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Some Geometric Properties of a Subclass of Multivalent Analytic Functions Defined by the First-Order Differential Subordination

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Dedicated to Professor H. M. Srivastava on the Occasion of his Eightieth Birth Anniversary

Abstract. A new class $\mathcal{T}_n(A,B,\lambda)$ of multivalent analytic functions defined by the first-order differential subordination is introduced. Some geometric properties of this new class are investigated. The sharp lower bound on $|z| = r < 1$ for the functional $\operatorname{Re}\left\{(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}}\right\}$ over the class $\mathcal{T}_n(A,B,0)$ is given.

1. Introduction

Throughout our present investigation, we assume that

$$n,\ p \in \mathbb{N},\ -1 \leq B < 1,\ B < A \ \text{ and } \lambda > 0. \tag{1.1}$$

Let $\mathcal{A}_n(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_k z^{k+p} \tag{1.2}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

For functions $f(z)$ and $g(z)$ analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ and write $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

If the function $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

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Definition. A function $f(z) \in \mathcal{A}_n(p)$ is said to be in the class $\mathcal{T}_n(A, B, \lambda)$ if it satisfies first-order differential subordination:

$$(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{1.3}$$

Recently, several authors (see, e.g., [1-7, 9, 11] and the references cited therein) introduced and investigated various subclasses of multivalent analytic functions. Some properties such as distortion bounds, inclusion relations and coefficient estimates were given. In [12] Srivastava made a systematic investigation of various analytic function classes associated with operators of basic (or q-) calculus and fractional q-calculus. Inspired by some recent works of Srivastava et al. [8, 12-18] the main object of the paper is to obtain inclusion relation, sharp bounds on $\mathrm{Re}\left(\frac{f(z)}{z^p}\right)$ and coefficient estimates for functions $f(z)$ belonging to the class $\mathcal{T}_n(A, B, \lambda)$. Furthermore, we study a new problem, that is, to find

$$\min_{|z|=r<1} \mathrm{Re}\left\{(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}}\right\},$$

where $f(z)$ varies in the class:

$$\mathcal{T}_n(A, B, 0) = \left\{f(z) \in \mathcal{A}_n(p) : \frac{f(z)}{z^p} \prec \frac{1+Az}{1+Bz}\right\}. \tag{1.4}$$

In order to derive our main results we need the following lemma.

Lemma [10]. Let $g(z)$ be analytic in $\mathbb{U}$ and $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z),$$

where $\mathrm{Re}\mu \geq 0$ and $\mu \neq 0$, then $g(z) \prec h(z)$.

2. Geometric properties of functions in the class $\mathcal{T}_n(A, B, \lambda)$

Theorem 1. Let $0 < \lambda_1 < \lambda_2$. Then $\mathcal{T}_n(A, B, \lambda_2) \subset \mathcal{T}_n(A, B, \lambda_1)$.

Proof. Let $0 < \lambda_1 < \lambda_2$ and suppose that

$$g(z) = \frac{f(z)}{z^p} \tag{2.1}$$

for $f(z) \in \mathcal{T}_n(A, B, \lambda_2)$. Then $g(z)$ is analytic in $\mathbb{U}$ and $g(0) = 1$. By using (1.3) and (2.1), we have

$$\begin{aligned}(1-\lambda_2)\frac{f(z)}{z^p} + \frac{\lambda_2 f'(z)}{pz^{p-1}} &= g(z) + \frac{\lambda_2}{p} z g'(z) \\ &\prec \frac{1+Az}{1+Bz}.\end{aligned} \tag{2.2}$$

An application of Lemma yields

$$g(z) \prec \frac{1+Az}{1+Bz}. \tag{2.3}$$

Noting that $0 < \frac{\lambda_1}{\lambda_2} < 1$ and that the function $\frac{1+Az}{1+Bz}$ is convex univalent in $\mathbb{U}$, it follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}&(1-\lambda_1)\frac{f(z)}{z^p} + \frac{\lambda_1 f'(z)}{pz^{p-1}} \\ &= \frac{\lambda_1}{\lambda_2}\left((1-\lambda_2)\frac{f(z)}{z^p} + \frac{\lambda_2 f'(z)}{pz^{p-1}}\right) + \left(1 - \frac{\lambda_1}{\lambda_2}\right) g(z) \\ &\prec \frac{1+Az}{1+Bz}.\end{aligned}$$

This shows that $f(z) \in \mathcal{T}_n(A, B, \lambda_1)$. The proof of Theorem 1 is completed.

Theorem 2. Let $f(z) \in \mathcal{T}_n(A, B, \lambda)$. Then for $|z| = r < 1$, we have

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) \geq 1 - p(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}r^{nm}}{\lambda nm + p}, \tag{2.4}$$

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) > 1 - p(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{\lambda nm + p}, \tag{2.5}$$

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) \leq 1 + p(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}r^{nm}}{\lambda nm + p} \tag{2.6}$$

and

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) < 1 + p(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}}{\lambda nm + p} \quad (B \neq -1). \tag{2.7}$$

All the bounds are sharp for the function $f_n(z)$ defined by

$$f_n(z) = z^p + p(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}z^{nm+p}}{\lambda nm + p} \quad (z \in \mathbb{U}). \tag{2.8}$$

Proof. It is known that for $|\xi| \leq \sigma$ $(\sigma < 1)$,

$$\left|\frac{1+A\xi}{1+B\xi} - \frac{1-AB\sigma^2}{1-B^2\sigma^2}\right| \leq \frac{(A-B)\sigma}{1-B^2\sigma^2} \tag{2.9}$$

and

$$\frac{1-A\sigma}{1-B\sigma} \leq \mathrm{Re}\left(\frac{1+A\xi}{1+B\xi}\right) \leq \frac{1+A\sigma}{1+B\sigma}. \tag{2.10}$$

Let $f(z) \in \mathcal{T}_n(A, B, \lambda)$. Then we can write

$$(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}} = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{U}), \tag{2.11}$$

where $w(z) = w_n z^n + w_{n+1}z^{n+1} + \cdots$ is analytic and $|w(z)| < 1$ for $z \in \mathbb{U}$. By the Schwarz lemma we know that $|w(z)| \leq |z|^n$ $(z \in \mathbb{U})$. It follows from (2.11) that

$$\frac{p(1-\lambda)}{\lambda}z^{\frac{p(1-\lambda)}{\lambda}-1}f(z) + z^{\frac{p(1-\lambda)}{\lambda}}f'(z) = \frac{p}{\lambda}z^{\frac{p}{\lambda}-1}\left(\frac{1+Aw(z)}{1+Bw(z)}\right),$$

which implies that

$$\left(z^{\frac{p(1-\lambda)}{\lambda}}f(z)\right)' = \frac{p}{\lambda}z^{\frac{p}{\lambda}-1}\left(\frac{1+Aw(z)}{1+Bw(z)}\right).$$

After integartion we get

$$\begin{aligned} f(z) &= \frac{p}{\lambda}z^{-\frac{p(1-\lambda)}{\lambda}}\int_0^z \xi^{\frac{p}{\lambda}-1}\left(\frac{1+Aw(\xi)}{1+Bw(\xi)}\right)d\xi \\ &= \frac{p}{\lambda}z^p\int_0^1 t^{\frac{p}{\lambda}-1}\left(\frac{1+Aw(tz)}{1+Bw(tz)}\right)dt. \end{aligned} \tag{2.12}$$

Since

$$|w(tz)| \leq t^n r^n \quad (|z| = r < 1;\ 0 \leq t \leq 1),$$

we have from (2.12) and left-hand inequality in (2.10) that for $|z| = r < 1$,

$$\begin{aligned} \mathrm{Re}\left(\frac{f(z)}{z^p}\right) &\geq \frac{p}{\lambda}\int_0^1 t^{\frac{p}{\lambda}-1}\left(\frac{1-At^n r^n}{1-Bt^n r^n}\right)dt \\ &= 1 - p(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}r^{nm}}{\lambda nm + p}, \end{aligned} \tag{2.13}$$

and for $z \in \mathbb{U}$,

$$\begin{aligned} \mathrm{Re}\left(\frac{f(z)}{z^p}\right) &> \frac{p}{\lambda}\int_0^1 t^{\frac{p}{\lambda}-1}\left(\frac{1-At^n}{1-Bt^n}\right)dt \\ &= 1 - p(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{\lambda nm + p}. \end{aligned}$$

Similarly, by using (2.12) and the right-hand inequality in (2.10), we have (2.6) and (2.7).

Furthermore, for the function $f_n(z)$ defined by (2.8), we find that $f_n(z) \in \mathcal{A}_n(p)$ and

$$(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}} = 1 + (A-B)\sum_{m=1}^{\infty}(-B)^{m-1}z^{nm} \prec \frac{1+Az}{1+Bz}.$$

Hence $f_n(z) \in \mathcal{T}_n(A, B, \lambda)$ and from (2.8) we conclude that the inequalities (2.4) to (2.7) are sharp. The proof of Theorem 2 is completed.

Theorem 3. Let $f(z) \in \mathcal{T}_1(A, B, \lambda)$ and

$$g(z) \in \mathcal{T}_1(A_1, B_1, \lambda_1) \quad (-1 \leq B_1 < 1;\ B_1 < A_1;\ \lambda_1 > 0).$$

If

$$p(A_1 - B_1)\sum_{m=1}^{\infty}\frac{B_1^{m-1}}{\lambda_1 m + p} \leq \frac{1}{2}, \tag{2.14}$$

then $(f * g)(z) \in \mathcal{T}_1(A, B, \lambda)$, where the symbol $*$ denotes the familiar Hadamard product of two analytic functions in $\mathbb{U}$.

Proof. Since $g(z) \in \mathcal{T}_1(A_1, B_1, \lambda_1)$, we have from the inequality (2.5) and (2.14) that

$$\mathrm{Re}\left(\frac{g(z)}{z^p}\right) > 1 - p(A_1 - B_1)\sum_{m=1}^{\infty}\frac{B_1^{m-1}}{\lambda_1 m + p} \geq \frac{1}{2} \quad (z \in \mathbb{U}).$$

Thus the function $\frac{g(z)}{z^p}$ has the Herglotz representation:

$$\frac{g(z)}{z^p} = \int_{|x|=1}\frac{d\mu(x)}{1-xz} \quad (z \in \mathbb{U}), \tag{2.15}$$

where $\mu(x)$ is a probability measure on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$.

For $f(z) \in \mathcal{T}_1(A, B, \lambda)$, we have

$$\frac{(f*g)(z)}{z^p} = \frac{f(z)}{z^p} * \frac{g(z)}{z^p}$$

and

$$\frac{(f*g)'(z)}{z^{p-1}} = \frac{f'(z)}{z^{p-1}} * \frac{g(z)}{z^p}.$$

Thus

$$\begin{aligned}
&(1-\lambda)\frac{(f*g)(z)}{z^p}+\frac{\lambda(f*g)'(z)}{pz^{p-1}} \\
&=(1-\lambda)\left(\frac{f(z)}{z^p}*\frac{g(z)}{z^p}\right)+\frac{\lambda}{p}\left(\frac{f'(z)}{z^{p-1}}*\frac{g(z)}{z^p}\right) \\
&=h(z)*\frac{g(z)}{z^p}, \qquad (2.16)
\end{aligned}$$

where

$$h(z):=(1-\lambda)\frac{f(z)}{z^p}+\frac{\lambda f'(z)}{pz^{p-1}}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}). \qquad (2.17)$$

In view of the function $\frac{1+Az}{1+Bz}$ is convex univalent in $\mathbb{U}$, it follows from (2.15) to (2.17) that

$$(1-\lambda)\frac{(f*g)(z)}{z^p}+\frac{\lambda(f*g)'(z)}{pz^{p-1}}=\int_{|x|=1}h(xz)d\mu(x)\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}).$$

This shows that $(f*g)(z)\in\mathcal{T}_1(A,B,\lambda)$. The proof of Theorem 3 is completed.

Theorem 4. Let

$$f(z)=z^p+\sum_{k=n}^{\infty}a_kz^{k+p}\in\mathcal{T}_n(A,B,\lambda). \qquad (2.18)$$

Then

$$|a_k|\le\frac{p(A-B)}{\lambda k+p}\quad(k\ge n). \qquad (2.19)$$

The result is sharp for each $k\ge n$.

Proof. It is known that, if

$$\varphi(z)=\sum_{j=1}^{\infty}c_jz^j\prec\psi(z)\quad(z\in\mathbb{U}),$$

where $\varphi(z)$ is analytic in $\mathbb{U}$ and $\psi(z)=z+\cdots$ is analytic and convex univalent in $\mathbb{U}$, then $|c_j|\le 1$ $(j\in\mathbb{N})$.

By (2.18) we have

$$\begin{aligned}
\frac{(1-\lambda)\frac{f(z)}{z^p}+\frac{\lambda f'(z)}{pz^{p-1}}-1}{p(A-B)}&=\frac{1}{p(A-B)}\sum_{k=n}^{\infty}(\lambda k+p)a_kz^k \\
&\prec\frac{z}{1+Bz}\quad(z\in\mathbb{U}). \qquad (2.20)
\end{aligned}$$

In view of the function $\frac{z}{1+Bz}$ is analytic and convex univalent in $\mathbb{U}$, it follows from (2.20) that

$$\frac{\lambda k+p}{p(A-B)}|a_k|\le 1\quad(k\ge n),$$

which gives (2.19).

Next we consider the function $f_k(z)$ defined by

$$f_k(z)=z^p+p(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}z^{km+p}}{\lambda km+p}\quad(z\in\mathbb{U};\ k\ge n).$$

Since

$$(1-\lambda)\frac{f_k(z)}{z^p}+\frac{\lambda f_k'(z)}{pz^{p-1}}=\frac{1+Az^k}{1+Bz^k}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U})$$

and

$$f_k(z)=z^p+\frac{p(A-B)}{\lambda k+p}z^{k+p}+\cdots$$

for each $k\geq n$, the proof of Theorem 4 is completed.

Theorem 5. Let $f(z)\in\mathcal{T}_n(A,B,0)$. Then for $|z|=r<1$,

(i) if $M_n(A,B,\lambda,r)\geq 0$, we have

$$\operatorname{Re}\left\{(1-\lambda)\frac{f(z)}{z^p}+\frac{\lambda f'(z)}{pz^{p-1}}\right\}\geq\frac{p-(p(A+B)+\lambda n(A-B))r^n+pABr^{2n}}{p(1-Br^n)^2};\tag{2.21}$$

(ii) if $M_n(A,B,\lambda,r)\leq 0$, we have

$$\operatorname{Re}\left\{(1-\lambda)\frac{f(z)}{z^p}+\frac{\lambda f'(z)}{pz^{p-1}}\right\}\geq\frac{4\lambda^2K_AK_B-L_n^2}{4\lambda p(A-B)r^{n-1}(1-r^2)K_B},\tag{2.22}$$

where

$$\begin{cases}K_A=1-A^2r^{2n}-nAr^{n-1}(1-r^2),\\ K_B=1-B^2r^{2n}-nBr^{n-1}(1-r^2),\\ L_n=2\lambda(1-ABr^{2n})-\lambda n(A+B)r^{n-1}(1-r^2)-p(A-B)r^{n-1}(1-r^2),\\ M_n(A,B,\lambda,r)=2\lambda K_B(1-Ar^n)-L_n(1-Br^n).\end{cases}\tag{2.23}$$

The results are sharp.

Proof. Equality in (2.21) occurs for $z=0$. Thus we assume that $0<|z|=r<1$.

For $f(z)\in\mathcal{T}_n(A,B,0)$, we can write

$$\frac{f(z)}{z^p}=\frac{1+Az^n\varphi(z)}{1+Bz^n\varphi(z)}\quad(z\in\mathbb{U}),\tag{2.24}$$

where $\varphi(z)$ is analytic and $|\varphi(z)|\leq 1$ in $\mathbb{U}$. It follows from (2.24) that

$$\begin{aligned}&(1-\lambda)\frac{f(z)}{z^p}+\frac{\lambda f'(z)}{pz^{p-1}}\\&=\frac{f(z)}{z^p}+\frac{\lambda(A-B)(nz^n\varphi(z)+z^{n+1}\varphi'(z))}{p(1+Bz^n\varphi(z))^2}\\&=\frac{f(z)}{z^p}+\frac{\lambda n}{p(A-B)}\left(\frac{f(z)}{z^p}-1\right)\left(A-B\frac{f(z)}{z^p}\right)+\frac{\lambda(A-B)z^{n+1}\varphi'(z)}{p(1+Bz^n\varphi(z))^2}.\end{aligned}\tag{2.25}$$

Making use of the Carathéodory inequality:

$$|\varphi'(z)|\leq\frac{1-|\varphi(z)|^2}{1-r^2},$$

we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{z^{n+1}\varphi'(z)}{(1+Bz^n\varphi(z))^2}\right\}&\geq-\frac{r^{n+1}(1-|\varphi(z)|^2)}{(1-r^2)|1+Bz^n\varphi(z)|^2}\\&=-\frac{r^{2n}|A-B\frac{f(z)}{z^p}|^2-|\frac{f(z)}{z^p}-1|^2}{(A-B)^2r^{n-1}(1-r^2)}.\end{aligned}\tag{2.26}$$

Put $\frac{f(z)}{z^p} = u + iv \quad (u, v \in \mathbb{R})$. Then (2.25) and (2.26) give

$$\begin{aligned}
&\mathrm{Re}\left\{(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}}\right\} \geq \left(1 + \frac{\lambda n(A+B)}{p(A-B)}\right)u - \frac{\lambda nA}{p(A-B)} \\
&- \frac{\lambda nB}{p(A-B)}(u^2 - v^2) - \frac{\lambda[r^{2n}((A-Bu)^2 + (Bv)^2] - ((u-1)^2 + v^2)]}{p(A-B)r^{n-1}(1-r^2)} \\
&= \left(1 + \frac{\lambda n(A+B)}{p(A-B)}\right)u - \frac{\lambda n}{p(A-B)}(A + Bu^2) - \frac{\lambda(r^{2n}(A-Bu)^2 - (u-1)^2)}{p(A-B)r^{n-1}(1-r^2)} \\
&+ \frac{\lambda}{p(A-B)}\left(nB + \frac{1 - B^2r^{2n}}{r^{n-1}(1-r^2)}\right)v^2.
\end{aligned} \tag{2.27}$$

Note that

$$\begin{aligned}
\frac{1 - B^2r^{2n}}{r^{n-1}(1-r^2)} &\geq \frac{1 - r^{2n}}{r^{n-1}(1-r^2)} = \frac{1}{r^{n-1}}\left(1 + r^2 + r^4 + \cdots + r^{2(n-2)} + r^{2(n-1)}\right) \\
&= \frac{1}{2r^{n-1}}\left[(1 + r^{2(n-1)}) + (r^2 + r^{2(n-2)}) + \cdots + (r^{2(n-1)} + 1)\right] \\
&\geq n \geq -nB.
\end{aligned} \tag{2.28}$$

Combining (2.27) and (2.28) we get

$$\begin{aligned}
\mathrm{Re}\left\{(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}}\right\} &\geq \left(1 + \frac{\lambda n(A+B)}{p(A-B)}\right)u - \frac{\lambda n}{p(A-B)}(A + Bu^2) \\
&+ \frac{\lambda((u-1)^2 - r^{2n}(A-Bu)^2)}{p(A-B)r^{n-1}(1-r^2)} \\
&=: \psi_n(u).
\end{aligned} \tag{2.29}$$

Also, (2.10) and (2.24) imply that

$$\frac{1 - Ar^n}{1 - Br^n} \leq u = \mathrm{Re}\left(\frac{f(z)}{z^p}\right) \leq \frac{1 + Ar^n}{1 + Br^n}.$$

Now we calculate the minimum value of $\psi_n(u)$ on the segment $\left[\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right]$. Obviously,

$$\psi_n'(u) = 1 + \frac{\lambda n(A+B)}{p(A-B)} - \frac{2\lambda nB}{p(A-B)}u + \frac{2\lambda((1 - B^2r^{2n})u - (1 - ABr^{2n}))}{p(A-B)r^{n-1}(1-r^2)},$$

$$\psi_n''(u) = \frac{2\lambda}{p(A-B)}\left(\frac{1 - B^2r^{2n}}{r^{n-1}(1-r^2)} - nB\right) \geq \frac{2\lambda n(1-B)}{p(A-B)} > 0 \quad \text{(see (2.28))} \tag{2.30}$$

and $\psi_n'(u) = 0$ if and only if

$$\begin{aligned}
u = u_n &= \frac{2\lambda(1 - ABr^{2n}) - \lambda n(A+B)r^{n-1}(1-r^2) - p(A-B)r^{n-1}(1-r^2)}{2\lambda(1 - B^2r^{2n} - nBr^{n-1}(1-r^2))} \\
&= \frac{L_n}{2\lambda K_B} \quad \text{(see (2.20))}.
\end{aligned} \tag{2.31}$$

Since

$$\begin{aligned}
&2\lambda K_B(1 + Ar^n) - L_n(1 + Br^n) \\
&= 2\lambda\left[(1 + Ar^n)(1 - B^2r^{2n}) - (1 + Br^n)(1 - ABr^{2n})\right] \\
&\quad + \lambda nr^{n-1}(1-r^2)\left[(A+B)(1 + Br^n) - 2B(1 + Ar^n)\right] + p(A-B)r^{n-1}(1-r^2)(1 + Br^n) \\
&= 2\lambda(A-B)r^n(1 + Br^n) + \lambda n(A-B)r^{n-1}(1-r^2)(1 - Br^n) + p(A-B)r^{n-1}(1-r^2)(1 + Br^n) \\
&> 0,
\end{aligned}$$

we see that

$$u_n < \frac{1+Ar^n}{1+Br^n}. \tag{2.32}$$

However, u_n is not always greater than $\frac{1-Ar^n}{1-Br^n}$. The following two cases arise.

(i) $u_n \le \frac{1-Ar^n}{1-Br^n}$, that is, $M_n(A,B,\lambda,r) \ge 0$ (see (2.23)). In view of $\psi_n'(u_n) = 0$ and (2.30), the function $\psi_n(u)$ is increasing on the segment $\left[\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right]$. Therefore we deduce from (2.29) that, if $M_n(A,B,\lambda,r) \ge 0$, then

$$\begin{aligned}
&\mathrm{Re}\left\{(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}}\right\} \ge \psi_n\left(\frac{1-Ar^n}{1-Br^n}\right)\\
&= \left(1+\frac{\lambda n(A+B)}{p(A-B)}\right)\left(\frac{1-Ar^n}{1-Br^n}\right) - \frac{\lambda n}{p(A-B)}\left(A + B\left(\frac{1-Ar^n}{1-Br^n}\right)^2\right)\\
&= \frac{1-Ar^n}{1-Br^n} - \frac{\lambda n}{p(A-B)}\left(1-\frac{1-Ar^n}{1-Br^n}\right)\left(A - B\frac{1-Ar^n}{1-Br^n}\right)\\
&= \frac{p-(p(A+B)+\lambda n(A-B))r^n + pABr^{2n}}{p(1-Br^n)^2}.
\end{aligned}$$

This proves (2.21).

Next we consider the function $f(z) \in \mathcal{T}_n(A,B,0)$ given by

$$\frac{f(z)}{z^p} = \frac{1-Az^n}{1-Bz^n} \quad (z \in \mathbb{U}).$$

It is easy to find that

$$(1-\lambda)\frac{f(r)}{r^p} + \frac{\lambda f'(r)}{pr^{p-1}} = \frac{p-(p(A+B)+\lambda n(A-B))r^n + pABr^{2n}}{p(1-Br^n)^2},$$

which shows that the inequality (2.21) is sharp.

(ii) $u_n \ge \frac{1-Ar^n}{1-Br^n}$, that is, $M_n(A,B,\lambda,r) \le 0$. In this case we easily have

$$\mathrm{Re}\left\{(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}}\right\} \ge \psi_n(u_n). \tag{2.33}$$

In view of (2.23), $\psi_n(u)$ in (2.29) can be written as

$$\psi_n(u) = \frac{\lambda K_B u^2 - L_n u + \lambda K_A}{p(A-B)r^{n-1}(1-r^2)}. \tag{2.34}$$

Therefore, if $M_n(A,B,\lambda,r) \le 0$, then it follows from (2.31), (2.33) and (2.34) that

$$\begin{aligned}
\mathrm{Re}\left\{(1-\lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}}\right\} &\ge \frac{\lambda K_B u_n^2 - L_n u_n + \lambda K_A}{p(A-B)r^{n-1}(1-r^2)}\\
&= \frac{4\lambda^2 K_A K_B - L_n^2}{4\lambda p(A-B)r^{n-1}(1-r^2)K_B}.
\end{aligned}$$

To show that the inequality (2.22) is sharp, we take

$$\frac{f(z)}{z^p} = \frac{1+Az^n\varphi(z)}{1+Bz^n\varphi(z)} \quad \text{and} \quad \varphi(z) = -\frac{z-c_n}{1-c_n z} \quad (z \in \mathbb{U}),$$

where $c_n \in \mathbb{R}$ is determined by

$$\frac{f(r)}{r^p} = \frac{1 + Ar^n\varphi(r)}{1 + Br^n\varphi(r)} = u_n \in \left[\frac{1 - Ar^n}{1 - Br^n}, \frac{1 + Ar^n}{1 + Br^n}\right).$$

Clearly, $-1 \le \varphi(r) < 1$, $-1 \le c_n < 1$, $|\varphi(z)| \le 1$ $(z \in \mathbb{U})$, and so $f(z) \in \mathcal{T}_n(A, B, 0)$. Since

$$\varphi'(r) = -\frac{1 - c_n^2}{(1 - c_n r)^2} = -\frac{1 - |\varphi(r)|^2}{1 - r^2},$$

from the above argument we obtain that

$$(1 - \lambda)\frac{f(r)}{r^p} + \frac{\lambda f'(r)}{pr^{p-1}} = \psi_n(u_n).$$

Now the proof of Theorem 5 is completed.

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