Sharp Z-Eigenvalue Inclusion Set-Based Method for Testing the Positive Definiteness of Multivariate Homogeneous Forms

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Abstract. In this paper, we establish a sharp Z-eigenvalue inclusion set for even-order real tensors by Z-identity tensor and prove that new Z-eigenvalue inclusion set is sharper than existing results. We propose some sufficient conditions for testing the positive definiteness of multivariate homogeneous forms via new Z-eigenvalue inclusion set. Further, we establish upper bounds on the Z-spectral radius of weakly symmetric nonnegative tensors and estimate the convergence rate of the greedy rank-one algorithms. The given numerical experiments show the validity of our results.

1. Introduction

Consider the following multivariate homogeneous forms with spherical constraint:

\[ f_\mathcal{A}(x) = \mathcal{A}x^m = \sum_{i_1,i_2,...,i_n=1}^n a_{i_1i_2...i_n}x_{i_1}x_{i_2}...x_{i_n} \]

s.t. \[ x^\top x = 1, \]

where \( x \in \mathbb{R}^n, m, n \geq 2, f_\mathcal{A}(x) \) is a multivariate homogeneous form of degree \( m \) with \( n \) variables, and \( \mathcal{A} \in \mathbb{R}^{[m,n]} \) is an \( m \)-order \( n \)-dimensional real tensor with entries \( a_{i_1...i_n} \in \mathbb{R}, i_j \in \mathbb{N} = \{1, ..., n\}, j = 1, ..., m. \)

Clearly, the critical points of (1) satisfy the following equations for some \( \lambda \in \mathbb{R} : \)

\[ \mathcal{A}x^{m-1} = \lambda x \text{ and } x^\top x = 1, \]

where \( (\mathcal{A}x^{m-1})_i = \sum_{i_2,...,i_n \in \mathbb{N}} a_{i_1i_2...i_n}x_{i_2}...x_{i_n}. \) The real number \( \lambda \) and the real vector \( x \) satisfying with (2) are called Z-eigenvalue and Z-eigenvector, respectively.

The multivariate homogeneous form \( f_\mathcal{A}(x) \) is positive definite, which plays important roles in signal processing [15] and the stability study of nonlinear autonomous systems via Lyapunov’s direct method in

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automatic control [3, 4, 13]. Note that $f_A(x)$ is positive definite if and only if tensor $A$ is positive definite, and that an even-order real symmetric tensor is positive definite if and only if all of its $Z$-eigenvalues are positive [14]. Some effective algorithms for finding $Z$-eigenvalue and the corresponding eigenvector have been implemented [5–9, 11, 16, 18, 21–26], but it is difficult to compute all the $Z$-eigenvalues and judge the positive definiteness of an even-order real symmetric tensor. Very recently, Li et al. [10] proposed Gershgorin-type $Z$-eigenvalue inclusion set with parameters by $Z$-identity tensor, which can identify the positive-definiteness of an even-order real symmetric tensor. It is remarkable that Brauer-type inclusion set is tighter than Gershgorin-type inclusion set [20]. As a continuation of the article [20], we shall establish the positive definiteness of an even-order real symmetric tensor.

To end this section, we introduce $Z$-identity tensor in [8, 10] and important results proposed in [10].

**Definition 1.1.** Assume that $m$ is even. We call $I_Z$ a $Z$-identity tensor if

$$I_Z x^{m-1} = x, \quad x^T x = 1, \quad \forall x \in \mathbb{R}^n.$$ 

It is worth noting that the even-order $n$ dimension $Z$-identity tensor is not unique in general. For instance, each even tensor in the following is a $Z$-identity tensor:

Case I: $(I_Z)_{|i_1|\leq \ldots \leq |i_m|} = 1, \forall k \in \mathbb{N}$ and $m = 2k$;

Case II (Property 2.4 of [8]): $(I_Z)_{|i_1|\leq \ldots \leq |i_m|} = \frac{1}{m!} \sum_{p \in \mathbb{P}_m} \delta_{p(1)} \delta_{p(2)} \ldots \delta_{p(m-1)} \delta_{p(m)}$, where $\delta$ is the standard Kronecker, i.e.,

$$\delta_{ij} = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{otherwise}.
\end{cases}$$

**Lemma 1.2.** (Theorem 2 of [10]) Let $A = (a_{i_1\ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a $Z$-identity tensor with $m$ being even. Let $\sigma_Z(A)$ be the set of all $Z$-eigenvalues of $A$. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$, then

$$\sigma_Z(A) \subseteq \mathcal{G}_1(A, \alpha) = \bigcup_{i \in \mathbb{N}} \mathcal{G}_i(A, \alpha) = \{ z \in \mathbb{R} : |z - \alpha_j| \leq R_i(A, \alpha_i) \},$$

where $R_i(A, \alpha_i) = \sum_{|i_1|\leq \ldots \leq |i_m|} |a_{i_1\ldots i_m} - \alpha_i(I_Z)_{i_1\ldots i_m}|$. Furthermore, $\sigma_Z(A) \subseteq \bigcap_{\alpha \in \mathbb{R}^n} \mathcal{G}(A, \alpha)$.

2. A sharp $Z$-eigenvalue inclusion set for even-order real tensors

In this section, we establish new $Z$-eigenvalue inclusion set for even-order tensors. To this end, we define

$$\Theta = \{ (i_2, i_3, \ldots, i_m) : i_k = j \text{ for some } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in \mathbb{N} \},$$

$$\overline{\Theta} = \{ (i_2, i_3, \ldots, i_m) : i_k \neq j \text{ all any } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in \mathbb{N} \},$$

$$r_j^\Theta(A, \alpha) = \sum_{(i_2, \ldots, i_m) \in \overline{\Theta}} |a_{i_1\ldots i_m} - \alpha_i(I_Z)_{i_1\ldots i_m}|, \quad r_j^\overline{\Theta}(A, \alpha) = \sum_{(i_2, \ldots, i_m) \in \overline{\Theta}} |a_{i_1\ldots i_m} - \alpha_i(I_Z)_{i_1\ldots i_m}|.$$

Obviously, $R_i(A, \alpha_i) = r_j^\Theta(A, \alpha_i) + r_j^\overline{\Theta}(A, \alpha_i)$.

**Theorem 2.1.** Let $A = (a_{i_1\ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a $Z$-identity tensor with $m$ being even. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$, then

$$\sigma_Z(A) \subseteq \mathcal{U}_j(A, \alpha) = \bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}, j \neq i} \mathcal{U}_j(A, \alpha),$$

where $U_{ij}(A, \alpha) = \{ z \in \mathbb{R} : |z - \alpha_j| - r_j^\overline{\Theta}(A, \alpha_i)|z - \alpha_i| \leq r_j^\Theta(A, \alpha_i)R_i(A, \alpha_i) \}$. Furthermore, $\sigma_Z(A) \subseteq \bigcap_{\alpha \in \mathbb{R}^n} \mathcal{U}(A, \alpha)$. 

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Proof. Let \((\lambda, x)\) be a \(Z\)-eigenpair of \(\mathcal{A}\) and \(I_Z \in \mathbb{R}^{[m,n]}\) be a \(Z\)-identity tensor, i.e.,
\[
\mathcal{A}x^{m-1} = \lambda x = \lambda I_Z x^{m-1}, \quad x^T x = 1.
\]
Assume \(|x_i| = \max_{i \in \mathbb{N}} |x_i|\), then \(0 < |x_i|^{m-1} \leq |x_i| \leq 1\).

On one hand, taking the \(t\)-th equation from (3), for any \(j \in \mathbb{N}, j \neq t\), we have
\[
\sum_{i_2, \ldots, i_m \in \mathbb{N}} \lambda(I_Z)_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m} = \sum_{i_2, \ldots, i_m \in \mathbb{N}} a_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}.
\]
Hence, for any real number \(a_t\), it follows that
\[
(\lambda - a_t)x_t = \sum_{i_2, \ldots, i_m \in \mathbb{N}} (\lambda - a_t)(I_Z)_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m} = \sum_{i_2, \ldots, i_m \in \mathbb{N}} (a_{i_2 \ldots i_m} - a_t(I_Z)_{i_2 \ldots i_m}) x_{i_2} \cdots x_{i_m}
\]
\[
= \sum_{i_2, \ldots, i_m \in \mathbb{E}_j} (a_{i_2 \ldots i_m} - a_t(I_Z)_{i_2 \ldots i_m}) x_{i_2} \cdots x_{i_m} + \sum_{i_2, \ldots, i_m \in \mathbb{E}_j} (a_{i_2 \ldots i_m} - a_t(I_Z)_{i_2 \ldots i_m}) x_{i_2} \cdots x_{i_m}
\]
Taking modulus in (5) and using the triangle inequality give
\[
|\lambda - a_t||x_t| \leq \sum_{i_2, \ldots, i_m \in \mathbb{E}_j} |a_{i_2 \ldots i_m} - a_t(I_Z)_{i_2 \ldots i_m}||x_{i_2}| \cdots |x_{i_m}| + \sum_{i_2, \ldots, i_m \in \mathbb{E}_j} |a_{i_2 \ldots i_m} - a_t(I_Z)_{i_2 \ldots i_m}||x_{i_2}| \cdots |x_{i_m}|
\]
\[
\leq r_j^\Theta(\mathcal{A}, a_t)|x_t| + r_j^\Theta(\mathcal{A}, a_t)|x_t|
\]
i.e.,
\[
\left(|\lambda - a_t| - r_j^\Theta(\mathcal{A}, a_t)\right)|x_t| \leq r_j^\Theta(\mathcal{A}, a_t)|x_t|.
\]
On the other hand, for \(t \neq j \in \mathbb{N}\), taking the \(j\)-th equation from (3), we obtain
\[
(\lambda - a_t)x_j = \sum_{i_2, \ldots, i_m \in \mathbb{N}} (\lambda - a_t)(I_Z)_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m} = \sum_{i_2, \ldots, i_m \in \mathbb{N}} (a_{i_2 \ldots i_m} - a_t(I_Z)_{i_2 \ldots i_m}) x_{i_2} \cdots x_{i_m}.
\]
Taking modulus in (8) and using the triangle inequality, one has
\[
|\lambda - a_t||x_j| \leq R_j(\mathcal{A}, a_t)|x_j|.
\]
If \(|x_j| = 0\), by (7), we obtain
\[
|\lambda - a_t| \leq r_j^\Theta(\mathcal{A}, a_t).
\]
Thus, \(\lambda \in \mathcal{U}_{ij}(\mathcal{A}, a) \subseteq \mathcal{U}(\mathcal{A}, a)\).

Otherwise, \(|x_j| > 0\). Multiplying (9) yields
\[
\left(|\lambda - a_t| - r_j^\Theta(\mathcal{A}, a_t)\right)|x_j||x_j| \leq r_j^\Theta(\mathcal{A}, a_t)R_j(\mathcal{A}, a_t)||x||x||x|,
\]
equivalently,
\[
\left(|\lambda - a_t| - r_j^\Theta(\mathcal{A}, a_t)\right)|\lambda - a_t||x||x| \leq r_j^\Theta(\mathcal{A}, a_t)R_j(\mathcal{A}, a_t),
\]
which implies \(\lambda \in \mathcal{U}_{ij}(\mathcal{A}, a)\). From the arbitrariness of \(j\), we have \(\lambda \in \bigcup_{i \in \mathbb{N}} \cap_{j \in \mathbb{N}, j \neq i} \mathcal{U}_{ij}(\mathcal{A}, a)\). Further, \(\sigma_Z(\mathcal{A}) \subseteq \bigcap_{a \in \mathbb{R}^n} \mathcal{U}(\mathcal{A}, a)\) by the arbitrariness of \(a\). \(\square\)

Corollary 2.2. Let \(\mathcal{A} = (a_{i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}\) with \(m\) being even. For any real vector \(\alpha = (a_1, \ldots, a_n)^T \in \mathbb{R}^n\), then \(\mathcal{U}(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)\).
Proof. For any \( \lambda \in \mathcal{U}(\mathcal{A}, a) \), without loss of generality, there exists \( t \in \mathbb{N} \) such that \( \lambda \in \mathcal{U}_{t,s}(\mathcal{A}) \), that is,

\[
(\lambda - a_{t}) - r_{t}^{\Theta}(\mathcal{A}, a_{s}) \leq r_{t}^{\Theta}(\mathcal{A}, a_{s})R_{t}(\mathcal{A}, a_{s}), \; \forall s \neq t.
\]  

(10)

Next, the following argument is divided into two cases.

Case I: \( r_{t}^{\Theta}(\mathcal{A}, a_{s})R_{t}(\mathcal{A}, a_{s}) = 0 \). Since \( |\lambda - a_{t}| \geq 0 \), from (10), we deduce \( |\lambda - a_{t}| - r_{t}^{\Theta}(\mathcal{A}, a_{s}) \leq 0 \). Further, it holds that

\[
|\lambda - a_{t}| \leq r_{t}^{\Theta}(\mathcal{A}, a_{s}) \leq R_{t}(\mathcal{A}, a_{s}),
\]

i.e., \( \lambda \in \mathcal{G}_{t}(\mathcal{A}, a) \). So, we have \( \mathcal{U}_{t,s}(\mathcal{A}, a) \subseteq \mathcal{G}_{t}(\mathcal{A}, a) \).

Case II: \( r_{t}^{\Theta}(\mathcal{A}, a_{s})R_{t}(\mathcal{A}, a_{s}) > 0 \). Then dividing both sides by \( r_{t}^{\Theta}(\mathcal{A}, a_{s})R_{t}(\mathcal{A}, a_{s}) \) in (10), we obtain

\[
\frac{|\lambda - a_{t}| - r_{t}^{\Theta}(\mathcal{A}, a_{s})}{r_{t}^{\Theta}(\mathcal{A}, a_{s})} \cdot \frac{|\lambda - a_{t}|}{R_{t}(\mathcal{A}, a_{s})} \leq 1,
\]

which implies

\[
\frac{|\lambda - a_{t}| - r_{t}^{\Theta}(\mathcal{A}, a_{s})}{r_{t}^{\Theta}(\mathcal{A}, a_{s})} \leq 1
\]

(12)

or

\[
\frac{|\lambda - a_{t}|}{R_{t}(\mathcal{A}, a_{s})} \leq 1.
\]

(13)

If (12) holds, then we have \( |\lambda - a_{t}| - r_{t}^{\Theta}(\mathcal{A}, a_{s}) \leq r_{t}^{\Theta}(\mathcal{A}, a_{s}) \), i.e,

\[
|\lambda - a_{t}| \leq r_{t}^{\Theta}(\mathcal{A}, a_{s}) + r_{t}^{\Theta}(\mathcal{A}, a_{s}) = R_{t}(\mathcal{A}, a_{s}).
\]

So, \( \lambda \in \mathcal{G}_{t}(\mathcal{A}, a) \). Otherwise, (13) holds, we can verify \( \lambda \in \mathcal{G}_{t}(\mathcal{A}, a) \).

From the above two cases, we can get \( \mathcal{U}_{t,s}(\mathcal{A}, a) \subseteq \mathcal{G}_{t}(\mathcal{A}, a) \cup \mathcal{G}_{s}(\mathcal{A}, a) \). Thus, \( \mathcal{U}(\mathcal{A}, a) \subseteq \mathcal{G}(\mathcal{A}, a) \) for a given parameter \( a \).

Next, we give a numerical comparison between Theorem 2.1 and Theorem 2 of [10].

**Example 2.3.** Consider \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{4 \times 2} \) defined by

\[
a_{ijkl} = \begin{cases} 
10; & a_{1111} = 10; a_{1122} = 9; a_{1211} = a_{2121} = -1; \\
5; & a_{2222} = 5; a_{2211} = 6; a_{2122} = a_{2221} = -1; \\
0, & a_{ijkl} = 0, \; \text{otherwise}.
\end{cases}
\]

All Z-eigenvalues of \( \mathcal{A} \) are 5.0000 and 10.0000. We choose different parameters \( a_{1} = [3, 8]^T, a_{2} = [10, 7]^T, a_{3} = [9, 5]^T \) and \( a_{4} = [9, 5, 5]^T \) respectively. Set \( a_{1} = [3, 8]^T \) and \( I_{Z} = (i_{ijkl}) \) as Case I of Definition 1.1

\[
i_{ijkl} = \begin{cases} 
1; & i_{1111} = i_{1122} = i_{2211} = i_{2222} = 1; \\
0, & \text{otherwise}.
\end{cases}
\]

Accordingly to Theorem 2.1, we obtain

\[
\mathcal{U}(\mathcal{A}, a_{1} = (3, 8)) = [-7.5917, 16.5498] \cup [-3.8102, 15.7178] = [-7.5917, 16.5498];
\]

Similarly, we can obtain the following table:

<table>
<thead>
<tr>
<th>( a )</th>
<th>( a = [3, 8]^T )</th>
<th>( a = [10, 7]^T )</th>
<th>( a = [9, 5]^T )</th>
<th>( a = [9, 5, 5]^T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{U}(\mathcal{A}, a) )</td>
<td>([-7.5917, 16.5498])</td>
<td>([-3.5949, 12.6533])</td>
<td>([-3.6277, 11])</td>
<td>([-3.6088, 10.6225])</td>
</tr>
<tr>
<td>( \mathcal{G}(\mathcal{A}, a) )</td>
<td>([-12, 18])</td>
<td>([2, 13])</td>
<td>([2, 12])</td>
<td>([2.5, 12])</td>
</tr>
</tbody>
</table>

Numerical results show that the bound of Theorem 2.1 is tighter than that of Theorem 2 of [10] and the suitable parameter \( a \) has a great influence on the numerical effect.
3. Positive definiteness of multivariate homogeneous forms

In this section, based on the inclusion set $\Omega(\mathcal{A}, \alpha)$ in Theorem 2.1, we propose a sufficient condition for the positive definiteness of even-order tensors. Before proceeding further, we introduce the results of [1, 10].

**Definition 3.1.** (i) We say that $\mathcal{A}$ is symmetric if

$$a_{i_1 \ldots i_m} = a_{i_{\pi(i_1) \ldots i_{\pi(m)}}}, \forall \pi \in \Gamma_m,$$

where $\Gamma_m$ is the permutation group of $m$ indices.

(ii) We say that $\mathcal{A}$ is weakly symmetric if the associated homogeneous polynomial $f_\mathcal{A}(x)$ satisfies

$$\nabla f_\mathcal{A}(x) = m\mathcal{A}x^{m-1}.$$ 

Obviously, if tensor $\mathcal{A}$ is symmetric, then $\mathcal{A}$ weakly symmetric. However, the converse result may not hold.

**Lemma 3.2.** (Theorem 3 of [10]) Let $\lambda$ be a Z-eigenvalue of $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_2 \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with $m$ being even. If there exists a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ such that

$$\alpha_j > R_j(\mathcal{A}, \alpha_i), \forall j \in N,$$

then $\lambda > 0$. Further, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_\mathcal{A}(x)$ defined in (1) is positive definite.

**Theorem 3.3.** Let $\lambda$ be a Z-eigenvalue of $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_2 \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with $m$ being even. For $i \in N$, if there exist a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ and $j \neq i$ such that

$$(\alpha_i - r_i^\Theta(\mathcal{A}, \alpha_i))\alpha_j > r_i^\Theta(\mathcal{A}, \alpha_i)R_j(\mathcal{A}, \alpha_j),$$

(14)

then $\lambda > 0$. Further, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_\mathcal{A}(x)$ defined in (1) is positive definite.

**Proof.** Suppose on the contrary that $\lambda \leq 0$. From Theorem 2.1, there exists $t \in N$ with $\lambda \in U_{t_i}(\mathcal{A}, \alpha_i)$, i.e.,

$$|\lambda - \alpha_i| - r_i^\Theta(\mathcal{A}, \alpha_i)|\lambda - \alpha_j| \leq r_i^\Theta(\mathcal{A}, \alpha_i)R_j(\mathcal{A}, \alpha_j), \forall j \neq t.$$

Further, it follows from $\alpha_i > 0$ and $\lambda \leq 0$ that

$$(\alpha_i - r_i^\Theta(\mathcal{A}, \alpha_i))\alpha_j \leq r_i^\Theta(\mathcal{A}, \alpha_i)R_j(\mathcal{A}, \alpha_j), \forall j \neq t,$$

which contradicts (14). Thus, $\lambda > 0$. When $\mathcal{A}$ is a symmetric tensor and all Z-eigenvalues are positive, $\mathcal{A}$ is positive definite and $f_\mathcal{A}(x)$ defined in (1) is positive definite. \(\square\)

The following example shows the validity of Theorem 3.3.

**Example 3.4.** Consider $f_\mathcal{A}(x) = \mathcal{A}x^m$ deduced by symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ as follows

$$a_{1111} = 1.4; a_{2222} = 3.2; a_{3333} = 2.6; a_{1112} = a_{1211} = a_{2112} = a_{2121} = a_{2211} = -0.1;$$
$$a_{1122} = a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2212} = a_{2221} = 0.8;$$
$$a_{1133} = a_{1313} = a_{1331} = a_{3113} = a_{3131} = a_{3311} = 1.1;$$
$$a_{1233} = a_{1323} = a_{2133} = a_{2313} = a_{2331} = a_{3213} = a_{3231} = a_{3321} = -0.1;$$
$$a_{1312} = a_{1322} = a_{1332} = a_{1213} = a_{1231} = a_{1312} = a_{1321} = a_{1332} = 0.1;$$
$$a_{2233} = a_{2323} = a_{2332} = a_{3223} = a_{3232} = a_{3322} = 0.1;$$
$$a_{2333} = a_{2332} = a_{3233} = a_{3332} = a_{3322} = 1.0; a_{ijkl} = 0, otherwise.$$
Taking $I_Z$ as Case II (Case I) of Definition 1.1, by simple computations, we cannot find positive real number $\alpha_1$ such that

$$\alpha_1 > R_3(\mathcal{A}, \alpha_1),$$

which shows that Theorem 3 of [10] cannot check the positive definiteness of $\mathcal{A}$ and $f_\mathcal{A}(x)$.

Set $\alpha = (2.85, 3.0, 2.7)$ and let $I_Z = (i_{ij})$ be Case II of Definition 1.1

$$I_{ij} = \begin{cases} 
I_{1111} = I_{2222} = I_{3333} = 1; \\
I_{1122} = I_{1212} = I_{1133} = I_{1313} = I_{3113} = I_{3333} = \frac{1}{3}; \\
I_{2132} = I_{2221} = I_{3223} = I_{2323} = I_{3323} = \frac{1}{3}; \\
0, \text{ otherwise.}
\end{cases}$$

From Theorem 3.3, we can calculate the following corresponding values

<table>
<thead>
<tr>
<th>$i = 1, j = 2$</th>
<th>$\frac{\alpha_i - r_f^\Theta(\mathcal{A}, \alpha_i))}{\alpha_j}$</th>
<th>$r_f^\Theta(\mathcal{A}, \alpha_i)R_f(\mathcal{A}, \alpha_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1, j = 3$</td>
<td>2.85</td>
<td>1.575</td>
</tr>
<tr>
<td>$i = 2, j = 1$</td>
<td>1.755</td>
<td>1.275</td>
</tr>
<tr>
<td>$i = 2, j = 3$</td>
<td>4.56</td>
<td>2.065</td>
</tr>
<tr>
<td>$i = 3, j = 1$</td>
<td>6.21</td>
<td>2.55</td>
</tr>
<tr>
<td>$i = 3, j = 2$</td>
<td>6.27</td>
<td>3.54</td>
</tr>
</tbody>
</table>

From the above table, we verify

$$(\alpha_i - r_f^\Theta(\mathcal{A}, \alpha_i))\alpha_j > r_f^\Theta(\mathcal{A}, \alpha_i)R_f(\mathcal{A}, \alpha_i), \forall i \neq j \in N,$$

which implies that $\mathcal{A}$ is positive definite and $f_\mathcal{A}(x)$ is positive definite.

4. Estimations of Z-spectral radius and convergence rate on the greedy rank-one algorithms

As we know, the best rank-one approximation which has numerous applications in wireless communication systems, image processing, data analysis [7, 15–17, 21]. The best rank-one approximation of $\mathcal{A} = (d_{i_1 i_2 ... i_n})$ is to find a rank-one tensor $kx^m = (x_{i_1} x_{i_2} ... x_{i_n})$ such that

$$\min_{x \in \mathbb{R}^n} \|\mathcal{A} - kx^m\|_f : x^T x = 1,$$

where $\|\mathcal{A}\|_f := \sqrt{\sum_{i_1, i_2, ..., i_n \in N} a_{i_1 i_2 ... i_n}^2}$. When $\mathcal{A}$ is nonnegative and weakly symmetric, $\rho(\mathcal{A})x_0^m$ is a best rank-one approximation of $\mathcal{A}$, i.e.,

$$\min_{x \in \mathbb{R}^n, x^T x = 1} \|\mathcal{A} - kx^m\|_f = \|\mathcal{A} - \rho(\mathcal{A})x_0^m\|_f = \sqrt{\|\mathcal{A}\|_f^2 - \rho(\mathcal{A})^2}.$$

Further, Qi [17] defined the quotient on the residual of the best rank-one approximation of tensor $\mathcal{A}$ as follows:

$$\omega = \frac{\|\mathcal{A} - \rho(\mathcal{A})x_0^m\|_f}{\|\mathcal{A}\|_f} = \sqrt{1 - \frac{\rho(\mathcal{A})^2}{\|\mathcal{A}\|_f^2}},$$

which can estimate the convergence rate of the greedy rank-one algorithm [2, 17, 18, 25]. Hence, we shall devote to finding sharp upper bounds of the Z-spectral radius of weakly symmetric nonnegative tensors to estimate the convergence rate of the greedy rank-one algorithms. We recall some fundamental results of nonnegative tensors [1].
Thus, the conclusion holds.

\[ \Lambda \]

where

Solving for (16), we obtain

Since

Lemma 4.2. (Corollary 4.10 of [1]) Assume \( \mathcal{A} \) is a weakly symmetric nonnegative tensor. Then,

\[ \rho(\mathcal{A}) \geq \max_{i \in \mathbb{N}} a_{i,i} \]

Theorem 4.3. Let \( \mathcal{A} = (a_{i_1, i_2, \ldots, i_n}) \in \mathbb{R}^{[m,n]} \) be a weakly symmetric nonnegative tensor and \( I_{\mathbb{Z}} \in \mathbb{R}^{[m,n]} \) be a Z-identity tensor (Case I or Case II) with \( m \) being even. For real vector \( \alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n \) with \( \alpha_i \leq \max_{i \in \mathbb{N}} a_{i,i} \), then

\[ \rho(\mathcal{A}) \leq \max_{i \in \mathbb{N}} \left\{ \min_{p \in \mathbb{N}, t \in \mathbb{R}^n} \frac{1}{2} (\alpha_j + \alpha_i + r_i^\Theta (\mathcal{A}, a_i)) + \Lambda_{i,j}(\mathcal{A}, a_i) \right\} \]

where \( \Lambda_{i,j}(\mathcal{A}) = (\alpha_i - \alpha_j + r_i^\Theta (\mathcal{A}, a_i))^2 + 4r_i^\Theta (\mathcal{A}, a_i)R_i(\mathcal{A}, a_j) \).

Proof. From Lemma 4.1, we assume that \( \rho(\mathcal{A}) = \lambda^* \) is the largest Z-eigenvalue. It follows from Theorem 2.1 that there exists \( t \in \mathbb{N} \) such that

\[ (|\rho(\mathcal{A}) - \alpha_i| - r_i^\Theta (\mathcal{A}, a_i))|\rho(\mathcal{A}) - \alpha_i| \leq r_i^\Theta (\mathcal{A}, a_i)R_i(\mathcal{A}, a_j), \forall j \neq t. \] \tag{15}

Since \( \mathcal{A} \) is nonnegative and Lemma 4.2 holds, for \( \alpha_i \leq \max_{i \in \mathbb{N}} a_{i,i} \), we have

\[ \rho(\mathcal{A}) \geq \alpha_i \text{ and } \rho(\mathcal{A}) \geq \alpha_j. \]

Thus, (15) is equivalent to

\[ (\rho(\mathcal{A}) - \alpha_i - r_i^\Theta (\mathcal{A}, a_i))(\rho(\mathcal{A}) - \alpha_j) \leq r_i^\Theta (\mathcal{A}, a_i)R_i(\mathcal{A}, a_j), \forall j \neq t. \] \tag{16}

Solving for (16), we obtain

\[ \rho(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{i \in \mathbb{N}} \left\{ \min_{p \in \mathbb{N}, t \in \mathbb{R}^n} \frac{1}{2} (\alpha_j + \alpha_i + r_i^\Theta (\mathcal{A}, a_i)) + \Lambda_{i,j}(\mathcal{A}, a_i) \right\} \]

\[ \text{and} \]

Consequently,

\[ \rho(\mathcal{A}) \leq \max_{i \in \mathbb{N}} \left\{ \min_{p \in \mathbb{N}, t \in \mathbb{R}^n} \frac{1}{2} (\alpha_j + \alpha_i + r_i^\Theta (\mathcal{A}, a_i)) + \Lambda_{i,j}(\mathcal{A}, a_i) \right\} \]

Thus, the conclusion holds. \( \Box \)

The following numerical experiment shows validity of Theorem 4.3 and gives an estimation for the convergence rate of the greedy rank-one algorithms.

Example 4.4. Consider tensor \( \mathcal{A} = (a_{i,j,k}) \in \mathbb{R}^{[1,2]} \) defined by

\[
\begin{align*}
a_{i,j,k} = \begin{cases} 
  a_{111} = 1; a_{222} = 3; a_{112} = a_{121} = a_{212} = a_{212} = a_{121} = a_{221} = 1; \\
a_{112} = a_{121} = a_{212} = a_{211} = 1; a_{j,k} = 0, \text{ otherwise.}
\end{cases}
\end{align*}
\]

By simple computation, we obtain \( (\rho(\mathcal{A}), \lambda) = (3, (0, 1)) \) and \( \|\mathcal{A}\|_F = 3.3166 \). For this tensor, set \( \alpha = (1, 1) \) and let \( I_{\mathbb{Z}} = (I_{\mathbb{Z}}) \) be Case II of Definition 1.1. The bounds via different estimations given in the literature are shown in the following table:
From the table above, it is easy to see that only the upper bound obtained by Theorem 4.1 is smaller than \|A\|_F. Consequently, we have

$$\min_{\kappa \in \mathbb{R}^n, \kappa \neq 0} \|A - \kappa x_m\|_F = \sqrt{\|A\|^2_F - \rho(A)^2} \geq 1.3559.$$ 

Further, we obtain that the quotient on the residual of the best rank-one approximation of \(A\) is

$$\omega = \frac{\|A - \rho(A) x_m\|_F}{\|A\|_F} = \sqrt{1 - \frac{\rho(A)^2}{\|A\|^2_F}} \geq 0.3511,$$

which implies the convergence rate of the greedy rank-one algorithm [2, 17, 18, 24, 25].

5. Conclusions

In this paper, we established a Brauer-type \(Z\)-eigenvalue inclusion set for even-order real tensors by \(Z\)-identity tensor and proposed some sufficient conditions for the positive definiteness of multivariate homogeneous forms. Note that the suitable parameter \(\alpha\) has a great influence on the numerical effects and positive definiteness of \(f_A(\alpha)\). Therefore, how to select the suitable parameter \(\alpha\) is our further research.

Competing Interests

The authors declare that they have no competing interests.

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