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Decomposability of Krein Space Operators

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Abstract. In this paper, we review some properties in the local spectral theory and various subclasses of decomposable operators. We prove that every Krein space selfadjoint operator having property (β) is decomposable, and clarify the relation between decomposability and property (β) for $\mathcal{J}$-selfadjoint operators. We prove the equivalence of these properties for $\mathcal{J}$-selfadjoint operators T and T^* by using their local spectra and local spectral subspaces.

1. Introduction

Let $\mathcal{K}$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and J be an involutive selfadjoint operator on $\mathcal{K}$. Then $\mathcal{K}$ becomes a Krein space with the indefinite inner product given by $\langle \xi, \eta \rangle_{\mathcal{J}} = \langle J\xi, \eta \rangle$ for any $\xi, \eta \in \mathcal{K}$. The operator J is called a *fundamental symmetry*, which is a bounded linear operator with $J = J^{-1} = J^*$. That is, a Krein space is a Hilbert space equipped with an indefinite inner product, which is a special kind of indefinite metric spaces. Nevertheless, Krein spaces share many characteristics of Hilbert spaces and the Krein space theory have proven to provide an effective tool in situations where the indefinite inner product has a useful property which the Hilbert space inner product lacks. In the past two decade, the Krein space theory has attracted increasing attention in mathematics, physics, and many areas including control and signal processing theory. For a detailed information of Krein spaces, we refer readers to [3, 5].

Let $\mathcal{K}$ and $\mathcal{H}$ be complex Krein spaces. We denote by $\mathcal{L}(\mathcal{K}, \mathcal{H})$ the set of all bounded linear operators from $\mathcal{K}$ to $\mathcal{H}$, and abbreviate $\mathcal{L}(\mathcal{K}) = \mathcal{L}(\mathcal{K}, \mathcal{K})$. If $T \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, we write $\ker(T)$ for the kernel of T; $\operatorname{ran}(T)$ for the range of T. Throughout this paper, * denotes the Hilbert space adjoint, whereas $^\#$ denotes the $\mathcal{J}$-adjoint with respect to the indefinite inner product. It is known that the $\mathcal{J}$-adjoint operator $T^\#$ of $T \in \mathcal{L}(\mathcal{K})$ is given by $T^\# = JT^*J$ for a fundamental symmetry J. We say that $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$*-selfadjoint* if $T = T^\#$, and $\mathcal{J}$*-unitary* if $T^{-1} = T^\#$. In [15], Langer introduced to the theory of linear operators in Krein spaces and studied their spectral properties. Ran and Wojtylak [16] analysed the spectra of the $\mathcal{J}$-selfadjoint (possibly unbounded) operators. We also studied several properties of $\mathcal{J}$-selfadjoint operators [1, 2].

The notion of a decomposable operator introduced by Foiaş [4] is of central importance in the spectral decomposition theory. Since that, many people gave an axiomatic description of the various kind of spectral

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decompositions that admit a useful and rich theory. In [14], the fundamental properties of decomposable operators was discussed and various subclasses of decomposable operators and their the relationships were studied. In this paper, we give several definitions concerning various decomposability and study various spectral decomposition property for $\mathcal{J}$-selfadjoint operators on Krein spaces.

We now give a brief overview of the organization of the paper. In section 2, we recall several special properties in local spectral theory such as decomposability, property (β), and the single valued extension property. We prove that every Krein space selfadjoint operator having property (β) is decomposable, and study the relation between decomposability and property (β) for $\mathcal{J}$-selfadjoint operators. We also consider the decomposability for upper triangular 2×2 operator matrices. In section 3, we briefly review various decomposability and their hierarchy whose structures are useful for the spectral decomposition problem. We prove the equivalence of these properties for $\mathcal{J}$-selfadjoint operators T and T^* by using their local spectra and local spectral subspaces.

2. Decomposability for operators and their $\mathcal{J}$-adjoints in Krein spaces

In this section, we concentrate on several properties in local spectral theory such as decomposability, the single valued extension property, property (β) and Dunford's property (C), which are useful for the Banach space theory.

We say that $T \in \mathcal{L}(\mathcal{H})$ has *the single valued extension property* (abbreviated SVEP) if for every open subset U of $\mathbb{C}$ and every $\mathcal{K}$-valued analytic function f on U such that $(T-\lambda)f(\lambda) \equiv 0$ on U, we have that $f(\lambda) \equiv 0$ on U. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *property* (β) if, for any open subset V of $\mathbb{C}$ and any sequence (f_n) of $\mathcal{K}$-valued analytic functions on V such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on every compact subset of V, f_n converges uniformly to 0 on every compact subset of V. It is well known that if T has property (β), then it has SVEP. We say that $T \in \mathcal{L}(\mathcal{K})$ is *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$, there are T-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{K}$ such that

$$\mathcal{K} = \mathcal{M} + \mathcal{N}, \quad \sigma(T|_{\mathcal{M}}) \subset U \text{ and } \sigma(T|_{\mathcal{N}}) \subset V.$$

In general, it is known that T and T^* have property (β) if and only if T is decomposable. In [9, Theorem 2.1], the relation between decomposability and property (β) for complex symmetric operators has been investigated. The following lemma investigate the equivalence of decomposability and property (β) for a $\mathcal{J}$-selfadjoint operator.

Lemma 2.1. *Let T be in $\mathcal{L}(\mathcal{K})$ where $(\mathcal{K}, J)$ is a Krein space.*

(i) If T is $\mathcal{J}$-selfadjoint, then either T or T^* has property (β) if and only if it is decomposable.

(ii) If T is $\mathcal{J}$-unitary, then T is decomposable if and only if T^{-1} is decomposable.

Proof. (i) We see from [12, Theorem 1.2.7] that every decomposable operator has property (β), so that we need only to prove the converse implication. We first assume that T has property (β). Let U be any open set in $\mathbb{C}$ and let (f_n) be a sequence of $\mathcal{K}$-valued analytic functions on U such that

$$\lim_{n\to\infty} \sup_{\lambda \in F} \|(T^* - \lambda) f_n(\lambda)\| = 0$$

where F is any compact subset in U. Then we have that

$$\begin{aligned} \lim_{n\to\infty} \sup_{\lambda \in F} \|(T-\lambda) J f_n(\lambda)\| &= \lim_{n\to\infty} \sup_{\lambda \in F} \|J(T^* - \lambda) f_n(\lambda)\| \\ &= \lim_{n\to\infty} \sup_{\lambda \in F} \|(T^* - \lambda) f_n(\lambda)\| = 0. \end{aligned}$$

Put $g_k(\lambda) := J(f_k(\lambda))$ for each $k \in \mathbb{N}$. If $f_k(\lambda)$ is analytic at μ for fixed k, then we can write $f_k(\lambda) = \sum_{n=0}^{\infty}(\lambda-\mu)^n \xi_n$ for any λ in some neighborhood of μ and $\xi_n \in \mathcal{K}$. We have that for any λ in some neighborhood of μ,

$$g_k(\lambda) = J(f_k(\lambda)) = J\left(\sum_{n=0}^{\infty}(\lambda-\mu)^n \xi_n\right) = \sum_{n=0}^{\infty}(\lambda-\mu)^n J\xi_n,$$

which implies that $g_k(\lambda)$ is analytic at μ. Hence, $g_k = J(f_k)$ is analytic at μ, so that $J(f_k(\lambda))$ is analytic on the whole U. Since T has property (β), $\{J(f_n)\}$ converges uniformly to 0 on every compact subset of U, that is,

$$\lim_{n\to\infty} \sup_{\lambda\in F} \|Jf_n(\lambda)\| = 0.$$

Hence T^* has property (β), so that T is decomposable.

In the case where T^* has property (β), it can be proved by the same way.

(ii) We note that T is decomposable if and only if T and T^* have property (β). However, T is $\mathcal{J}$-unitary if and only if so is T^{-1}, hence $(T^{-1})^*$ and T^{-1} have property (β) in the same way as in (i). Equivalently, T^{-1} is decomposable. □

The *Aluthge transform* of T is defined by $\widetilde{T} := |T|^{1/2}U|T|^{1/2}$ where $T = U|T|$ is a polar decomposition of T for a partial isometry U and a non-negative operator $|T| = (T^*T)^{1/2}$. We also define the sequence $\{\widetilde{T}^{(n)}\}$ of iterated Aluthge transforms of T by $\widetilde{T}^{(1)} = \widetilde{T}$ and $\widetilde{T}^{(n+1)} = \widetilde{(\widetilde{T}^{(n)})}$ for every positive integer $n \geq 1$. For some properties of Aluthge transforms of $\mathcal{J}$-selfadjoint operators, we refer to [2]. We note that every hyponormal $\mathcal{J}$-selfadjoint operator is normal. Indeed, it follows from $T = T^\#$ and hyponormality that

$$TT^* = JT^*JT^* = JT^*TJ \geq JTT^*J = T^*JT^*J = T^*T.$$

From this observation, we obtain the following corollary.

Corollary 2.2. *Let $T \in \mathcal{L}(\mathcal{K})$ be $\mathcal{J}$-selfadjoint. If $\widetilde{T}^{(n)}$ is hyponormal for some positive integer n, then T is decomposable and it has a nontrivial invariant subspace.*

Proof. Since $\widetilde{T}^{(n)}$ is hyponormal for some positive integer n, $\widetilde{T}^{(n)}$ has property (β). Then it follows from [10] that T also has property (β). Since T is $\mathcal{J}$-selfadjoint, Lemma 2.1 shows that it is also decomposable. Moreover, since $\widetilde{T}^{(n)}$ is hyponormal and $\mathcal{J}$-selfadjoint, it is normal. Thus, $\widetilde{T}^{(n)}$ has a nontrivial invariant subspace. It follows from [6, Remark 1.21] that T has a nontrivial invariant subspace. □

In the following theorem, we now discuss when $\mathcal{J}$-selfadjoint 2×2 operator matrices can be decomposable.

Theorem 2.3. *Let $(\mathcal{K}, J)$ be a Krein space and $T \in \mathcal{L}(\mathcal{K}\oplus\mathcal{K})$ be an operator matrix of the form*

$$T = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where A is a $\mathcal{J}$-selfadjoint operator with property (β). If B is a nilpotent of order 2 and $\mathcal{J}$-selfadjoint operator commuting with A and J, then T is decomposable.

Proof. We first show that T has property (β). Let $\{f_n^{(1)}\}$ and $\{f_n^{(2)}\}$ be two sequences of $\mathcal{K}$-valued analytic functions defined on open sets U_1 and U_2 in $\mathbb{C}$, respectively. Set $U := U_1 \cap U_2$. Suppose that $f_n := f_n^{(1)} \oplus f_n^{(2)}$ is a $\mathcal{K}\oplus\mathcal{K}$-valued analytic function on the open set U such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 on every compact subset of U. Then we have that

$$\lim_{n\to\infty}\left[(A-\lambda)f_n^{(1)}(\lambda) + Bf_n^{(2)}(\lambda)\right] = 0 \tag{1}$$

and

$$\lim_{n\to\infty}\left[Bf_n^{(1)}(\lambda)+(A-\lambda)f_n^{(2)}(\lambda)\right]=0 \tag{2}$$

where the convergence is uniform on every compact subset of U.

Since B is nilpotent of order 2 and commutes with A, it follows from (1) that the sequence $\{(A-\lambda)Bf_n^{(1)}(\lambda)\}$ is uniformly convergent to 0 on every compact subset of U. Since A has property (β),

$$\lim_{n\to\infty} Bf_n^{(1)}(\lambda)=0 \tag{3}$$

uniformly on every compact subset of U. By (2) and (3), the sequence $\{(A-\lambda)f_n^{(2)}(\lambda)\}$ also converges uniformly to 0 on compact subsets of U. Since A has property (β), $\{f_n^{(2)}(\lambda)\}$ is uniformly convergent to 0 on every compact subset of U. It follows from (1) that

$$\lim_{n\to\infty}(A-\lambda)f_n^{(1)}(\lambda)=0$$

where the convergence is uniform on every compact subset of U.

Since A has property (β), it follows that $\{f_n^{(1)}(\lambda)\}$ converges uniformly to 0 on every compact subset of U. Thus, the sequence $\{f_n(\lambda)\}$ converges uniformly to 0 on every compact subset of U, which implies that T has property (β). Moreover, we see that T is $\mathcal{J}$-selfadjoint because B commutes with J. By Lemma 2.1, T is decomposable. □

In the following remarks, we discuss when 2×2 operator matrices can be $\mathcal{J}$-selfadjoint.

Remark 2.4. Let $(\mathcal{K}_i, J_i)$ $(i=1,2)$ be a Krein space. Suppose that $T\in\mathcal{L}(\mathcal{K}_1\oplus\mathcal{K}_2)$ is an operator matrix of the form

$$T=\begin{pmatrix}A_1 & B\\ B^* & A_2\end{pmatrix}.$$

Then T is $\mathcal{J}$-selfadjoint with respect to $J:=J_1\oplus J_2$ if and only if A_1 and A_2 are $\mathcal{J}$-selfadjoint operators defined on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, and $J_1B=BJ_2$.

Indeed, it is obvious that $J:=J_1\oplus J_2$ is a fundamental symmetry on $\mathcal{K}_1\oplus\mathcal{K}_2$. Then we have that

$$TJ=\begin{pmatrix}A_1J_1 & BJ_2\\ B^*J_1 & A_2J_2\end{pmatrix} \text{ and } JT^*=\begin{pmatrix}J_1A_1^* & J_1B\\ J_2B^* & J_2A_2^*\end{pmatrix}.$$

Hence T is $\mathcal{J}$-selfadjoint if and only if A_1 and A_2 are $\mathcal{J}$-selfadjoint operators on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, and $J_1B=BJ_2$. □

Remark 2.5. Let $(\mathcal{K}, J)$ be a Krein space and let $T=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathcal{L}(\mathcal{K}\oplus\mathcal{K})$.

(i) For a fundamental symmetry $J_o:=\begin{pmatrix}0 & J\\ J & 0\end{pmatrix}\in\mathcal{L}(\mathcal{K}\oplus\mathcal{K})$, T is $\mathcal{J}$-selfadjoint if and only if B and C are $\mathcal{J}$-selfadjoint operators defined on $\mathcal{K}$ and $D=A^\#$.

(ii) For a fundamental symmetry $J_d:=\begin{pmatrix}J & 0\\ 0 & J\end{pmatrix}\in\mathcal{L}(\mathcal{K}\oplus\mathcal{K})$, T is $\mathcal{J}$-selfadjoint if and only if A and D are $\mathcal{J}$-selfadjoint operators defined on $\mathcal{K}$ and $B=C^\#$.

Lemma 2.6. *Let $(\mathcal{K}, J)$ be a Krein space and $T\in\mathcal{L}(\mathcal{K})$. Then T is decomposable if and only if $T^\#$ is decomposable.*

Proof. We first assume that T is decomposable. Both T and T^* have property (β) by [12, Theorem 1.2.29]. Let U be an open subset in $\mathbb{C}$ and $\{f_n\}$ be a sequence of analytic functions with values in $\mathcal{K}$ such that

$$\lim_{n\to\infty} \|(T^{\#} - \lambda)f_n(\lambda)\| = 0 \tag{4}$$

where the convergence is uniform on every compact set in U. Since J is a fundamental symmetry on $\mathcal{K}$, the equation (4) implies that the sequence $\{\|(T^* - \lambda)Jf_n(\lambda)\|\}$ converges uniformly on every compact set in U. We observe that if each $\mathcal{K}$-valued function $f_n(\lambda)$ is analytic at λ_0, then $Jf_n(\lambda)$ is also analytic at λ_0. Since T^* has property (β), we have

$$\lim_{n\to\infty} \|Jf_n(\lambda)\| = 0,$$

so that $\lim_{n\to\infty} \|f_n(\lambda)\| = 0$ uniformly on every compact set in U. This implies that $T^{\#}$ has property (β). Moreover, we have that $JTJ = (T^{\#})^*$ has property (β) since T has property (β). Therefore, $T^{\#}$ is decomposable.

Conversely, suppose that $T^{\#}$ is decomposable. Then JT^*J and JTJ have property (β). Let $\{f_n\}$ be a sequence of $\mathcal{K}$-valued analytic functions such that $\{\|(T - \lambda)f_n(\lambda)\|\}$ converges uniformly on every compact set in an open set U of $\mathbb{C}$. Then we have that

$$\lim_{n\to\infty} \|(JTJ - \lambda)Jf_n(\lambda)\| = 0$$

where the convergence is uniform on every compact set in U. Since JTJ has property (β), we have that $\lim_{n\to\infty} \|Jf_n(\lambda)\| = 0$ uniformly on compact set in U, so that $\|f_n(\lambda)\| \to 0$ uniformly on every compact set in U. Hence T has property (β). Since JT^*J has property (β), we see that T^* has property (β). This implies that T is decomposable. □

In the next proposition, we discuss the decomposability of 2×2 upper triangular operator matrices.

Proposition 2.7. *Let* $T = \begin{pmatrix} A & B \\ 0 & A^{\#} \end{pmatrix}$ *be an operator matrix in* $\mathcal{L}(\mathcal{K} \oplus \mathcal{K})$.

(i) If A is decomposable, then T is decomposable.

(ii) If A is $\mathcal{J}$-selfadjoint, then T is decomposable if and only if A is decomposable.

Proof. (i) Suppose that A is decomposable. By Lemma 2.6, we have that $A^{\#}$ is also decomposable. It is well known that property (β) holds for a 2×2 upper triangular operator matrix whenever each of its diagonal entries has property (β) (see [8, Theorem 3.7] for more details). This means that T is decomposable.

(ii) If T is decomposable, then T has property (β) [12, Theorem 1.2.29]. Since

$$0 = \lim_{n\to\infty} \|f_n(\lambda) \oplus 0\| = \lim_{n\to\infty} \|f_n(\lambda)\|$$

uniformly on every compact set in an open set U, A has property (β). Since A is $\mathcal{J}$-selfadjoint, it follows from Lemma 2.1 that A is decomposable. □

3. Property (C) and various decomposability for $\mathcal{J}$-selfadjoint operators

We recall some definitions for the local spectral theory and refer the reader to [12] for a complete account of definitions and properties considered in this section.

The *local resolvent set* $\rho_T(\xi)$ of $T \in \mathcal{L}(\mathcal{K})$ at $\xi \in \mathcal{K}$ is defined as the union of all open subsets U of $\mathbb{C}$ such that there exists an analytic function $f : U \to \mathcal{K}$ satisfying $(T - \lambda)f(\lambda) = \xi$ for all $\lambda \in U$. The *local spectrum* $\sigma_T(\xi)$ of T at ξ is the set defined by $\sigma_T(\xi) := \mathbb{C} \setminus \rho_T(\xi)$. It is obvious that $\sigma_T(\xi)$ is a (possibly empty) closed subset of $\sigma(T)$ where $\sigma(T)$ denotes the spectrum of T. For any subset $F \subseteq \mathbb{C}$, we define the *local spectral subspace* of $T \in \mathcal{L}(\mathcal{K})$ at F by

$$\mathcal{K}_T(F) := \{\xi \in \mathcal{K} : \sigma_T(\xi) \subset F\}.$$

By the definition, we see that $\mathcal{K}_T(F)$ is a T-invariant linear manifold of $\mathcal{K}$. For every closed subset $F \subseteq \mathbb{C}$, we have that

$$(T-\lambda)\mathcal{K}_T(F) = \mathcal{K}_T(F) \quad \text{for all } \lambda \in \mathbb{C} \setminus F$$

(see [12, Proposition 1.2.16]). Moreover, it is well known that SVEP is equivalent to the condition that $\mathcal{K}_T(\emptyset) = \{0\}$. We refer [12, Proposition 1.2.16 and 3.3.2] for more information. These linear manifolds, while generally not closed, play a significant role in the theory of spectral decompositions.

It is reasonable to expect that these are useful in the case of an operator T for which $\mathcal{K}_T(F)$ is closed for every closed set $F \subset \mathbb{C}$. It was introduced by Dunford, played a large role in the development of the theory of spectral operators. From this notations, we say that $T \in \mathcal{L}(\mathcal{K})$ has *Dunford's property* (*C*) if $\mathcal{K}_T(F)$ is closed for every closed set $F \subset \mathbb{C}$. We see from [12, Proposition 1.2.19] the following implications;

$$\text{property } (\beta) \Longrightarrow \text{Dunford's property } (C) \Longrightarrow \text{SVEP}.$$

We now discuss Dunford's property (C) of two $\mathcal{J}$-selfadjoint operators T and T^*. To do this, we first begin with the basic lemma.

Lemma 3.1. *If $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-selfadjoint, then the following equalities hold for any $\xi \in \mathcal{K}$;*

$$\sigma_{T^*}(J\xi) = \sigma_T(\xi) \text{ and } \sigma_T(J\xi) = \sigma_{T^*}(\xi).$$

Proof. Since the $\mathcal{J}$-selfadjointness of T implies that of T^*, we need only to show the first equality. If $\lambda_0 \in \rho_T(\xi)$, then there exists a $\mathcal{K}$-valued analytic function f in a neighborhood U of λ_0 such that $(T-\lambda)f(\lambda) = \xi$ for every $\lambda \in U$. Thus, we have that

$$(T^*-\lambda)Jf(\lambda) = J(T-\lambda)f(\lambda) \equiv J\xi \quad \text{on } U.$$

Since $Jf(\lambda)$ is also analytic at λ_0, we have that $\lambda_0 \in \rho_{T^*}(J\xi)$. Hence, we obtain the inclusion $\rho_T(\xi) \subseteq \rho_{T^*}(J\xi)$.

For the reverse inclusion, we assume that $\lambda_0 \in \rho_{T^*}(J\xi)$. Then there exists a $\mathcal{K}$-valued analytic function f in a neighborhood U of λ_0 such that $(T^*-\lambda)f(\lambda) = J\xi$ for every $\lambda \in U$. Since T is $\mathcal{J}$-selfadjoint, we obtain that

$$(T-\lambda)Jf(\lambda) = J(T^*-\lambda)f(\lambda) \equiv \xi \text{ on } U.$$

This implies that $\lambda_0 \in \rho_T(\xi)$, so that we have $\rho_{T^*}(J\xi) \subseteq \rho_T(\xi)$. Therefore, we obtain the first equality $\rho_{T^*}(J\xi) = \rho_T(\xi)$, which means that $\sigma_{T^*}(J\xi) = \sigma_T(\xi)$. □

Theorem 3.2. *For a $\mathcal{J}$-selfadjoint operator T, T has Dunford's property* (*C*) *if and only if so does T^*.*

Proof. By interchanging roles of T and T^*, we need only to prove the necessary condition. Suppose that T has Dunford's property (C). We note that T has SVEP, so that T^* also has SVEP. We first show that $\overline{\mathcal{K}_{T^*}(F)} = \mathcal{K}_{T^*}(F)$ for every closed subset F in $\mathbb{C}$ where $\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$. If $\eta \in \overline{\mathcal{K}_{T^*}(F)}$, then there exists a sequence $\{\eta_n\}$ in $\mathcal{K}_{T^*}(F)$ converging to η. Hence it follows that

$$\lim_{n\to\infty} \|J\eta_n - J\eta\| = \lim_{n\to\infty} \|\eta_n - \eta\| = 0.$$

Since $\{\eta_n\}$ is a sequence in $\mathcal{K}_{T^*}(F)$, it follows from Lemma 3.1 that $\sigma_T(J\eta_n) = \sigma_{T^*}(\eta_n) \subset F$ for each n. This implies that $\{J\eta_n\} \subset \mathcal{K}_{T^*}(F)$. By Dunford's property (C) of T, we have that $J\eta \in \mathcal{K}_{T^*}(F)$.

Now, we observe that $\mathcal{K}_T(E) = J\mathcal{K}_{T^*}(E)$ for any subset E of $\mathbb{C}$. Indeed, if $\xi \in \mathcal{K}_T(E)$, then we have that $\sigma_{T^*}(J\xi) = \sigma_T(\xi) \subset E$. Thus, we obtain that $J\xi \in \mathcal{K}_{T^*}(E)$, so that $J\mathcal{K}_T(E) \subset \mathcal{K}_{T^*}(E)$. We similarly get the converse inclusion.

Hence we have that $J\eta \in J\mathcal{K}_{T^*}(F)$, so that this inclusion $J(\overline{\mathcal{K}_{T^*}(F)}) \subseteq J\mathcal{K}_{T^*}(F)$ holds. Furthermore, we have that

$$\overline{\mathcal{K}_{T^*}(F)} = J(J(\overline{\mathcal{K}_{T^*}(F)})) \subset J(J\mathcal{K}_{T^*}(F)) = \mathcal{K}_{T^*}(F).$$

Therefore, $\mathcal{K}_{T^*}(F)$ is closed for any closed subset F of $\mathbb{C}$, which implies that T^* has Dunford's property (C). □

Laursen and Neumann [11] introduced super-decomposability to study questions about multipliers on Banach algebras. We say that $T \in \mathcal{L}(\mathcal{K})$ is *super-decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$, there exists some operator $R \in \mathcal{L}(\mathcal{K})$ such that

$$RT = TR, \quad \sigma(T|_{\overline{\mathrm{ran}(R)}}) \subset U, \quad \text{and} \quad \sigma(T|_{\overline{\mathrm{ran}(I-R)}}) \subset V.$$

The condition $RT = TR$ implies that $\overline{\mathrm{ran}(R)}$ and $\overline{\mathrm{ran}(I-R)}$ are invariant subspaces under T, so that the definition makes sense. Obviously, a super-decomposable operator is decomposable. The following corollary shows the equivalence of supper-decomposability of T and T^* for a $\mathcal{J}$-selfadjoint operator T.

Corollary 3.3. *Suppose T is a $\mathcal{J}$-selfadjoint operator. Then T is supper-decomposable if and only if T^* is.*

Proof. By interchanging roles of T and T^*, we need only to prove the necessary condition. Suppose that T is supper-decomposable. Then T has Dunford's property (C) and for any open cover $\{U, V\}$ of $\mathbb{C}$, there exists an operator $R \in \mathcal{L}(\mathcal{K})$ such that

$$RT = TR, \quad \mathrm{ran}(R) \subset \mathcal{K}_T(\overline{U}), \quad \text{and} \quad \mathrm{ran}(I-R) \subset \mathcal{K}_T(\overline{V}).$$

It follows from Theorem 3.2 that T^* has Dunford's property (C).

Since $RT = TR$ and $T^* = JTJ$, the operator $S := JRJ$ commutes with T^*. Indeed, we have

$$ST^* = J(TR)J = (JTJ)(JRJ) = T^*S.$$

For any $\xi \in \mathcal{K}$, we have that

$$\begin{aligned} S\xi &= JRJ\xi \in J\mathcal{K}_T(\overline{U}) = \mathcal{K}_{T^*}(\overline{U}), \\ (I-S)\xi &= J(I-R)J\xi \in J\mathcal{K}_T(\overline{V}) = \mathcal{K}_{T^*}(\overline{V}). \end{aligned}$$

Therefore, T^* is supper decomposable. $\square$

Lemma 3.4. *For a $\mathcal{J}$-selfadjoint operator T, T has SVEP if and only if T^* has SVEP.*

Proof. We assume that T has SVEP. Let U be an open set in $\mathbb{C}$ and f be a $\mathcal{K}$-valued analytic function such that $(T^* - \lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since $TJ = JT^*$, we have that

$$(T-\lambda)Jf(\lambda) = J(T^* - \lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in U.$$

By SVEP of T, we see that $Jf(\lambda) = 0$ for all $\lambda \in U$. Since J is invertible, $f \equiv 0$ on U, which implies that T^* has SVEP. By symmetry, the converse is true. $\square$

Recall that $T \in \mathcal{L}(\mathcal{K})$ with SVEP has *Dunford's boundedness condition* (B) if there exists a constant $k > 0$ such that for every $\xi, \eta \in \mathcal{K}$ with $\sigma_T(\xi) \cap \sigma_T(\eta) = \emptyset$,

$$\|\xi\| \leq k\|\xi + \eta\|$$

where k is independent of ξ and η. We say that T is *hypercyclic* if there is a vector $\xi \in \mathcal{K}$ such that the orbit $\{T^n\xi : n \in \mathbb{N}\}$ is dense in $\mathcal{K}$. In this case, ξ is called a *hypercyclic vector* for T. We call T *hypertransitive* if every nonzero vector in $\mathcal{K}$ is hypercyclic for T.

Proposition 3.5. *For a $\mathcal{J}$-selfadjoint operator T, the following statements hold.*

(i) T has Dunford's boundedness condition (B) if and only if so does T^*.

(ii) T is hypercyclic if and only if T^* is.

(iii) For any positive integer n, T^n is non-hypertransitive if and only if $(T^*)^n$ is.

Proof. (i) Suppose that a $\mathcal{J}$-selfadjoint operator T has Dunford's boundedness condition (B). Since T has SVEP, it follows from Lemma 3.4 that T^* also has SVEP. Take arbitrary vectors $\xi, \eta \in \mathcal{K}$ with $\sigma_{T^*}(\xi) \cap \sigma_{T^*}(\eta) = \emptyset$. By Lemma 3.1, we have that $\sigma_{T^*}(\xi) = \sigma_T(J\xi)$. Thus, we see that $\sigma_T(J\xi) \cap \sigma_T(J\eta) = \emptyset$. Since T has Dunford's boundedness condition (B), there exists a constant $k > 0$, independent of ξ and η, such that

$$\|J\xi\| \leq k\|J\xi + J\eta\|,$$

which is equivalent to the inequality $\|\xi\| \leq k\|\xi + \eta\|$. Hence, T^* has Dunford's boundedness condition (B). By replacing T with T^*, we get the reverse implication.

(ii) Suppose that a $\mathcal{J}$-selfadjoint operator T is hypercyclic. There exists $\xi \in \mathcal{K}$ such that the orbit $\{T^n\xi : n \in \mathbb{N}\}$ is dense in $\mathcal{K}$. Thus, $\{JT^n\xi : n \in \mathbb{N}\}$ is also dense in $\mathcal{K}$. Since $JT^n = (T^*)^n J$ for any positive integer n, we have $\mathcal{K} = \overline{\{(T^*)^n J\xi : n \in \mathbb{N}\}}$. Hence T^* is hypercyclic. The same argument shows that the converse is also true.

(iii) By [7, Theorem 1.7], T is non-hypertransitive if and only if T^n is non-hypertransitive for any $n \in \mathbb{N}$. Thus, it is enough to show that T is non-hypertransitive if and only if T^* is. However, we observe that $\xi \in \mathcal{K}$ is a hypercyclic vector for T if and only if $J\xi$ is a hypercyclic vector for T^*. Since J is surjective, $\mathrm{ran}(J) = \mathcal{K}$. This completes the proof. □

In the remaining of this section, we discuss several kinds of spectral decomposition properties for $\mathcal{J}$-selfadjoint operators. We denote the interior of F by $\mathrm{int}(F)$. We first recall some definitions related to decomposability. A decomposable operator $T \in \mathcal{L}(\mathcal{K})$ is *strongly decomposable* if, for every T-spectral maximal subspace $\mathcal{K}'$, the operator $T|_{\mathcal{K}'}$ is decomposable, equivalently, if for arbitrary closed sets F_1 and F_2 in $\mathbb{C}$ with $\sigma(T) \subset \mathrm{int}(F_1) \cup \mathrm{int}(F_2)$, we have

$$\mathcal{K}_T(C) = \mathcal{K}_T(C \cap F_1) + \mathcal{K}_T(C \cap F_2) \tag{5}$$

where C is any closed set in $\mathbb{C}$. A decomposable operator T is *quasi-strongly decomposable* if the restriction $T|_{\mathcal{K}_T(\overline{G \cap \sigma(T)})}$ is also decomposable for each open set G of $\mathbb{C}$. By [14, Theorem 1.3.11], we note that T is quasi-strongly decomposable if and only if T is decomposable and for arbitrary open sets U and V in $\mathbb{C}$, the inclusion

$$\mathcal{K}_T(\overline{\sigma(T) \cap (U \cup V)}) \subseteq \mathcal{K}_T(\overline{\sigma(T) \cap U_\epsilon}) + \mathcal{K}_T(\overline{\sigma(T) \cap V}) \tag{6}$$

holds where U_ε denotes the ε-neighborhood of $\overline{U}$.

We say that $T \in \mathcal{L}(\mathcal{K})$ has *asymptotic spectral decomposition* if, for any finite open cover $\{U_i\}_{i=1}^n$ of $\mathbb{C}$, there exist T-invariant subspaces $\{\mathcal{M}_i\}_{i=1}^n$ in $\mathcal{K}$ such that

$$\mathcal{K} = \bigvee_{i=1}^{n} \mathcal{M}_i \quad \text{and} \quad \sigma(T|_{\mathcal{M}_i}) \subset U_i \;\; (i = 1, 2, \ldots, n).$$

A T-invariant subspace $\mathcal{M}$ is *analytically T-invariant* if, for any region G in $\mathbb{C}$ and any $\mathcal{K}$-valued analytic function f on G with $(T - \lambda)f(\lambda) \in \mathcal{M}$ for all $\lambda \in G$, it follows that $f(\lambda) \in \mathcal{M}$ for all $\lambda \in G$. We say that T is *quasi-decomposable* if T has asymptotic spectral decomposition and Dunford's property (C), and *analytically decomposable* if T has asymptotic spectral decomposition consisting of analytically invariant subspaces. In [14, Chapter 1], Lange and Wang gave a hierarchy of several decomposability as follows;

strongly decomposable $\Longrightarrow$ *quasi-strongly decomposable* $\Longrightarrow$ *decomposable*
$\Longrightarrow$ *quasi-decomposable* $\Longrightarrow$ *analytically decomposable.*

We now study spectral decomposition properties for $\mathcal{J}$-selfadjoint operators.

Lemma 3.6. *If T is a $\mathcal{J}$-selfadjoint operator on a Krein space $(\mathcal{K}, J)$, then the following assertions hold.*

(i) For any subset F of $\mathbb{C}$, $J\mathcal{K}_T(F) = \mathcal{K}_{T^*}(F)$ and $J\mathcal{K}_{T^*}(F) = \mathcal{K}_T(F)$.

(ii) $\mathcal{M}$ is a T-invariant subspace if and only if $J\mathcal{M}$ is a T^*-invariant subspace.

(iii) $\mathcal{M}$ is an analytically T-invariant subspace if and only if $J\mathcal{M}$ is an analytically T^*-invariant subspace.

(iv) For any subset F of $\mathbb{C}$ and any T-invariant subspace $\mathcal{M}$, if $\sigma(T|_{\mathcal{M}}) \subseteq F$, then $\sigma(T^*|_{J\mathcal{M}}) \subseteq F$

Proof. (i) Since J is a fundamental symmetry on $\mathcal{K}$, we need only to prove the first equality. If $\xi \in \mathcal{K}_T(F)$, then $\sigma_T(\xi) \subseteq F$. By Lemma 3.1, we have $\sigma_T(\xi) = \sigma_{T^*}(J\xi)$. Hence we have that $\sigma_{T^*}(J\xi) \subseteq F$, so that $J\xi \in \mathcal{K}_{T^*}(F)$. This shows the inclusion $J\mathcal{K}_T(F) \subseteq \mathcal{K}_{T^*}(F)$. By the same way, we can see the reverse inclusion $J\mathcal{K}_{T^*}(F) \subseteq \mathcal{K}_T(F)$.

(ii) We first assume that $\mathcal{M}$ is a T-invariant subspace. It is clear that $J\mathcal{M}$ is a closed subspace of $\mathcal{K}$. Since $\mathcal{M}$ is T-invariant and $T^*J = JT$, we obtain that $T^*J\mathcal{M} = JT\mathcal{M} \subseteq J\mathcal{M}$, which implies that $J\mathcal{M}$ is a T^*-invariant subspace. By replacing T with T^*, we can see that the converse is also true.

(iii) Suppose that $\mathcal{M}$ is an analytically T-invariant subspace. It follows from (ii) that $J\mathcal{M}$ is a T^*-invariant subspace. Let G be any region in $\mathbb{C}$ and f be any $\mathcal{K}$-valued analytic function on G such that $(T^* - \lambda)f(\lambda) \in J\mathcal{M}$ for all $\lambda \in G$. Since T is $\mathcal{J}$-selfadjoint, we have that

$$(T - \lambda)Jf(\lambda) = J(T^* - \lambda)f(\lambda) \in \mathcal{M}$$

for all $\lambda \in G$. Since $Jf(\lambda)$ is analytic on G and $\mathcal{M}$ is analytically T-invariant, we have that $Jf(\lambda) \in \mathcal{M}$ for all $\lambda \in G$. Thus, we see that $f(\lambda) \in J\mathcal{M}$ for all $\lambda \in G$, which shows that $J\mathcal{M}$ is analytically T^*-invariant.

(iv) Let F be a subset of $\mathbb{C}$ and $\mathcal{M}$ be any T-invariant subspace. Suppose that $\sigma(T|_{\mathcal{M}}) \subset F$. If $\lambda \notin F$, then the operator $T|_{\mathcal{M}} - \lambda$ is invertible. We see from (ii) that $J\mathcal{M}$ is a T^*-invariant subspace. First, we claim that $T^*|_{J\mathcal{M}} - \lambda$ is injective. Indeed, if $\xi \in \ker(T^*|_{J\mathcal{M}} - \lambda) = \ker(T^* - \lambda) \cap J\mathcal{M}$, then there exists a vector η in $\mathcal{M}$ such that $\xi = J\eta$ and $T^*\xi = \lambda\xi$. Thus, we have that $\lambda J\eta = T^*J\eta = JT\eta$, so that $J(T - \lambda)\eta = 0$. This yields that

$$\eta \in \ker(T - \lambda) \cap \mathcal{M} = \ker(T|_{\mathcal{M}} - \lambda).$$

Since $T|_{\mathcal{M}} - \lambda$ is injective, we have $\eta = 0$, so that $\xi = 0$. Hence $T^*|_{J\mathcal{M}} - \lambda$ is injective.

To prove the subjectivity of $T^*|_{J\mathcal{M}} - \lambda$, we take any $\xi \in J\mathcal{M}$. For some $\eta \in \mathcal{M}$, we have $\xi = J\eta$. Since $T|_{\mathcal{M}} - \lambda$ is surjective, there is a vector $\zeta \in \mathcal{M}$ such that $\eta = (T - \lambda)\zeta$. Then we have that

$$\xi = J\eta = J(T - \lambda)\zeta = (T^* - \lambda)J\zeta \in (T^* - \lambda)(J\mathcal{M}),$$

which shows that $J\mathcal{M} \subseteq (T^* - \lambda)(J\mathcal{M})$. Thus, the operator $T^*|_{J\mathcal{M}} - \lambda$ is invertible, so that $\lambda \notin \sigma(T^*|_{J\mathcal{M}})$. This shows that $\sigma(T^*|_{J\mathcal{M}}) \subseteq F$. $\square$

Motivated by Dunford's approach to the spectral decomposition, Wang [17] introduced the class of operators that have the spectral decomposition property with respect to the identity, which is equivalent to that of the super-decomposable operators. A decomposable operator $T \in \mathcal{L}(\mathcal{K})$ is *decomposable relative to the identity* if, for each finite open cover $\{U_i\}_{i=1}^n$ of $\mathbb{C}$, there exist a corresponding system $\{\mathcal{M}_i\}_{i=1}^n$ of T-invariant subspaces and bounded operators $\{P_i\}_{i=1}^n \subseteq \{T\}'$ such that

$$\sigma(T|_{\mathcal{M}_i}) \subset U_i,\ \ P_i(\mathcal{K}) \subset \mathcal{M}_i,\ (i = 1, \ldots, n) \quad \text{and} \quad \sum_{i=1}^{n} P_i = I$$

where $\{T\}'$ denotes the commutant of T. Lange and Wang [13] proved that T is decomposable relative to the identity if and only if for any open cover $\{U, V\}$ of $\mathbb{C}$, there exists an operator $P \in \{T\}'$ such that

$$\gamma(P\xi, T) \subset \overline{U} \ \text{ and } \ \gamma((I - P)\xi, T) \subset \overline{V} \ \text{ for all } \xi \in \mathcal{K},$$

where $\gamma(\eta, T) := \bigcap \{F \subset \mathbb{C} : \eta \in \mathcal{K}_T(F)\}$ for $\eta \in \mathcal{K}$. They showed that these operators form a proper subclass of the strongly decomposable operators.

Theorem 3.7. *For a $\mathcal{J}$-selfadjoint operator T on $\mathcal{K}$, the following statements hold.*

(i) T is decomposable relative to the identity if and only if T^* is decomposable relative to the identity.

(ii) T is strongly decomposable if and only if T^* is strongly decomposable.

(iii) T is quasi-strongly decomposable if and only if T^* is quasi-strongly decomposable.

(iv) T is quasi-decomposable if and only if T^* is quasi-decomposable.

(v) T is analytically decomposable if and only if T^* is analytically decomposable.

Proof. By symmetry, it suffices to prove the forward direction in all above statements.

(i) We first assume that T is decomposable relative to the identity. Let $\{U, V\}$ be any open cover of $\mathbb{C}$. Then there is a bounded linear operator P commuting with T such that

$$\gamma(P\xi, T) \subset \overline{U} \text{ and } \gamma((I-P)\xi, T) \subset \overline{V}$$

for all $\xi \in \mathcal{K}$. By setting $Q = JPJ$, we have that $T^*Q = JTPJ = JPTJ = JPJT^* = QT^*$, which shows that Q commutes with T^*. Furthermore, it follows from (i) of Lemma 3.6 that

$$\begin{aligned}\gamma(\xi, T^*) &= \bigcap \{F \in \mathbb{C} : \xi \in \mathcal{K}_{T^*}(F)\}\\ &= \bigcap \{F \in \mathbb{C} : \xi \in J\mathcal{K}_T(F)\}\\ &= \bigcap \{F \in \mathbb{C} : J\xi \in J\mathcal{K}_T(F)\}\\ &= \gamma(J\xi, T)\end{aligned}$$

for all $\xi \in \mathcal{K}$. It follows from [12] that T^* is decomposable relative to the identity.

(ii) Suppose that T is strongly decomposable. Then it is obvious that T^* is decomposable. Let F_1 and F_2 be two closed sets in $\mathbb{C}$ such that $\sigma(T^*) \subseteq \mathrm{int}(F_1) \cup \mathrm{int}(F_2)$. Since T is $\mathcal{J}$-selfadjoint, we have $\sigma(T) = \sigma(T^*)$, so that $\sigma(T) \subset \mathrm{int}(F_1) \cup \mathrm{int}(F_2)$. Moreover, since T is strongly decomposable, the identity $\mathcal{K}_T(C) = \mathcal{K}_T(C \cap F_1) + \mathcal{K}_T(C \cap F_2)$ holds for any closed set C in $\mathbb{C}$. By (i) in Lemma 3.6, we have that

$$\begin{aligned}\mathcal{K}_{T^*}(C) &= J\mathcal{K}_T(C) = J\mathcal{K}_T(C \cap F_1) + J\mathcal{K}_T(C \cap F_2)\\ &= \mathcal{K}_{T^*}(C \cap F_1) + \mathcal{K}_{T^*}(C \cap F_2)\end{aligned}$$

where C is any closed set in $\mathbb{C}$. Therefore, T^* is strongly decomposable.

(iii) Suppose that T is quasi-strongly decomposable. Since T^* is decomposable, we need only to prove the inclusion in (6). Let U and V be two open sets in $\mathbb{C}$ and U_ε be an ε-neighborhood of $\overline{U}$. By (i) in Lemma 3.6, we have that

$$\begin{aligned}\mathcal{K}_{T^*}(\overline{\sigma(T^*) \cap (U \cup V)}) &= J\mathcal{K}_T(\overline{\sigma(T) \cap (U \cup V)})\\ &\subseteq J\mathcal{K}_T(\overline{\sigma(T) \cap U_\varepsilon}) + J\mathcal{K}_T(\overline{\sigma(T) \cap V})\\ &= \mathcal{K}_{T^*}(\overline{\sigma(T^*) \cap U_\varepsilon}) + \mathcal{K}_{T^*}(\overline{\sigma(T^*) \cap V}).\end{aligned}$$

Therefore, T^* is quasi-strongly decomposable.

(iv) Assume that T is quasi-decomposable. Since T has Dunford's property (C), T^* also has Dunford's property (C) by Theorem 3.2. Thus, it is enough to show that T^* has asymptotic spectral decomposition. Let $\{U_i\}_{i=1}^n$ be any finite open cover of $\mathbb{C}$. Since T is quasi-decomposable, there exist a finite sequence $\{\mathcal{M}_i\}_{i=1}^n$ of T-invariant subspaces such that

$$\mathcal{K} = \bigvee_{i=1}^n \mathcal{M}_i \text{ and } \sigma(T|_{\mathcal{M}_i}) \subset U_i \ (i = 1, \ldots, n).$$

It follows from (ii) in Lemma 3.6 that each subspace $J\mathcal{M}_i$ is T^*-invariant. Clearly, we have $\mathcal{K} = J\mathcal{K} = \bigvee_{i=1}^n J\mathcal{M}_i$. Furthermore, it follows from (iv) in Lemma 3.6 that $\sigma(T^*|_{J\mathcal{M}_i}) \subset U_i$ for all $i = 1, \ldots, n$. Therefore, T^* is quasi-decomposable.

(v) By (iii) in Lemma 3.6 and the proof of (iv), we can get the proof. $\square$

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