Some Geometric Properties of a Subclass of Multivalent Analytic Functions Defined by the First-Order Differential Subordination

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Dedicated to Professor H. M. Srivastava on the Occasion of his Eightieth Birth Anniversary

Abstract. A new class $T_n(A, B, \lambda)$ of multivalent analytic functions defined by the first-order differential subordination is introduced. Some geometric properties of this new class are investigated. The sharp lower bound on $|z| = r < 1$ for the functional $\Re \left\{ (1 - \lambda) \frac{f^n}{f^n'} + \frac{f'}{f^n} \right\}$ over the class $T_n(A, B, 0)$ is given.

1. Introduction

Throughout our present investigation, we assume that

$$n, p \in \mathbb{N}, -1 \leq B < 1, B < A \text{ and } \lambda > 0. \quad (1.1)$$

Let $\mathcal{A}_n(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_k z^{k+p} \quad (1.2)$$

which are analytic in the open unit disk $\mathbb{U} = \{ z : |z| < 1 \}.$

For functions $f(z)$ and $g(z)$ analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ and write $f(z) \prec g(z)$ ($z \in \mathbb{U}$), if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$|w(z)| \leq |z| \text{ and } f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

If the function $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

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Definition. A function \( f(z) \in \mathcal{A}_n(p) \) is said to be in the class \( \mathcal{T}_n(A, B, \lambda) \) if it satisfies first-order differential subordination:

\[
(1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).
\]

(1.3)

Recently, several authors (see, e.g., [1-7, 9, 11] and the references cited therein) introduced and investigated various subclasses of multivalent analytic functions. Some properties such as distortion bounds, inclusion relations and coefficient estimates were given. In [12] Srivastava made a systematic investigation of various analytic function classes associated with operators of basic (or \( q \)-) calculus. Inspired by some recent works of Srivastava et al. [8, 12-18] the main object of the paper is to obtain inclusion relation, sharp bounds on \( \text{Re} \left( \frac{f(z)}{z^p} \right) \) and coefficient estimates for functions \( f(z) \) belonging to the class \( \mathcal{T}_n(A, B, \lambda) \). Furthermore, we study a new problem, that is, to find

\[
\min_{\|f\|_{r<1}} \text{Re} \left\{ (1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}} \right\},
\]

where \( f(z) \) varies in the class:

\[
\mathcal{T}_n(A, B, 0) = \left\{ f(z) \in \mathcal{A}_n(p) : \frac{f(z)}{z^p} < \frac{1 + Az}{1 + Bz} \right\}.
\]

(1.4)

In order to derive our main results we need the following lemma.

Lemma [10]. Let \( g(z) \) be analytic in \( \mathbb{U} \) and \( h(z) \) be analytic and convex univalent in \( \mathbb{U} \) with \( h(0) = g(0) \). If

\[
g(z) + \frac{1}{\mu} z g'(z) < h(z),
\]

where \( \text{Re} \mu \geq 0 \) and \( \mu \neq 0 \), then \( g(z) < h(z) \).

2. Geometric properties of functions in the class \( \mathcal{T}_n(A, B, \lambda) \)

Theorem 1. Let \( 0 < \lambda_1 < \lambda_2 \). Then \( \mathcal{T}_n(A, B, \lambda_2) \subset \mathcal{T}_n(A, B, \lambda_1) \).

Proof. Let \( 0 < \lambda_1 < \lambda_2 \) and suppose that

\[
g(z) = \frac{f(z)}{z^p}.
\]

(2.1)

for \( f(z) \in \mathcal{T}_n(A, B, \lambda_2) \). Then \( g(z) \) is analytic in \( \mathbb{U} \) and \( g(0) = 1 \). By using (1.3) and (2.1), we have

\[
(1 - \lambda_2) \frac{f(z)}{z^p} + \frac{\lambda_2 f'(z)}{pz^{p-1}} = g(z) + \frac{\lambda_2}{p} z g'(z)
\]

\[
< \frac{1 + Az}{1 + Bz}.
\]

(2.2)

An application of Lemma yields

\[
g(z) < \frac{1 + Az}{1 + Bz}.
\]

(2.3)

Noting that \( 0 < \frac{\lambda_1}{\lambda_2} < 1 \) and that the function \( \frac{1 + Az}{1 + Bz} \) is convex univalent in \( \mathbb{U} \), it follows from (2.1), (2.2) and (2.3) that

\[
(1 - \lambda_1) \frac{f(z)}{z^p} + \frac{\lambda_1 f'(z)}{pz^{p-1}}
\]

\[
= \frac{\lambda_1}{\lambda_2} \left( (1 - \lambda_2) \frac{f(z)}{z^p} + \frac{\lambda_2 f'(z)}{pz^{p-1}} \right) + \left( 1 - \frac{\lambda_1}{\lambda_2} \right) g(z)
\]

\[
< \frac{1 + Az}{1 + Bz}.
\]
This shows that \( f(z) \in \mathcal{T}_n(A, B, \lambda_1) \). The proof of Theorem 1 is completed.

**Theorem 2.** Let \( f(z) \in \mathcal{T}_n(A, B, \lambda) \). Then for \( |z| = r < 1 \), we have

\[
\text{Re}\left(\frac{f(z)}{z^p}\right) \geq 1 - p(A - B) \sum_{m=1}^{\infty} \frac{B^{m-1}p^m}{\lambda nm + p'}, (2.4)
\]

\[
\text{Re}\left(\frac{f(z)}{z^p}\right) > 1 - p(A - B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{\lambda nm + p'}, (2.5)
\]

\[
\text{Re}\left(\frac{f(z)}{z^p}\right) \leq 1 + p(A - B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1}p^m}{\lambda nm + p} (2.6)
\]

and

\[
\text{Re}\left(\frac{f(z)}{z^p}\right) < 1 + p(A - B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1}p^m}{\lambda nm + p} (B \neq -1). (2.7)
\]

All the bounds are sharp for the function \( f_n(z) \) defined by

\[
f_n(z) = z^p + p(A - B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1}z^{nm+p}}{\lambda nm + p} (z \in \mathbb{D}). (2.8)
\]

**Proof.** It is known that for \( |\xi| \leq \sigma (\sigma < 1)\),

\[
\frac{1 + A\xi}{1 + B\xi} - \frac{1 - AB\sigma^2}{1 - B^2\sigma^2} \leq (A - B)\sigma\frac{1}{1 - B^2\sigma^2} (2.9)
\]

and

\[
\frac{1 - A\sigma}{1 - B\sigma} \leq \text{Re}\left(\frac{1 + A\xi}{1 + B\xi}\right) \leq \frac{1 + A\sigma}{1 + B\sigma}. (2.10)
\]

Let \( f(z) \in \mathcal{T}_n(A, B, \lambda) \). Then we can write

\[
(1 - \lambda)\frac{f(z)}{z^p} + \frac{\lambda f'(z)}{pz^{p-1}} = \frac{1 + Aw(z)}{1 + Bw(z)} (z \in \mathbb{D}), (2.11)
\]

where \( w(z) = w_nz^n + w_{n+1}z^{n+1} + \cdots \) is analytic and \(|w(z)| < 1\) for \( z \in \mathbb{D} \). By the Schwarz lemma we know that \(|w(z)| \leq |z|^n (z \in \mathbb{D})\). It follows from (2.11) that

\[
\frac{p(1 - \lambda)}{\lambda} z^{n-1} f(z) + z^{n-1} f'(z) = \frac{p}{\lambda} z^{n-1} \left(\frac{1 + Aw(z)}{1 + Bw(z)}\right),
\]

which implies that

\[
\left(z^{n-1} f(z)\right)' = \frac{p}{\lambda} z^{n-1} \left(\frac{1 + Aw(z)}{1 + Bw(z)}\right).
\]

After integration we get

\[
f(z) = \frac{p}{\lambda} z^{n-1} \int_0^1 \xi^{n-1} \left(\frac{1 + Aw(\xi)}{1 + Bw(\xi)}\right) d\xi = \frac{p}{\lambda} z^p \int_0^1 t^{n-1} \left(\frac{1 + Aw(t)}{1 + Bw(t)}\right) dt. (2.12)
\]
Since
\[ |w(tz)| \leq t^n r^n \quad (|z| = r < 1; \ 0 \leq t \leq 1), \]
we have from (2.12) and the left-hand inequality in (2.10) that for \(|z| = r < 1,\)
\[
\text{Re}\left(\frac{f(z)}{z^p}\right) \geq \frac{p}{\lambda} \int_0^1 t^{\frac{p-1}{2}} \left(1 - \frac{At^n}{1 + Bt^n}\right) dt
= 1 - p(A - B) \sum_{m=1}^{\infty} \frac{B_{m-1}nm}{\lambda nm + p},
\]
and for \(z \in \mathbb{U},\)
\[
\text{Re}\left(\frac{f(z)}{z^p}\right) > \frac{p}{\lambda} \int_0^1 t^{\frac{p-1}{2}} \left(1 - \frac{At^n}{1 + Bt^n}\right) dt
= 1 - p(A - B) \sum_{m=1}^{\infty} \frac{B_{m-1}}{\lambda nm + p}.
\]
Similarly, by using (2.12) and the right-hand inequality in (2.10), we have (2.6) and (2.7).
Furthermore, for the function \(f_n(z)\) defined by (2.8), we find that \(f_n(z) \in \mathcal{A}_n(p)\) and
\[
(1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda f'(z)}{p z^{p-1}} = 1 + (A - B) \sum_{m=1}^{\infty} (-B)^{m-1} z^m < \frac{1 + Az}{1 + Bz}.
\]
Hence \(f_n(z) \in \mathcal{T}_n(A, B, \lambda)\) and from (2.8) we conclude that the inequalities (2.4) to (2.7) are sharp. The proof of Theorem 2 is completed.

**Theorem 3.** Let \(f(z) \in \mathcal{T}_1(A, B, \lambda)\) and
\[
g(z) \in \mathcal{T}_1(A_1, B_1, \lambda_1) \quad (-1 \leq B_1 < 1;\ B_1 < A_1;\ \lambda_1 > 0).\]
If
\[
p(A_1 - B_1) \sum_{m=1}^{\infty} \frac{B_{m-1}}{\lambda_1 m + p} \leq \frac{1}{2},
\]
then \((f \ast g)(z) \in \mathcal{T}_1(A, B, \lambda),\) where the symbol \(\ast\) denotes the familiar Hadamard product of two analytic functions in \(\mathbb{U}.

**Proof.** Since \(g(z) \in \mathcal{T}_1(A_1, B_1, \lambda_1),\) we have from the inequality (2.5) and (2.14) that
\[
\text{Re}\left(\frac{g(z)}{z^p}\right) > 1 - p(A_1 - B_1) \sum_{m=1}^{\infty} \frac{B_{m-1}}{\lambda_1 m + p} \geq \frac{1}{2} \quad (z \in \mathbb{U}).
\]
Thus the function \(\frac{g(z)}{z^p}\) has the Herglotz representation:
\[
\frac{g(z)}{z^p} = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in \mathbb{U}),
\]
where \(\mu(x)\) is a probability measure on the unit circle \(|x| = 1\) and \(\int_{|x|=1} d\mu(x) = 1.

For \(f(z) \in \mathcal{T}_1(A, B, \lambda),\) we have
\[
\frac{(f \ast g)(z)}{z^p} = \frac{f(z)}{z^p} \ast \frac{g(z)}{z^p}
\]
and
\[
\frac{(f \ast g')(z)}{z^{p-1}} = \frac{f'(z)}{z^{p-1}} \ast \frac{g(z)}{z^p}.
\]
Thus

\[
(1 - \lambda) \left( \frac{f \ast g(z)}{z^p} \right) + \frac{\lambda (f \ast g)'(z)}{p z^{p-1}} = (1 - \lambda) \left( \frac{f(z)}{z^p} \ast \frac{g(z)}{z^p} \right) + \frac{\lambda}{p} \left( \frac{f'(z)}{2z^{p-1}} \ast \frac{g(z)}{2^p} \right)
\]

\[
= h(z) \ast \frac{g(z)}{z^p},
\]

where

\[
h(z) := (1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda f'(z)}{p z^{p-1}} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).
\]

(2.17)

In view of the function \( \frac{1 + Az}{1 + Bz} \) is convex univalent in \( \mathbb{U} \), it follows from (2.15) to (2.17) that

\[
(1 - \lambda) \left( \frac{f \ast g(z)}{z^p} \right) + \frac{\lambda (f \ast g)'(z)}{p z^{p-1}} = \int_{|\mu|=1} h(xz) d\mu(x) < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).
\]

This shows that \( f \ast g(z) \in \mathcal{T}_1(A, B, \lambda) \). The proof of Theorem 3 is completed.

**Theorem 4.** Let

\[
f(z) = z^p + \sum_{k=n}^{\infty} a_k z^{k+p} \in \mathcal{T}_n(A, B, \lambda).
\]

(2.18)

Then

\[
|a_k| \leq \frac{p(A - B)}{\lambda k + p} \quad (k \geq n).
\]

(2.19)

The result is sharp for each \( k \geq n \).

**Proof.** It is known that, if

\[
\varphi(z) = \sum_{j=1}^{\infty} c_j z^j < \psi(z) \quad (z \in \mathbb{U}),
\]

where \( \varphi(z) \) is analytic in \( \mathbb{U} \) and \( \psi(z) = z + \cdots \) is analytic and convex univalent in \( \mathbb{U} \), then \( |c_j| \leq 1 \) \((j \in \mathbb{N})\).

By (2.18) we have

\[
\frac{(1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda f'(z)}{p z^{p-1}} - 1}{p(A - B)} = \frac{1}{p(A - B)} \sum_{k=n}^{\infty} (\lambda k + p)a_k z^k
\]

\[
< \frac{z}{1 + Bz} \quad (z \in \mathbb{U}).
\]

(2.20)

In view of the function \( \frac{z}{1 + Bz} \) is analytic and convex univalent in \( \mathbb{U} \), it follows from (2.20) that

\[
\frac{\lambda k + p}{p(A - B)} |a_k| \leq 1 \quad (k \geq n),
\]

which gives (2.19).

Next we consider the function \( f_k(z) \) defined by

\[
f_k(z) = z^p + p(A - B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1} z^{km+p}}{\lambda km + p} \quad (z \in \mathbb{U}; \ k \geq n).
\]
Since
\[ (1 - \lambda) \frac{f_k(z)}{z^p} + \frac{1 + A_{\lambda}^k}{p(1 + B_{\lambda}^k)} = 1 + \frac{1 + A_{\lambda}}{1 + B_{\lambda}} (z \in \mathbb{U}) \]
and
\[ f_k(z) = z^p + \frac{p(A - B)}{\lambda k + p} z^{k+1} + \ldots \]
for each \( k \geq n \), the proof of Theorem 4 is completed.

**Theorem 5.** Let \( f(z) \in \mathcal{T}_n(A, B, 0) \). Then for \( |z| = r < 1 \),

(i) if \( M_n(A, B, \lambda, r) \geq 0 \), we have
\[ \text{Re} \left\{ (1 - \lambda) \frac{f(z)}{z^p} + \frac{1 + A_{\lambda}^n}{p(1 + B_{\lambda})} \right\} \geq \frac{p - (p(A + B) + \lambda n(A - B)r^n + pABz^n)}{p(1 - Br^n)^2}; \tag{2.21} \]

(ii) if \( M_n(A, B, \lambda, r) \leq 0 \), we have
\[ \text{Re} \left\{ (1 - \lambda) \frac{f(z)}{z^p} + \frac{1 + A_{\lambda}^n}{p(1 + B_{\lambda})} \right\} \geq \frac{4\lambda^2 K_{\lambda} A_{\lambda} - L_{\lambda}^2}{4\lambda p(A - B)r^{n-1}(1 - r^2) K_{\lambda}}, \tag{2.22} \]

where
\[
\begin{align*}
K_{\lambda} &= 1 - A_{\lambda}^2 r^n - nA_{\lambda} r^{n-1}(1 - r^2), \\
K_{\lambda} &= 1 - B_{\lambda}^2 r^n - nB_{\lambda} r^{n-1}(1 - r^2), \\
L_{\lambda} &= 2\lambda(1 - AB_{\lambda}^2) - n(A + B)n r^{n-1}(1 - r^2) - p(A - B)r^{n-1}(1 - r^2), \\
M_n(A, B, \lambda, r) &= 2\lambda K_{\lambda} (1 - Ar^n) - L_{\lambda}(1 - Br^n).
\end{align*}
\]

The results are sharp.

**Proof.** Equality in (2.21) occurs for \( z = 0 \). Thus we assume that \( 0 < |z| = r < 1 \).

For \( f(z) \in \mathcal{T}_n(A, B, 0) \), we can write
\[ \frac{f(z)}{z^p} = \frac{1 + Az^n \varphi(z)}{1 + Bz^n \varphi(z)} \quad (z \in \mathbb{U}), \tag{2.24} \]

where \( \varphi(z) \) is analytic and \( |\varphi(z)| \leq 1 \) in \( \mathbb{U} \). It follows from (2.24) that
\[
\begin{align*}
(1 - \lambda) \frac{f(z)}{z^p} + \frac{1 + A_{\lambda}^n}{p(1 + B_{\lambda}^n)} &= \frac{f(z)}{z^p} + \frac{\lambda(A - B)(nz^n \varphi(z) + z^{n+1} \varphi'(z))}{p(1 + Bz^n \varphi(z))^2} \\
&= \frac{f(z)}{z^p} + \frac{\lambda n p(A - B)}{p(1 + Bz^n \varphi(z))^2} \left( \frac{f(z)}{z^p} - 1 \right) \left( A - B \frac{f(z)}{z^p} + \frac{\lambda(A - B)z^{n+1} \varphi'(z)}{p(1 + Bz^n \varphi(z))^2} \right). \tag{2.25}
\end{align*}
\]

Making use of the Carathéodory inequality:
\[ |\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - r^2}, \]
we obtain
\[
\begin{align*}
\text{Re} \left\{ \frac{z^{n+1} \varphi'(z)}{(1 + Bz^n \varphi(z))^2} \right\} &\geq -\frac{r^{n+1}(1 - |\varphi(z)|^2)}{(1 - r^2)(1 + Bz^n \varphi(z))^2} \\
&= -\frac{r^2 n |A - B \frac{f(z)}{z^p} - \frac{f(z)}{z^p} - 1|^2}{(A - B)^2 r^{n-1}(1 - r^2)}. \tag{2.26}
\end{align*}
\]
Note that

and

Also, (2.10) and (2.24) imply that

\[ f(u) = \frac{\lambda n(A + B)}{p(A - B)} (A + Bu^2) \]

\[ + \frac{\lambda}{p(A - B)} \left( nB + \frac{1 - B^2 r^2 n}{r^{n-1}(1 - r^2)} \right) v^2. \]  

(2.27)

Note that

\[ \frac{1 - B^2 r^2 n}{r^{n-1}(1 - r^2)} \geq \frac{1}{r^{n-1}(1 - r^2)} = \frac{1}{r^{n-1}} \left( 1 + r^2 + r^4 + \ldots + r^{2(n-2)} + r^{2(n-1)} \right) \]

\[ = \frac{1}{2r^{n-1}} \left[ (1 + r^2(n-1)) + (r^2 + r^{2(n-2)}) + \ldots + (r^{2(n-1)} + 1) \right] \]

\[ \geq n \geq -nB. \]  

(2.28)

Combining (2.27) and (2.28) we get

\[ \text{Re} \left\{ (1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda f'(z)}{p z^{p-1}} \right\} \geq \left( 1 + \frac{\lambda n(A + B)}{p(A - B)} \right) u - \frac{\lambda n}{p(A - B)} (A + Bu^2) \]

\[ + \frac{\lambda((u - 1)^2 - r^2 n(A - Bu)^2)}{p(A - B)r^{n-1}(1 - r^2)} \]

\[ =: \psi_n(u). \]  

(2.29)

Also, (2.10) and (2.24) imply that

\[ \frac{1}{1 - Br^n} \leq u = \text{Re} \left\{ \frac{f(z)}{z^p} \right\} \leq \frac{1 + Ar^n}{1 + Br^n}. \]

Now we calculate the minimum value of \( \psi_n(u) \) on the segment \( \left[ \frac{1}{1 - Br^n}, \frac{1 + Ar^n}{1 + Br^n} \right] \). Obviously,

\[ \psi'_n(u) = 1 + \frac{\lambda n(A + B)}{p(A - B)} - \frac{2\lambda nB}{p(A - B)} u + \frac{2\lambda((1 - B^2 r^2 n)u - (1 - Br^{2n}))}{p(A - B)r^{n-1}(1 - r^2)}, \]

\[ \psi''_n(u) = \frac{2\lambda}{p(A - B)} \left[ \frac{1 - B^2 r^2 n}{r^{n-1}(1 - r^2)} - nB \right] \geq \frac{2\lambda n(1 - B r^{2n})}{p(A - B)} > 0 \]  

(see (2.28))

(2.30)

and \( \psi'_n(u) = 0 \) if and only if

\[ u = u_n = \frac{2\lambda((1 - Ar^n) - \lambda n(A + B)r^{n-1}(1 - r^2) - p(A - B)r^{n-1}(1 - r^2))}{2\lambda(1 - B^2 r^2 n - nBr^{2n-1}(1 - r^2))} \]

\[ = \frac{L_n}{2\lambda K_B} \]  

(see (2.20)).

(2.31)

Since

\[ 2\lambda K_B(1 + Ar^n) - L_n(1 + Br^n) \]

\[ = 2\lambda \left[ (1 + Ar^n)(1 - B^2 r^{2n}) - (1 + Br^n)(1 - AB r^{2n}) \right] \]

\[ + \lambda n r^{n-1}(1 - r^2) [(A + B)(1 + Br^n) - 2B(1 + Ar^n)] + p(A - B)r^{n-1}(1 - r^2)(1 + Br^n) \]

\[ = 2\lambda(A - B)r^n(1 + Br^n) + \lambda n(A - B)r^{n-1}(1 - r^2)(1 - Br^n) + p(A - B)r^{n-1}(1 - r^2)(1 + Br^n) > 0, \]
we see that
\[ u_n < \frac{1 + Ar^n}{1 + Br^n}. \]  \hspace{1cm} (2.32)

However, \( u_n \) is not always greater than \( \frac{1 - Ar^n}{1 - Br^n} \). The following two cases arise.

(i) \( u_n \leq \frac{1 - Ar^n}{1 - Br^n} \), that is, \( M_n(A, B, \lambda, r) \geq 0 \) (see (2.23)). In view of \( \psi_n(u_n) = 0 \) and (2.30), the function \( \psi_n(u) \) is increasing on the segment \( \left[ \frac{1 - Ar^n}{1 - Br^n}, \frac{1 + Ar^n}{1 + Br^n} \right] \). Therefore we deduce from (2.29) that, if \( M_n(A, B, \lambda, r) \geq 0 \), then

\[
\text{Re} \left\{ (1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda f'(z)}{p z^{p-1}} \right\} \geq \psi_n\left( \frac{1 - Ar^n}{1 - Br^n} \right)
= \left( 1 + \frac{\lambda n(A + B)}{p(A - B)} \right) \left( \frac{1 - Ar^n}{1 - Br^n} \right) - \frac{\lambda n}{p(A - B)} \left( A + B \left( \frac{1 - Ar^n}{1 - Br^n} \right)^2 \right)
= \frac{1 - Ar^n}{1 - Br^n} - \frac{\lambda n}{p(A - B)} \left( A - B \frac{1 - Ar^n}{1 - Br^n} \right)
= \frac{p - (p(A + B) + \lambda n(A - B))r^n + pABr^{2n}}{p(1 - Br^n)^2}.
\]

This proves (2.21).

Next we consider the function \( f(z) \in T_n(A, B, 0) \) given by
\[
\frac{f(z)}{z^p} = \frac{1 - Az^n}{1 - Bz^n} \quad (z \in \mathbb{U}).
\]

It is easy to find that
\[
(1 - \lambda) \frac{f(r)}{r^p} + \frac{\lambda f'(r)}{p r^{p-1}} = \frac{p - (p(A + B) + \lambda n(A - B))r^n + pABr^{2n}}{p(1 - Br^n)^2},
\]
which shows that the inequality (2.21) is sharp.

(ii) \( u_n \geq \frac{1 - Ar^n}{1 - Br^n} \), that is, \( M_n(A, B, \lambda, r) \leq 0 \). In this case we easily have

\[
\text{Re} \left\{ (1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda f'(z)}{p z^{p-1}} \right\} \geq \psi_n(u_n).
\] \hspace{1cm} (2.33)

In view of (2.23), \( \psi_n(u) \) in (2.29) can be written as

\[
\psi_n(u) = \frac{\lambda K_B u^2 - L_n u + \lambda K_A}{p(A - B)r^{n-1}(1 - r^2)}. \hspace{1cm} (2.34)
\]

Therefore, if \( M_n(A, B, \lambda, r) \leq 0 \), then it follows from (2.31), (2.33) and (2.34) that

\[
\text{Re} \left\{ (1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda f'(z)}{p z^{p-1}} \right\} \geq \frac{\lambda K_B u_n^2 - L_n u_n + \lambda K_A}{p(A - B)r^{n-1}(1 - r^2)}
= \frac{4\lambda^2 K_A K_B - L_n^2}{4\lambda p(A - B)r^{n-1}(1 - r^2)K_B}.
\]

To show that the inequality (2.22) is sharp, we take
\[
\frac{f(z)}{z^p} = \frac{1 + Az^n}{1 + Bz^n} \quad \text{and} \quad \varphi(z) = \frac{z - c_n}{1 - c_n z} \quad (z \in \mathbb{U}),
\]
where \( c_n \in \mathbb{R} \) is determined by
\[
\frac{f(r)}{r^p} = \frac{1 + Ar^p \varphi(r)}{1 + Br^p \varphi(r)} = u_n \in \left[ \frac{1 - Ar^n}{1 - Br^n}, \frac{1 + Ar^n}{1 + Br^n} \right].
\]
Clearly, \(-1 \leq \varphi(r) < 1, -1 \leq c_n < 1, |\varphi(z)| \leq 1 (z \in \mathbb{U})\), and so \( f(z) \in \mathcal{T}_n(A, B, 0) \). Since
\[
\varphi'(r) = -\frac{1 - c_n^2}{(1 - c_n r)^2} = -\frac{1 - |\varphi(r)|^2}{1 - r^2},
\]
from the above argument we obtain that
\[
(1 - \lambda) \frac{f(r)}{r^p} + \frac{\lambda f'(r)}{pr^{p-1}} = \psi_n(u_n).
\]
Now the proof of Theorem 5 is completed.

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References