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Perturbation of the Spectra of Complex Symmetric Operators

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Abstract. An operator T on a complex Hilbert space $\mathcal{H}$ is called complex symmetric if T has a symmetric matrix representation relative to some orthonormal basis for $\mathcal{H}$. This paper focuses on the perturbation theory for the spectra of complex symmetric operators. We prove that each complex symmetric operator on a complex separable Hilbert space has a small compact perturbation being complex symmetric and having the single-valued extension property. Also it is proved that each complex symmetric operator on a complex separable Hilbert space has a small compact perturbation being complex symmetric and satisfying generalized Weyl's theorem.

1. Introduction

Throughout this paper, $\mathcal{H}$ will always denote a complex separable infinite dimensional Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$. We let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ so that $CTC = T^*$. Recall that a conjugate-linear map C on $\mathcal{H}$ is called a *conjugation* if C is invertible with $C^{-1} = C$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. The term "complex symmetric" stems from the fact that an operator $T \in \mathcal{B}(\mathcal{H})$ is *complex symmetric* if and only if there exists an orthonormal basis $\{e_n\}$ with respect to which T admits a complex symmetric matrix representation, that is,

$$\langle Te_i, e_j \rangle = \langle Te_j, e_i \rangle \quad \text{for all} \quad i, j.$$

We denote by $\mathcal{S}(\mathcal{H})$ the set of all complex symmetric operators on $\mathcal{H}$.

The general study of complex symmetric operators was initiated by Garcia, Putinar and Wogen in [13, 14], and has many motivations in function theory, matrix analysis and other areas. Normal operators, Hankel operators, binormal operators and truncated Toeplitz operators are important examples of complex symmetric operators. For more results concerning complex symmetric operators, the reader is referred to [15, 16, 20, 25, 29, 32].

The aim of the present paper is to explore the perturbation theory for the spectra of complex symmetric operators. This is inspired by a recent paper by S. Zhu [30], in which Weyl's theorem for complex symmetric operators was studied.

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1.1. Generalized Weyl's theorem

For $T \in \mathcal{B}(\mathcal{H})$, we let $\sigma(T)$ denote the spectrum of T. The *Weyl spectrum* of T is the set

$$\sigma_w(T) = \cap\{\sigma(T+K) : K \in \mathcal{K}(\mathcal{H})\},$$

where $\mathcal{K}(\mathcal{H})$ denotes the ideal of compact operators in $\mathcal{B}(\mathcal{H})$. We denote by $\sigma_p(T)$ the point spectrum of T. Denote by $\ker T$ and $\operatorname{ran} T$ the kernel of T and the range of T respectively. T is called a *semi-Fredholm* operator, if $\operatorname{ran} T$ is closed and either $\operatorname{nul} T$ or $\operatorname{nul} T^*$ is finite, where $\operatorname{nul} T := \dim\ker T$ and $\operatorname{nul} T^* := \dim\ker T^*$; in this case, $\operatorname{ind} T := \operatorname{nul} T - \operatorname{nul} T^*$ is called the *index* of T. In particular, if $-\infty < \operatorname{ind} T < \infty$, then T is called a *Fredholm operator*. It is well known that if T is semi-Fredholm and $K \in \mathcal{K}(\mathcal{H})$, then $T+K$ is also semi-Fredholm and $\operatorname{ind}(T+K) = \operatorname{ind} T$.

Given a subset σ of $\mathbb{C}$, denote by $\operatorname{iso}\sigma$ the set of all isolated points of σ. For $T \in \mathcal{B}(\mathcal{H})$, we denote

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \dim\ker(\lambda - T) < \infty\}.$$

If $A \in \mathcal{B}(\mathcal{H})$ is normal, a theorem of H. Weyl [27] states that $\sigma_w(A)$ consists of all spectral points except isolated eigenvalues of finite multiplicity, that is, $\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A)$. Coburn [7] proved that Weyl's theorem holds for two classes of nonnormal operators, the hyponormal operators and the Toeplitz operators. Inspired by the results, many works are devoted to the study of Weyl's theorem for more classes of operators, such as [2, 9, 10, 17]. In particular, it is proved in [23] that Weyl's theorem holds for operators in a dense subset of $\mathcal{B}(\mathcal{H})$. In a recent paper [30], S. Zhu proved that Weyl's theorem holds for operators in a dense subset of the set of complex symmetric operators on $\mathcal{H}$. Inspired by these results, we wish to study a variant of Weyl's theorem for complex symmetric operators.

For $T \in \mathcal{B}(\mathcal{H})$ and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $\operatorname{ran} T^n$. If for some n the range space $\operatorname{ran} T^n$ is closed and $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm operator*. In this case, from [5, Proposition 2.1], $T_{[m]}$ is Fredholm and $\operatorname{ind}(T_{[m]}) = \operatorname{ind}(T_{[n]})$ for all $m \geq n$. This enables us to define the index of a B-Fredholm operator T as the index of the Fredholm operator $T_{[n]}$, where n is any nonnegative integer such that $\operatorname{ran} T^n$ is closed and $T_{[n]}$ is Fredholm. T is called a *B-Weyl operator* if it is a B-Fredholm operator of index 0. The *B-Weyl spectrum* of T, denoted by $\sigma_{BW}(T)$, is defined as $\{\lambda \in \mathbb{C} : T - \lambda \text{ is not B-Weyl}\}$. For more details, the reader is referred to [5].

Following Berkani and Koliha [4], we say that generalized Weyl's theorem holds for $T \in \mathcal{B}(\mathcal{H})$, denoted by $T \in$ (gW), if there is the equality

$$\sigma_{BW}(T) = \sigma(T) \setminus E(T),$$

where $E(T) := \sigma_p(T) \cap \operatorname{iso}\sigma(T)$. Operators satisfying generalized Weyl's theorem always satisfy Weyl's theorem (see [4]).

The first result of this paper is the following theorem, which strengthens S. Zhu's result in [30].

Theorem 1.1. *Given a complex symmetric operator T on $\mathcal{H}$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that (a) $T + K \in \mathcal{S}(\mathcal{H})$, and (b) $T + K \in$ (gW).*

Here we provide an example of complex symmetric operator T for which Weyl's theorem holds and $T \notin$ (gW).

Example 1.2. *Let V be the classical Volterra integration operator on $\mathcal{H} := L^2([0,1])$ defined by*

$$(Vf)(t) = \int_0^t f(s)\mathrm{d}s, \quad \forall t \in [0,1].$$

Define a conjugation J on $L^2([0,1])$ as $(Jf)(t) = \overline{f(1-t)}$, $\forall t \in [0,1]$. Then one can check that $JVJ = V^*$ and V is complex symmetric. It is well known that $\sigma(V) = \{0\}$ and $\sigma_p(V) = \emptyset$. Set

$$T = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}.$$

Clearly, V is complex symmetric. Also it is easy to check that $\sigma(T) \setminus \sigma_w(T) = \emptyset = \pi_{00}(T)$. Thus Weyl's theorem holds for T. On the other hand, by [3, Thm. 4.2], we have $\sigma_{BW}(T) = \{0\}$. It follows that

$$\sigma(T) \setminus \sigma_{BW}(T) = \emptyset \neq \{0\} = E(T).$$

The above example shows that Theorem 1.1 strengthens S. Zhu's result in [30].

1.2. The single-valued extension property

The other aim of the present paper focuses on the single-valued extension property (SVEP, for short) of complex symmetric operators.

Recall that an operator T on a complex Banach space $\mathcal{X}$ is said to have the *single-valued extension property*, denoted by $T \in (\textsc{svep})$, if, for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f(\cdot) : U \to \mathcal{X}$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U. Here $\mathbb{C}$ denotes the set of complex numbers. The SVEP was introduced by N. Dunford in the study of spectral operators (see [11]).

The notion of SVEP plays a key role in the local spectra theory. In fact, given an operator T on $\mathcal{X}$ and a vector $x \in \mathcal{X}$, people are often interested in the existence and the uniqueness of analytic solution $f(\cdot) : U \to \mathcal{X}$ of the local resolvent equation

$$(T - \lambda)f(\lambda) = x$$

on suitable open subset U of $\mathbb{C}$. Obviously, if T has SVEP, then the existence of analytic solution to any local resolvent equation(related to T) implies the uniqueness of its analytic solution.

We notice that Jung, Ko and Lee [22] provided a sufficient condition for a complex symmetric operator to have SVEP. There are some other works devoted to the stability of SVEP (see [1, 6, 31]). It was proved in [31] that each operator in $\mathcal{B}(\mathcal{H})$ has a compact perturbation having SVEP.

The second result of this paper is the following theorem.

Theorem 1.3. *Given $T \in \mathcal{S}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in \mathcal{S}(\mathcal{H})$ and $T + K \in (\textsc{svep})$.*

Thus Theorem 1.3 is an analogue of the result of [31] in the setting of complex symmetric operators.

Also it is natural for one to ask whether $\mathcal{S}(\mathcal{H})$ has a dense subclass of operators having no SVEP. The following result provides a positive answer.

Theorem 1.4. *Given $T \in \mathcal{S}(\mathcal{H})$ and $\varepsilon > 0$, there exists $E \in \mathcal{B}(\mathcal{H})$ with $\|E\| < \varepsilon$ such that $T + E \in \mathcal{S}(\mathcal{H})$ and $T + E \notin (\textsc{svep})$.*

In view of the result stated in Theorem 1.3, one may ask whether it can be required in addition that the operator E in Theorem 1.4 satisfies $E \in \mathcal{K}(\mathcal{H})$. The answer is negative. In fact, if $T = 0$, then T is complex symmetric and, by [31, Theorem 1.3], $T + K \in (\textsc{svep})$ for all $K \in \mathcal{K}(\mathcal{H})$. So the result of Theorem 1.4 is sharp.

The proof of Theorem 1.1 will be provided in Section 2. And Section 3 is devoted to the proofs of Theorems 1.3 and 1.4.

2. Proof of Theorem 1.1

The aim of this section is to give the proof of Theorem 1.1.

2.1. Preparation

In this subsection we make some preparation.

Throughout this paper, $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of natural numbers respectively.

Let $T \in \mathcal{B}(\mathcal{H})$. The *Wolf spectrum* $\sigma_{lre}(T)$ and the *essential spectrum* $\sigma_e(T)$ of T are defined as

$$\sigma_{lre}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}$$

and

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

respectively. The set $\rho_{s-F}(T) := \mathbb{C} \setminus \sigma_{lre}(T)$ is called the *semi-Fredholm domain* of T.

Let $T \in \mathcal{B}(\mathcal{H})$. If σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$. We let $E(\sigma; T)$ denote the *Riesz idempotent* of T corresponding to σ, that is,

$$E(\sigma; T) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda - T)^{-1} \mathrm{d}\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we denote $\mathcal{H}(\sigma; T) = \operatorname{ran} E(\sigma; T)$. If $\lambda \in \operatorname{iso}\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$ and we simply write $\mathcal{H}(\lambda; T)$ instead of $\mathcal{H}(\{\lambda\}; T)$; if, in addition, $\dim \mathcal{H}(\lambda; T) < \infty$, then λ is called a *normal eigenvalue* of T. A normal eigenvalue of T is also called a *Riesz point* of T. The set of all normal eigenvalues of T will be denoted by $\sigma_0(T)$.

We denote

$$\rho^0_{s-F}(T) := \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(T - \lambda) = 0\},$$

$$\rho^+_{s-F}(T) := \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(T - \lambda) > 0\}$$

and

$$\rho^-_{s-F}(T) := \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(T - \lambda) < 0\}.$$

Obviously, $\rho_{s-F}(T) = \rho^-_{s-F}(T) \cup \rho^0_{s-F}(T) \cup \rho^+_{s-F}(T)$.

Lemma 2.1 ([28, Cor. 3.2]). *Let $T \in \mathcal{B}(\mathcal{H})$. If $[\sigma(T) \setminus \sigma_w(T)] \subset \sigma_0(T)$ and $E(T) \subset \sigma_0(T)$, then $T \in$ (gW).*

Lemma 2.2 ([21, Lemma 3.2.6]). *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\emptyset \neq \Gamma \subseteq \sigma_{lre}(T)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that*

$$T + K = \begin{bmatrix} N & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where N is a diagonal normal operator of uniformly infinite multiplicity, $\sigma(N) = \sigma_{lre}(N) = \overline{\Gamma}$, $\sigma(T) = \sigma(A), \sigma_{lre}(T) = \sigma_{lre}(A)$ and $\operatorname{ind}(T - \lambda) = \operatorname{ind}(A - \lambda)$ for all $\lambda \in \rho_{s-F}(T)$.

Now we prove a key lemma.

Lemma 2.3. *Let $R \in \mathcal{B}(\mathcal{H})$ and assume that $\sigma(R) = \sigma_{lre}(R)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ satisfying*

(i) $\operatorname{iso}\sigma(R + K) \subset \sigma_0(R + K)$, *and*
(ii) $\sigma(R + K) = \sigma_{lre}(R + K) \cup \sigma_0(R + K)$.

Proof. Without loss of generality, we assume that $\operatorname{iso}\sigma(R) \neq \emptyset$ and $\operatorname{iso}\sigma(R) = \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$. It is clear that $\{\lambda_1, \lambda_2, \lambda_3, \cdots\} \subset \sigma_{lre}(R)$.

By Lemma 2.2, for given $\varepsilon > 0$, there exists a compact operator K_1 with $\|K_1\| < \varepsilon/2$ such that

$$R + K_1 = \begin{bmatrix} \lambda_1 I_1 & & & * \\ & \lambda_2 I_2 & & * \\ & & \ddots & \vdots \\ & & & A \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_0 \end{matrix},$$

where $\mathcal{H} = \oplus_{i=0}^{\infty} \mathcal{H}_i$, $\dim \mathcal{H}_i = \infty$, I_i is the identity on $\mathcal{H}_i$ $(i \geq 0)$, $\sigma(R) = \sigma(A)$, $\sigma_{lre}(R) = \sigma_{lre}(A)$ and $\operatorname{ind}(R - \lambda) = \operatorname{ind}(A - \lambda)$ for all $\lambda \in \rho_{s-F}(R)$.

Since each λ_i is an isolated point of $\sigma(R)$, we can find distinct $\{\lambda_{i,j} : i, j \geq 1\} \subset \mathbb{C} \setminus \sigma(R)$ such that

$$|\lambda_{i,j} - \lambda_i| < \frac{\varepsilon}{2^{i+j+2}}.$$

For each $i \geq 1$, assume that $\{e_{i,j}\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_i$. Define $K_{2,i} \in \mathcal{B}(\mathcal{H}_i)$ as

$$K_{2,i} = \sum_{j=1}^{\infty} (\lambda_{i,j} - \lambda_i) e_{i,j} \otimes e_{i,j}$$

Then there exists a compact operator K_2 with $\|K_2\| < \varepsilon/2$ such that

$$R + K_1 + K_2 = \begin{bmatrix} \lambda_1 I_1 + K_{2,1} & & & * \\ & \lambda_2 I_2 + K_{2,2} & & * \\ & & \ddots & \vdots \\ & & & A \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_0 \end{matrix}.$$

Set $K = K_1 + K_2$. Then K is compact with $\|K\| < \varepsilon/2$.

Now it remains to check that $R + K$ satisfies statements (i) and (ii).

Note that $\{\lambda_i : i \geq 1\} = \operatorname{iso}\sigma(R) = \operatorname{iso}\sigma(A)$, $\{\lambda_{i,j} : i, j \geq 1\} \cap \sigma(A) = \emptyset$ and

$$\{\lambda_i : i, j \geq 1\} \subset \{\lambda_{i,j} : i, j \geq 1\}^- \subset \{\lambda_{i,j} : i, j \geq 1\} \cup \sigma(A).$$

It follows that

$$\sigma(R + K) = \sigma(R) \cup \{\lambda_{i,j} : i, j \geq 1\} = \sigma(A) \cup \{\lambda_{i,j} : i, j \geq 1\}.$$

Note that $\{\lambda_{i,j} : i, j \geq 1\}$ are pairwise distinct. It follows that $\{\lambda_{i,j} : i, j \geq 1\} \subset \sigma_0(R + K)$. Since $\sigma(A) = \sigma(R) = \sigma_{lre}(R) = \sigma_{lre}(A) = \sigma_{lre}(R + K)$, it follows that $\sigma_0(R + K) = \{\lambda_{i,j} : i, j \geq 1\}$. Thus statement (ii) holds.

On the other hand, if $z \in \operatorname{iso}\sigma(R + K)$, then either $z \in \operatorname{iso}\sigma_0(R + K)$ or $z \in \operatorname{iso}\sigma_{lre}(R + K)$. In the former case, we are done. In the latter case, we have $z \in \operatorname{iso}\sigma_{lre}(R) = \operatorname{iso}\sigma(R)$. Hence $z = \lambda_i$ for some i. Note that $\lambda_{i,j} \in \sigma(R + K)$ and $\lambda_{i,j} \to \lambda_i$ as $j \to \infty$. So λ_i is not an isolated point of $\sigma(R + K)$, a contradiction. This proves statement (i). □

2.2. *Proof of Theorem 1.1*

We first introduce some useful lemmas.

Lemma 2.4 ([30, Prop. 2.7]). *If $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric, then, given $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ is complex symmetric and $\sigma(T + K) = \sigma_{lre}(T + K) \cup \sigma_0(T + K)$.*

Recall that two operators $A_i \in \mathcal{B}(\mathcal{H}_i)(i = 1, 2)$ are *approximately unitarily equivalent*, denoted as $A_1 \cong_a A_2$, if there exist unitary operators $U_n : \mathcal{H}_1 \to \mathcal{H}_2$ $(n \geq 1)$ such that $U_n A_1 U_n^* \to A_2$ as $n \to \infty$. By a consequence of Voiculescu's Theorem (see [26] or [8, Theorem 41.12]), if $A_1 \cong_a A_2$, then, given $\varepsilon > 0$, there exists compact K with $\|K\| < \varepsilon$ such that $A_1 + K$ and A_2 are unitarily equivalent.

Lemma 2.5 ([24, Cor. 3.4]). *If $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric, then there exists a complex symmetric operator R satisfying*

(i) $T \cong_a T \oplus R \oplus R$, *and*
(ii) $\sigma(R) = \sigma_{lre}(R) = \sigma_{lre}(T)$.

Now we are going to prove Theorem 1.1.

Proof. [Proof of Theorem 1.1] In view of Lemma 2.4, we may directly assume that $\sigma(T) = \sigma_{lre}(T) \cup \sigma_0(T)$.
By Lemma 2.5, there exists a complex symmetric operator $R \in \mathcal{B}(\mathcal{H})$ satisfying

(i) $T \cong_a R \oplus T \oplus R$, and
(ii) $\sigma(R) = \sigma_{lre}(R) = \sigma_{lre}(T)$.

We denote $W = T \oplus R \oplus R$. Then it suffices to prove the result for W.
Assume that C, C_0 are two conjugations on $\mathcal{H}$ such that $C_0TC_0 = T^*$ and $CRC = R^*$. Write $W = R \oplus T \oplus R$ and set

$$D = \begin{bmatrix} 0 & 0 & C \\ 0 & C_0 & 0 \\ C & 0 & 0 \end{bmatrix}.$$

Then D is a conjugation on $\mathcal{H}^{(3)} := \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ and $DWD = W^*$.
Now fix an $\varepsilon > 0$. In view of Lemma 2.3, we can find a compact operator K_0 on $\mathcal{H}$ with $\|K_0\| < \varepsilon$ satisfying

(iii) $\operatorname{iso}\sigma(R + K_0) \subset \sigma_0(R + K_0)$, and
(iv) $\sigma(R + K_0) = \sigma_{lre}(R) \cup \sigma_0(R + K_0)$.

Define a compact operator K on $\mathcal{H}^{(3)}$ as

$$K = \begin{bmatrix} K_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & CK_0^*C \end{bmatrix}.$$

Then it is easy to see $\|K\| < \varepsilon$ and $DKD = K^*$. So $W + K$ is complex symmetric with respect to D and

$$W + K = \begin{bmatrix} R + K_0 & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & R + CK_0^*C \end{bmatrix}.$$

It remains to check that $W + K \in (\mathrm{gW})$.
The rest of the proof relies heavily on two claims.
Claim 1. $[\rho^-_{s-F}(W + K) \cup \rho^0_{s-F}(W + K)] \cap \sigma_p(W + K) = \sigma_0(W + K)$.
The inclusion "$\supset$" is obvious.
"$\subset$". Note that $D(W + K - z)D = (W + K - z)^*$ for $z \in \rho_{s-F}(W + K)$. Thus $\operatorname{ind}(W + K - z) = 0$ for all $z \in \rho_{s-F}(W+K)$, that is, $\rho_{s-F}(W+K) = \rho^0_{s-F}(W+K)$. Thus it suffices to prove $\rho^0_{s-F}(W+K)\cap\sigma_p(W+K) \subset \sigma_0(W+K)$.
Assume that $z \in \rho^0_{s-F}(W + K) \cap \sigma_p(W + K)$. In view of statements (ii), we have $\sigma_{lre}(W + K) = \sigma_{lre}(T) = \sigma_{lre}(R) = \sigma_{lre}(R + K_0)$. Thus, by (ii), (iv) and the hypothesis, we have $z \in \sigma_0(R + K_0) \cup \sigma_0(T)$. It follows immediately that $z \in \sigma_0(W + K)$.
Claim 2. $\operatorname{iso}\sigma(W + K) \subset \sigma_0(W + K)$.
Assume that $\lambda \in \operatorname{iso}\sigma(W + K)$. Note that $\sigma(W + K) = \sigma(R + K_0) \cup \sigma(T)$. Thus the proof is divided into two cses.
Case 1. $\lambda \in \sigma(R + K_0)$.
This means that $\lambda \in \operatorname{iso}\sigma(R + K_0)$. In view of statement (iii), we have $\lambda \in \sigma_0(R + K_0)$. Note that $\sigma_{lre}(T) = \sigma_{lre}(R) = \sigma_{lre}(R + K_0)$. This means that $\lambda \notin \sigma_{lre}(T)$. Recall that $\sigma(T) = \sigma_{lre}(T) \cup \sigma_0(T)$. Then either $\lambda \notin \sigma(T)$ or $\lambda \in \sigma_0(T)$. Each of them implies $\lambda \in \sigma_0(W + K)$.

Case 2. $\lambda \notin \sigma(R + K_0)$.

This implies that $\lambda \in \operatorname{iso}\sigma(T)$ and $\lambda \notin \sigma_{lre}(R + K_0) = \sigma_{lre}(T)$. Note that $\sigma(T) = \sigma_{lre}(T) \cup \sigma_0(T)$. We obtain $\lambda \in \sigma_0(T)$, which implies $\lambda \in \sigma_0(W + K)$. This proves Claim 2.

Now we shall show that $W + K \in (\mathrm{gW})$.

One can easily check that $\sigma(W + K) \setminus \sigma_w(W + K) = \rho^0_{s-F}(W + K) \cap \sigma_p(W + K)$. Thus Claim 1 implies that $[\sigma(W + K) \setminus \sigma_w(W + K)] \subset \sigma_0(W + K)$. On the other hand, it follows from Claim 2 that $E(W + K) \subset \sigma_0(W + K)$. In view of Lemma 2.1, we conclude that $W + K \in (\mathrm{gW})$. Therefore the proof is complete. □

3. Proofs of Theorems 1.3 and 1.4

Before we give the proofs of main results, we make some preparation.

Lemma 3.1 ([18, Theorem 6.1]). *Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) = \sigma_{lre}(T)$. Then, given $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $\sigma_p(T + K) = \sigma_p(T^* + K^*) = \emptyset$.*

Lemma 3.2 ([24, Theorem 3.3]). *If $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric, then there exists a complex symmetric operator R satisfying*

(i) $T \cong_a T \oplus R^{(\infty)}$, and
(ii) $\sigma(R) = \sigma_{lre}(R) = \sigma_{lre}(T)$.

Proof. [Proof of Theorem 1.3] By Lemma 2.4, we may directly assume that $\sigma(T) = \sigma_{lre}(T) \cup \sigma_0(T)$. So

$$\sigma_p(T) = \sigma_0(T) \cup [\sigma_p(T) \cap \sigma_{lre}(T)].$$

If $\sigma_p(T) \cap \sigma_{lre}(T)$ is finite or empty, then $\sigma_p(T)$ is at most denumerable, from which it follows readily that T satisfies SVEP. So, in the sequel, we assume that $\sigma_p(T) \cap \sigma_{lre}(T)$ is an infinite set.

Choose a countably infinite, dense subset $\{z_n : n \geq 1\}$ of $\sigma_p(T) \cap \sigma_{lre}(T)$. Noting that $\sigma_0(T) \subset \operatorname{iso}\sigma(T)$, it follows that $\sigma_p(T) \setminus \{z_n : n \geq 1\}$ has no interior point.

By Lemma 3.2, we can find a complex symmetric operator R satisfying

(i) $T \cong_a T \oplus R^{(\infty)}$, and
(ii) $\sigma(R) = \sigma_{lre}(R) = \sigma_{lre}(T)$.

It follows that $T \cong_a R^{(\infty)} \oplus T \oplus R^{(\infty)}$. So it suffices to prove that $W := R^{(\infty)} \oplus T \oplus R^{(\infty)}$ has a small compact perturbation being complex symmetric and satisfying SVEP.

We fix an $\varepsilon > 0$. Note that $\sigma(R) = \sigma_{lre}(R)$. By Lemma 3.1, we can find for each $n \geq 1$ an operator $K_n \in \mathcal{K}(\mathcal{H})$ with $\|K_n\| < \frac{\varepsilon}{4^n}$ such that

$$\sigma_p(R + K_n) = \sigma_p(R^* + K_n^*) = \emptyset.$$

For each $n \geq 1$, denote $R_n = R + K_n$. Thus $\sigma(R_n) = \sigma_{lre}(R_n)$, $n \geq 1$. Also we note that $\operatorname{ran}(R_n - z_n)$ is not closed, since $z_n \in \sigma_{lre}(T) = \sigma_{lre}(R_n)$ and $R_n - z_n$ is injective. Then, by [19, Lemma 2.1], there exists a subspace M_n of $\mathcal{H}$ with $\dim M_n = \infty$ such that $M_n \cap \operatorname{ran}(R_n - z_n) = \{0\}$.

For each $n \geq 1$, denote $\mathcal{K}_n = \ker(T - z_n)$. Since $\dim M_n = \infty$, we can find $E_n \in \mathcal{K}(\mathcal{H})$ with $\|E_n\| < \frac{\varepsilon}{4^n}$ such that $E_n(\mathcal{K}_n) \subset M_n$ and $\ker E_n \cap \mathcal{K}_n = \{0\}$. Set

$$V = \left[\begin{array}{cccc|c} R_1 & & & & E_1 \\ & R_2 & & & E_2 \\ & & R_3 & & E_3 \\ & & & \ddots & \vdots \\ \hline & & & & T \end{array}\right] \begin{array}{l} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \vdots \\ \mathcal{H}_0 \end{array},$$

where $\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathcal{H}$. Clearly, V is a compact perturbation of $R^{(\infty)} \oplus T$, since

$$V = \left[\begin{array}{cccc|c} R & & & & \\ & R & & & \\ & & R & & \\ & & & \ddots & \\ \hline & & & & T \end{array}\right] + \left[\begin{array}{cccc|c} K_1 & & & & E_1 \\ & K_2 & & & E_2 \\ & & K_3 & & E_3 \\ & & & \ddots & \vdots \\ \hline & & & & 0 \end{array}\right].$$

Claim. $\{z_n : n \geq 1\} \cap \sigma_p(V) = \emptyset$.

Fix an n. Assume that $x \in (\oplus_{i=1}^{\infty} \mathcal{H}_i) \oplus \mathcal{H}_0$ such that $(V - z_n)x = 0$. Thus there exists $x_i \in \mathcal{H}_i$ such that $x = (x_1, x_2, x_3, \cdots, x_0)^t$. Thus we have $(T - z_n)x_0 = 0$ and $(R_i - z_n)x_i + E_i x_0 = 0$ for $i \geq 1$. So $x_0 \in \mathcal{K}_n$. For each $i \geq 1$, since $E_i(\mathcal{K}_n) \cap \operatorname{ran}(R_i - z_n) = \{0\}$, we obtain $(R_i - z_n)x_i = E_i x_0 = 0$. Recall that $\ker E_n \cap \mathcal{K}_n = \{0\}$ and $\sigma_p(R_i) = \emptyset$. We deduce that $x_0 = 0$ and $x_i = 0$. Thus $x = 0$, which implies $z_n \notin \sigma_p(V)$. This proves Claim.

Denote

$$\widetilde{R} = \begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & R_3 & \\ & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3, \\ \vdots \end{matrix} \quad \widetilde{E} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ \vdots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3, \\ \vdots \end{matrix} \quad \widetilde{C} = \begin{bmatrix} C & & & \\ & C & & \\ & & C & \\ & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3. \\ \vdots \end{matrix}$$

Then $\widetilde{E} : \mathcal{H}_0 \to \oplus_{i=1}^{\infty} \mathcal{H}_i$ is a bounded linear operator, $\widetilde{C}$ is a conjugation on $\oplus_{i=1}^{\infty} \mathcal{H}_i$ and

$$V = \begin{bmatrix} \widetilde{R} & \widetilde{E} \\ 0 & T \end{bmatrix} \begin{matrix} \oplus_{i=1}^{\infty} \mathcal{H}_i \\ \mathcal{H}_0 \end{matrix}.$$

Set

$$W_0 = \begin{bmatrix} \widetilde{R} & \widetilde{E} & 0 \\ 0 & T & C_0 \widetilde{E}^* \widetilde{C} \\ 0 & 0 & \widetilde{C}\widetilde{R}^*\widetilde{C} \end{bmatrix} \begin{matrix} \oplus_{i=1}^{\infty} \mathcal{H}_i \\ \mathcal{H}_0 \\ \oplus_{i=1}^{\infty} \mathcal{H}_i \end{matrix}, \quad D = \begin{bmatrix} 0 & 0 & \widetilde{C} \\ 0 & C_0 & 0 \\ \widetilde{C} & 0 & 0 \end{bmatrix} \begin{matrix} \oplus_{i=1}^{\infty} \mathcal{H}_i \\ \mathcal{H}_0 \\ \oplus_{i=1}^{\infty} \mathcal{H}_i \end{matrix} \tag{1}$$

Note that D is a conjugation and a straightforward calculation shows that $DW_0D = W^*$. That is, W_0 is complex symmetric.

One can see that $\widetilde{E}$ is compact with $\|\widetilde{E}\| < \varepsilon/3$. Also we compute to see

$$\widetilde{R} - R^{(\infty)} = \oplus_{i=1}^{\infty} K_i, \quad \widetilde{C}\widetilde{R}^*\widetilde{C} - R^{(\infty)} = \oplus_{i=1}^{\infty} C K_i^* C.$$

Then

$$W_0 - W = W_0 - R^{(\infty)} \oplus T \oplus R^{(\infty)} = \begin{bmatrix} \oplus_{i=1}^{\infty} K_i & \widetilde{E} & 0 \\ 0 & 0 & C_0 \widetilde{E}^* \widetilde{C} \\ 0 & 0 & \oplus_{i=1}^{\infty} C K_i^* C \end{bmatrix} \begin{matrix} \oplus_{i=1}^{\infty} \mathcal{H}_i \\ \mathcal{H}_0 \\ \oplus_{i=1}^{\infty} \mathcal{H}_i \end{matrix}$$

is compact with norm less than ε, since $\oplus_{i=1}^{\infty} K_i$ is compact with norm less than $\varepsilon/4$. Now, since W_0 is complex symmetric, it remains to check that $W_0 \in$ (SVEP).

Note that

$$\sigma_p(\widetilde{R}) = \cup_{n \geq 1} \sigma_p(R_n) = \emptyset = \cup_{n \geq 1} \sigma_p(R_n^*) = \sigma_p(\widetilde{R}^*).$$

Thus $\sigma_p(\widetilde{C}\widetilde{R}^*\widetilde{C}) = \emptyset$. In view of (1), it follows that $\sigma_p(W_0) = \sigma_p(V) \subset \sigma_p(T)$. By the hypothesis $\sigma(T) = \sigma_0(T) \cup \sigma_{lre}(T)$, we have

$$\sigma_p(T) = \sigma_0(T) \cup [\sigma_p(T) \cap \sigma_{lre}(T)].$$

By Claim, each z_n does not lie in $\sigma_p(V)$. Thus

$$\sigma_p(W_0) \subset [\sigma_p(T) \setminus \{z_n : n \geq 1\}].$$

Noting that $\sigma_p(T) \setminus \{z_n : n \geq 1\}$ has no interior point, so does $\sigma_p(W_0)$. Hence we conclude that W_0 has SVEP. $\square$

The proof of Theorem 1.4 follows a similar line as that of [30, Theorem 1.5].

Proof. [Proof of Theorem 1.4] By the proof of [30, Theorem 1.5], for given $\varepsilon > 0$, we can find $z_0 \in \mathbb{C}$, $\delta > 0$, $E \in \mathcal{B}(\mathcal{H})$ with $\|E\| < \varepsilon$ and an infinite-dimensional invariant subspace M of $T + E$ such that $T + E$ is complex symmetric and

$$T + E = \begin{bmatrix} z_0 + \delta S^* & * \\ 0 & * \end{bmatrix} \begin{matrix} M \\ M^{\perp} \end{matrix},$$

where S is the backward unilateral shift of multiplicity one on M. Since $\rho^{+}_{s-F}(S^*) = \{z \in \mathbb{C} : |z| < 1\}$, it follows from [31, Theorem 1.1] or [12, Theorems 9/10] that $S^* \notin$ (SVEP). This implies $T + E \notin$ (SVEP). □

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