. Gutiérrez García, I. Mardones Pérez, M. H. Burton, The relationship between various filter notionsons on a **GL**-monoid, J. Math. Anal. and Appl. **230**(2)(1999) 291–302.
[25] H. Herrlich, D. Zhang, Categorical properties of probabilistic convergence spaces, Appl. Categor. Struct. **6**(4)(1998) 495–513.
[26] D. Hofmann, G. J. Seal, W. Tholen, Monoidal Topology - A categorical Approach to Order, Metric, and Topology, Cambridge, UK: Cambridge University Press, 2014.
[27] U. Höhle, Commutative, residuated ℓ-monoids, Chapter IV, In: Klement PE, Höhle U.(editor), Nonclassical Logics and their Applications to Fuzzy Subsets, Dordrecht, The Netherlands: Kluwer Academic Publishers, 1995, pp. 53–106.
[28] U. Höhle, A. P. Šostak, Axiomatic foundations of fixed basis fuzzy topology, Chapter 3, In: Höhle U, Rodabaugh SE. (editor), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1999, pp. 123–272.
[29] J. Horváth, Topological Vector Spaces and Distributions, Vol. I, Addison-Wesley Company, Reading, Massachusets, 1966.
[30] G. Jäger, A category of L-fuzzy convergence spaces, Quaest. Math. **24**(4)(2001) 501–517.
[31] G. Jäger, Convergence approach spaces and approach spaces as lattice-valued convergence spaces, Iranian J. Fuzzy Systems **9**(4)(2012) 1–16.
[32] G. Jäger, Subcategories of lattice-valued convergence spaces, Fuzzy Sets and Systems **156**(2005) 1–24.
[33] G. Jäger, Lattice-valued categories of lattice-valued convergence spaces, Iranian J. Fuzzy Systems **8**(2)(2011) 67–89.
[34] P. E. Klement, R. Mesiar, E. Pap, Triangular Norms, Dordrecht, The Netherlands: Kluwer Academic Publishers, 2000.
[35] Lingqiang Li, Qiu Jin, On stratified L-convergence spaces: pretopological axioms and diagonal axioms, Fuzzy Sets and Systems **204**(1)(2012) 40–52.
[36] R. Lowen, Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad, New York, USA: Oxford University Press, 1997.
[37] E. Lowen, R. Lowen, A quasitopos containing CONV and MET as full subcategories, Internat. J. Math. and Math. Sci. **11**(3)(1998) 417–438.
[38] R. Lowen, B. Windels, Approach groups. Rocky Mountain Journal of Mathematics **30**(3)(2000) 1057–1073.
[39] R. A. McCoy, I. Ntantu, Topological Properties of Spaces of Continuous Functions, Berlin, Germany: Springer, 1988.
[40] D. Montgomery, L. Zippin, Topological Transformation Groups, New York, USA: Dover Publications, Inc., 2018.
[41] D. Orpen, G. Jäger, Lattice-valued convergence spaces: Extending lattice contex, Fuzzy Sets and Systems **190**(1)(2012) 1–20.
[42] W. R. Park, Convergence structures on homeomorphism groups, Math. Annal. **199**(1)(1972) 45–54.
[43] P. S. Palais, The Classification of G-space, Memoir 36. Rhodes Island, USA: Amer. Math. Soc., 1960.
[44] G. Preuss, Semiuniform convergence space, Math. Japonica **41**(3)(1995) 465–491.
[45] G. Preuss, Foundations of Topology: An Approach to Convenient Topology, Kluwer Academic Publishers, Dordrecht, 2002.
[46] H. Poppe, Compactness in General Function Spaces. Berlin, Germany: VEB Deutscher Verlag der Wissenschaften, 1974.
[47] N. Rath, Action of convergence groups, Topology Procedings **27**(2)(2003) 601–612.
[48] N. Rath, Applications of convergence spaces to group actions, Internat. J. Pure and Appl. Math. **48**(3)(2008) 415–433.
[49] K. I. Rosenthal, Quantales and their Applications, Burnt, Longman, London, UK: Pitman Research Notes in Mathematics, Vol. **234** 1990.
[50] A. Wilansky, Topology for Analysis, Dover Publications Inc., New York, 2008.
[51] L. Zhang, J. Fang, W. Wang, Monoidal closedness of L-generalized convergence spaces, Iranian J. Fuzzy Systems **16**(5)(2019) 139–153.