



General Viscosity Implicit Midpoint Rule For Nonexpansive Mapping

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Abstract. In this work, we suggest a general viscosity implicit midpoint rule for nonexpansive mapping in the framework of Hilbert space. Further, under the certain conditions imposed on the sequence of parameters, strong convergence theorem is proved by the sequence generated by the proposed iterative scheme, which, in addition, is the unique solution of the variational inequality problem. Furthermore, we provide some applications to variational inequalities, Fredholm integral equations, and nonlinear evolution equations and give a numerical example to justify the main result. The results presented in this work may be treated as an improvement, extension and refinement of some corresponding ones in the literature.

1. Introduction

Throughout the paper unless otherwise stated, H denotes a real Hilbert space, we denote the norm and inner product of H by $\|\cdot\|$, and $\langle \cdot, \cdot \rangle$ respectively. Let K be a nonempty, closed and convex subset of H . Let $\{x_n\}$ be any sequence in H , then $x_n \rightarrow x$ (respectively, $x_n \rightharpoonup x$) will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$.

A mapping $S : H \rightarrow H$ is said to be contraction mapping if there exists a constant $\alpha \in (0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha \|x - y\|,$$

for all $x, y \in H$. If $\alpha = 1$ then $S : H \rightarrow H$ is said to be nonexpansive mapping i.e., $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in H$. We use $\text{Fix}(S)$ to denote the set of fixed points of S . An operator $B : H \rightarrow H$ is said to be strongly positive bounded linear operator, if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Bx, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H.$$

The viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [1] in the framework of a Hilbert space, which generates the sequence $\{x_n\}$ by the following iterative scheme:

$$x_{n+1} = \alpha_n Q(x_n) + (1 - \alpha_n) Sx_n, \quad n \geq 0, \tag{1}$$

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where $\{\alpha_n\} \subset [0, 1]$ and Q is a contraction mapping on H . Note that the iterative scheme (1) generalize the results of Browder [2] and Halpern [3] in another direction. The convergence of the explicit iterative scheme (1) has been the subject of many authors because under suitable conditions these iteration converge strongly to the unique solution $q \in \text{Fix}(S)$ of the variational inequality

$$\langle (I - Q)q, x - q \rangle \geq 0, \quad \forall x \in \text{Fix}(S). \quad (2)$$

This fact allows us to apply this method to convex optimization, linear programming and monotone inclusions. In 2004, Xu [4] extended the result of Moudafi [1] to uniformly smooth Banach spaces and obtained strong convergence theorem. For related work, see [5–8].

In 2006, Marino and Xu [9] introduced the following iterative scheme based on viscosity approximation method, for fixed point problem for a nonexpansive mapping S on H :

$$x_{n+1} = \alpha_n \gamma Q(x_n) + (I - \alpha_n B)Sx_n, \quad n \geq 0, \quad (3)$$

where Q is a contraction mapping on H with constant $\alpha > 0$, B is a strongly positive self-adjoint bounded linear operator on H with constant $\bar{\gamma} > 0$ and $\gamma \in (0, \frac{\bar{\gamma}}{\alpha})$. They proved that the sequence $\{x_n\}$ generated by (3) converge strongly to the unique solution of the variational inequality

$$\langle (B - \gamma Q)z, x - z \rangle \geq 0, \quad \forall x \in \text{Fix}(S), \quad (4)$$

which is the optimality condition for the minimization problem

$$\min_{x \in \text{Fix}(S)} \frac{1}{2} \langle Bx, x \rangle - h(x),$$

where h is the potential function for γQ .

The implicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, we refer to [10–17] and the references cited therein. For instance, consider the initial value problem for the differential equation $y'(t) = f(y(t))$ with the initial condition $y(0) = y_0$, where f is a continuous function from R^d to R^d . The implicit midpoint rule in which generates a sequence $\{y_n\}$ by the following the recurrence relation

$$\frac{1}{h}(y_{n+1} - y_n) = f\left(\frac{y_{n+1} + y_n}{2}\right).$$

In 2014, implicit midpoint rule has been extended by Alghamdi *et al.* [18] to nonexpansive mappings, which generates a sequence $\{x_n\}$ by the following implicit iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \quad (5)$$

Recently, Xu *et al.* [19] extended and generalized the results of Alghamdi *et al.* [18] and presented the following viscosity implicit midpoint rule for nonexpansive mapping, which generates a sequence $\{x_n\}$ by the following implicit iterative scheme:

$$x_{n+1} = \alpha_n Q(x_n) + (1 - \alpha_n)S\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \quad (6)$$

where $\{\alpha_n\} \subset [0, 1]$ and S is a nonexpansive mapping. They proved that under some mild conditions, the sequence generated by (6) converge in norm to fixed point of nonexpansive mapping, which, in addition, solves the variational inequality (2). Further related work, see [20, 21].

Motivated by the work of Moudafi [1], Xu [4], Marino and Xu [9], Alghamdi *et al.* [18] and Xu *et al.* [19], and by the ongoing research in this direction, we suggest and analyze general viscosity implicit

midpoint iterative scheme for fixed point of nonexpansive mapping in real Hilbert space. Based on the general viscosity implicit midpoint iterative scheme, we prove the strong convergence theorems for a nonexpansive mapping. Further, some consequences from these theorems are also derived. Furthermore, we provide some applications to variational inequalities, Fredholm integral equations, and nonlinear evolution equations and also give a numerical example to justify the main result. The results and methods presented here extend and generalize the corresponding results and methods given in [1, 4, 9, 18, 19].

2. Preliminaries

We recall some concepts and results which are needed in sequel.

For every point $x \in H$, there exists a unique nearest point in K denoted by P_Kx such that

$$\|x - P_Kx\| \leq \|x - y\|, \quad \forall y \in K. \tag{7}$$

Remark 2.1. [22] It is well known that P_K is nonexpansive mapping and satisfies

$$\langle x - y, P_Kx - P_Ky \rangle \geq \|P_Kx - P_Ky\|^2, \quad \forall x, y \in H. \tag{8}$$

Moreover, P_Kx is characterized by the fact $P_Kx \in C$ and

$$\langle x - P_Kx, y - P_Kx \rangle \leq 0. \tag{9}$$

The following Lemma is the well known demiclosedness principles for nonexpansive mappings.

Lemma 2.2. [22, 23] Assume that S be a nonexpansive self mapping of a closed and convex subset K of a Hilbert space H . If S has a fixed point, then $I - S$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in K converging weakly to some $x \in K$ and the sequence $\{(I - S)x_n\}$ converges strongly to some y , it follows that $(I - S)x = y$.

Lemma 2.3. [22, 23] In real Hilbert space H , the following hold:

(i)
$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H; \tag{10}$$

(ii)
$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \tag{11}$$

for all $x, y \in H$ and $\lambda \in (0, 1)$.

Lemma 2.4. [9] Assume that B is a strongly positive self-adjoint bounded linear operator on a Hilbert space H with constant $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.5. [4]. Let $\{a_n\}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \beta_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in \mathbb{R} such that

- (i) $\sum_{n=1}^{\infty} \beta_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\beta_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. General Viscosity Implicit Midpoint Rule

In this section, we prove a strong convergence theorem based on the general viscosity implicit midpoint rule for fixed point of nonexpansive mapping.

Theorem 3.1. *Let H be a real Hilbert space and $B : H \rightarrow H$ be a strongly positive bounded linear operator with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$ and $Q : H \rightarrow H$ be a contraction mapping with constant $\alpha \in (0, 1)$. Let $S : H \rightarrow H$ be a nonexpansive mapping such that $\text{Fix}(S) \neq \emptyset$. Let the iterative sequence $\{x_n\}$ be generated by the following general viscosity implicit midpoint iterative schemes:*

$$x_{n+1} = \alpha_n \gamma Q(x_n) + (1 - \alpha_n B) S \left(\frac{x_n + x_{n+1}}{2} \right), \quad n \geq 0, \tag{12}$$

where $\{\alpha_n\}$ is the sequence in $(0, 1)$ and satisfying the following conditions

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ or $\lim_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then the sequence $\{x_n\}$ converge strongly to $z \in \text{Fix}(S)$, where $z = P_{\text{Fix}(S)} Q(z)$. In other words, which is also unique solution of variational inequality (4).

Proof. Note that from condition (i), we may assume without loss of generality that $\alpha_n \leq (1 - \beta_n) \|B\|^{-1}$ for all n . From Lemma 2.4, we know that if $0 < \rho \leq \|B\|^{-1}$, then $\|I - \rho B\| \leq 1 - \rho \bar{\gamma}$. We will assume that $\|I - B\| \leq 1 - \bar{\gamma}$.

Since B is strongly positive self-adjoint bounded linear operator on H , then

$$\|B\| = \sup\{|\langle Bu, u \rangle| : u \in H, \|u\| = 1\}.$$

Observe that

$$\begin{aligned} \langle (I - \alpha_n B)u, u \rangle &= 1 - \alpha_n \langle Bu, u \rangle \\ &\geq 1 - \alpha_n \|B\| \geq 0, \end{aligned}$$

which implies that $(I - \alpha_n B)$ is positive. It follows that

$$\begin{aligned} \|(I - \alpha_n B)\| &= \sup\{\langle (I - \alpha_n B)u, u \rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \alpha_n \langle Bu, u \rangle : u \in H, \|u\| = 1\} \\ &\leq 1 - \alpha_n \bar{\gamma}. \end{aligned}$$

Let $q = P_{\text{Fix}(S)}$. Since Q is a contraction mapping with constant $\alpha \in (0, 1)$. It follows that

$$\begin{aligned} \|q(I - B + \gamma Q)(x) - q(I - B + \gamma Q)(y)\| &\leq \|(I - B + \gamma Q)(x) - (I - B + \gamma Q)(y)\| \\ &\leq \|I - B\| \|x - y\| + \gamma \|Q(x) - Q(y)\| \\ &\leq (1 - \bar{\gamma}) \|x - y\| + \gamma \alpha \|x - y\| \\ &\leq (1 - (\bar{\gamma} - \gamma \alpha)) \|x - y\|, \end{aligned}$$

for all $x, y \in H$. Therefore, the mapping $q(I - B + \gamma Q)$ is a contraction mapping from H into itself. It follows from Banach contraction principle that there exists an element $z \in H$ such that $z = q(I - B + \gamma Q)z = P_{\text{Fix}(S)}(I - B + \gamma Q)(z)$.

Let $p \in \text{Fix}(S)$, we compute

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| \alpha_n \gamma Q(x_n) + (1 - \alpha_n B) S\left(\frac{x_n + x_{n+1}}{2}\right) - p \right\| \\ &\leq \alpha_n \|\gamma Q(x_n) - Bp\| + (1 - \alpha_n \bar{\gamma}) \left\| S\left(\frac{x_n + x_{n+1}}{2}\right) - p \right\| \\ &\leq \alpha_n [\|\gamma Q(x_n) - Q(p)\| + \|\gamma Q(p) - Bp\|] + (1 - \alpha_n \bar{\gamma}) \left\| \left(\frac{x_n + x_{n+1}}{2}\right) - p \right\| \\ &\leq \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma Q(p) - Bp\| + \frac{(1 - \alpha_n \bar{\gamma})}{2} (\|x_n - p\| + \|x_{n+1} - p\|), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{(1 + \alpha_n \bar{\gamma})}{2} \|x_{n+1} - p\| &\leq \left[\alpha_n \gamma \alpha + \frac{(1 - \alpha \bar{\gamma})}{2} \right] \|x_n - p\| + \alpha_n \|\gamma Q(p) - Bp\| \\ \|x_{n+1} - p\| &\leq \left[\frac{1 + 2(\gamma \alpha - \bar{\gamma}) \alpha_n}{1 + \alpha_n \bar{\gamma}} \right] \|x_n - p\| + \frac{2\alpha_n}{1 + \alpha_n \bar{\gamma}} \|\gamma Q(p) - Bp\| \\ &\leq \left[1 - \frac{2(\bar{\gamma} - \gamma \alpha) \alpha_n}{1 + \alpha_n \bar{\gamma}} \right] \|x_n - p\| + \frac{2\alpha_n}{1 + \alpha_n \bar{\gamma}} \|\gamma Q(p) - Bp\| \\ &\leq \left[1 - \frac{2(\bar{\gamma} - \gamma \alpha) \alpha_n}{1 + \alpha_n \bar{\gamma}} \right] \|x_n - p\| + \frac{2\alpha_n (\bar{\gamma} - \gamma \alpha)}{1 + \alpha_n \bar{\gamma}} \frac{\|\gamma Q(p) - Bp\|}{(\bar{\gamma} - \gamma \alpha)}. \end{aligned}$$

Consequently, we get

$$\|x_{n+1} - p\| \leq \max \left\{ \|x_n - p\|, \frac{\|\gamma Q(p) - Bp\|}{\bar{\gamma} - \gamma \alpha} \right\}.$$

Therefore by using induction, we obtain

$$\|x_{n+1} - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|\gamma Q(p) - Bp\|}{\bar{\gamma} - \gamma \alpha} \right\}. \tag{13}$$

Hence the sequence $\{x_n\}$ is bounded.

Next, we show that the sequence $\{x_n\}$ is asymptotically regular, i.e., $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. It follows from (12) that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \left\| \alpha_n \gamma Q(x_n) + (1 - \alpha_n B) S\left(\frac{x_n + x_{n+1}}{2}\right) \right. \\ &\quad \left. - \left[\alpha_{n-1} \gamma Q(x_{n-1}) + (1 - \alpha_{n-1} B) S\left(\frac{x_{n-1} + x_n}{2}\right) \right] \right\| \\ &= \left\| (1 - \alpha_n B) \left[S\left(\frac{x_n + x_{n+1}}{2}\right) - S\left(\frac{x_{n-1} + x_n}{2}\right) \right] \right. \\ &\quad \left. + (\alpha_{n-1} B - \alpha_n B) \left[S\left(\frac{x_{n-1} + x_n}{2}\right) - \gamma Q(x_{n-1}) \right] + \alpha_n (\gamma Q(x_n) - \gamma Q(x_{n-1})) \right\| \\ &\leq (1 - \alpha_n \bar{\gamma}) \left\| S\left(\frac{x_n + x_{n+1}}{2}\right) - S\left(\frac{x_{n-1} + x_n}{2}\right) \right\| + M |\alpha_{n-1} - \alpha_n| \\ &\quad + \alpha_n \gamma \|Q(x_n) - Q(x_{n-1})\| \\ &\leq \frac{(1 - \alpha_n \bar{\gamma})}{2} [\|x_{n+1} - x_n\| + \|x_n - x_{n-1}\|] + M |\alpha_{n-1} - \alpha_n| + \alpha_n \gamma \alpha \|x_n - x_{n-1}\|, \end{aligned}$$

where $M := \sup \left\{ S \left(\frac{x_n + x_{n+1}}{2} \right) + \gamma \|Q(x_n)\| : n \in \mathbb{N} \right\}$. It follows that

$$\begin{aligned} \frac{(1 + \alpha_n \bar{\gamma})}{2} \|x_{n+1} - x_n\| &\leq \frac{(1 - \alpha_n \bar{\gamma})}{2} \|x_n - x_{n-1}\| + M |\alpha_{n-1} - \alpha_n| + \alpha_n \gamma \alpha \|x_n - x_{n-1}\| \\ \|x_{n+1} - x_n\| &\leq \frac{1 + 2(\gamma \alpha - \bar{\gamma}) \alpha_n}{1 + \alpha_n \bar{\gamma}} \|x_n - x_{n-1}\| + \frac{2M}{1 + \alpha_n \bar{\gamma}} |\alpha_{n-1} - \alpha_n| \\ &\leq \left(1 - \frac{2(\gamma \alpha - \bar{\gamma}) \alpha_n}{1 + \alpha_n \bar{\gamma}} \right) \|x_n - x_{n-1}\| + \frac{2M}{1 + \alpha_n \bar{\gamma}} |\alpha_{n-1} - \alpha_n|. \end{aligned}$$

By using the conditions (i)-(iii) of Lemma 2.5, we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{14}$$

Next, we show that

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0.$$

We can write

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - x_{n+1}\| + \left\| x_{n+1} - S \left(\frac{x_n + x_{n+1}}{2} \right) \right\| \\ &\quad + \left\| S \left(\frac{x_n + x_{n+1}}{2} \right) - Sx_n \right\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \left\| \gamma Q(x_n) + (1 - \gamma B) S \left(\frac{x_n + x_{n+1}}{2} \right) \right\| + \frac{1}{2} \|x_{n+1} - x_n\| \\ &\leq \frac{3}{2} \|x_{n+1} - x_n\| + \alpha_n \left\| \gamma Q(x_n) - S \left(\frac{x_n + x_{n+1}}{2} \right) \right\| \\ &\leq \frac{3}{2} \|x_{n+1} - x_n\| + \alpha_n M. \end{aligned}$$

It follows from condition (i) and (14), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow \hat{x}$ say. Next, we claim that $\limsup_{n \rightarrow \infty} \langle Q(z) - z, x_n - z \rangle \leq 0$, where $z = P_{\text{Fix}(S)}(I - B + \gamma Q)z$. To show this inequality, we consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow \hat{x}$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (B - \gamma Q)z - z, x_n - z \rangle &= \limsup_{n \rightarrow \infty} \langle (B - \gamma Q)z - z, x_n - z \rangle \\ &= \limsup_{k \rightarrow \infty} \langle (B - \gamma Q)z - z, x_{n_k} - z \rangle \\ &= \langle (B - \gamma Q)z - z, \hat{x} - z \rangle \leq 0. \end{aligned} \tag{15}$$

Finally, we show that $x_n \rightarrow z$. It follows from Lemma 2.3 that

$$\begin{aligned}
 \|x_{n+1} - z\|^2 &= \left\| \alpha_n \gamma Q(x_n) + (I - \alpha_n B) S \left(\frac{x_n + x_{n+1}}{2} \right) - z \right\|^2 \\
 &= \left\| \alpha_n (\gamma Q(x_n) - Bz) + (I - \alpha_n B) S \left(\frac{x_n + x_{n+1}}{2} \right) - z \right\|^2 \\
 &\leq \left\| (I - \alpha_n B) S \left(\frac{x_n + x_{n+1}}{2} \right) - z \right\|^2 + 2\alpha_n \langle \gamma Q(x_n) - Bz, x_{n+1} - z \rangle \\
 &\leq (1 - \alpha_n \bar{\gamma})^2 \left\| S \left(\frac{x_n + x_{n+1}}{2} \right) - z \right\|^2 + 2\alpha_n \gamma \|Q(x_n) - Q(z)\| \|x_{n+1} - z\| \\
 &\quad + 2\alpha_n \langle \gamma Q(z) - Bz, x_{n+1} - z \rangle \\
 &\leq (1 - \alpha_n \bar{\gamma})^2 \left\| \frac{x_n + x_{n+1}}{2} - z \right\|^2 + 2\alpha_n \gamma \alpha \|x_n - z\| \|x_{n+1} - z\| \\
 &\quad + 2\alpha_n \langle \gamma Q(z) - Bz, x_{n+1} - z \rangle \\
 &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - z\|^2 + 2\alpha_n \gamma \alpha \|x_n - z\| \|x_{n+1} - z\| \\
 &\quad + 2\alpha_n \langle \gamma Q(z) - Bz, x_{n+1} - z \rangle \\
 &\leq (1 - \alpha_n \bar{\gamma})^2 \left[\frac{1}{2} \|x_n - z\|^2 + \frac{1}{2} \|x_{n+1} - z\|^2 - \frac{1}{4} \|x_{n+1} - x_n\|^2 \right] \\
 &\quad + 2\alpha_n \gamma \alpha \left[\|x_n - z\|^2 + \|x_{n+1} - z\|^2 \right] + 2\alpha_n \langle \gamma Q(z) - Bz, x_{n+1} - z \rangle \\
 &\leq \left[\frac{(1 - \alpha_n \bar{\gamma})^2}{2} + \alpha_n \gamma \alpha \right] (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
 &\quad + 2\alpha_n \langle \gamma Q(z) - Bz, x_{n+1} - z \rangle \\
 &\leq \frac{1 - 2\alpha_n \bar{\gamma} + 2\alpha_n \gamma \alpha}{2} (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n^2 \bar{\gamma}^2 M_1 \\
 &\quad + 2\alpha_n \langle \gamma Q(z) - Bz, x_{n+1} - z \rangle.
 \end{aligned}$$

This implies that

$$\begin{aligned}
 \|x_{n+1} - z\|^2 &\leq \frac{1 - 2(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 + 2(\bar{\gamma} - \gamma\alpha)\alpha_n} \|x_n - z\|^2 + \frac{2\alpha_n \bar{\gamma}^2}{1 + 2(\bar{\gamma} - \gamma\alpha)\alpha_n} M_1 \\
 &\quad + \frac{4\alpha_n}{1 + 2(\bar{\gamma} - \gamma\alpha)\alpha_n} \langle \gamma Q(z) - Bz, x_{n+1} - z \rangle \\
 &= \left[1 - \frac{4(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 + 2(\bar{\gamma} - \gamma\alpha)\alpha_n} \right] \|x_n - z\|^2 + \frac{2\alpha_n \bar{\gamma}^2}{1 + 2(\bar{\gamma} - \gamma\alpha)\alpha_n} M_1 \\
 &\quad + \frac{4\alpha_n}{1 + 2(\bar{\gamma} - \gamma\alpha)\alpha_n} \langle \gamma Q(z) - Bz, x_{n+1} - z \rangle \\
 &= (1 - \delta_n) \|x_n - z\|^2 + \delta_n \sigma_n,
 \end{aligned} \tag{16}$$

where $M_1 := \sup\{\|x_n - z\|^2 : n \geq 1\}$,

$\delta_n = \frac{4(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 + 2(\bar{\gamma} - \gamma\alpha)\alpha_n}$ and $\sigma_n = \frac{(\alpha_n \bar{\gamma}^2)M_1}{1 + 2(\bar{\gamma} - \gamma\alpha)\alpha_n} + \frac{4\alpha_n}{1 + 2(\bar{\gamma} - \gamma\alpha)\alpha_n} \langle \gamma Q(z) - Bz, x_{n+1} - z \rangle$. Since $\lim_{n \rightarrow \infty} \alpha_n = 0$

and $\sum_{n=0}^{\infty} \alpha_n = \infty$, it is easy to see that $\lim_{n \rightarrow \infty} \delta_n = 0$, $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$. Hence from (15), (16) and Lemma 2.5, we deduce that $x_n \rightarrow z$.

This completes the proof. \square

As a direct consequences of Theorem 3.1, we obtain the following result due to Xu *et al.* [19] for fixed point of nonexpansive mapping. Take $\gamma := 1$ and $B := I$ in Theorem 3.1 then the following Corollary is obtained.

Corollary 3.2. [19] Let H be a real Hilbert space and $Q : H \rightarrow H$ be a contraction mapping with constant $\alpha \in (0, 1)$. Let $S : H \rightarrow H$ be a nonexpansive mapping such that $\text{Fix}(S) \neq \emptyset$. Let the iterative sequence $\{x_n\}$ be generated by the following general viscosity implicit midpoint iterative schemes:

$$x_{n+1} = \alpha_n Q(x_n) + (1 - \alpha_n) S\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \quad (17)$$

where $\{\alpha_n\}$ is the sequence in $(0, 1)$ and satisfying the conditions (i)-(iii) of Theorem 3.1. Then the sequence $\{x_n\}$ converge strongly to $z \in \text{Fix}(S)$, which, in addition also solves variational inequality (2).

The following Corollary is due to Alghamdi *et al.* [18] for fixed point problem of nonexpansive mapping. Take $\gamma := 1$ and $Q, B := I$ in Theorem 3.1 then the following Corollary is obtained.

Corollary 3.3. [18] Let H be a real Hilbert space and $Q : H \rightarrow H$ be a contraction mapping with constant $\alpha \in (0, 1)$. Let $S : H \rightarrow H$ be a nonexpansive mapping such that $\text{Fix}(S) \neq \emptyset$. Let the iterative sequence $\{x_n\}$ be generated by the following general viscosity implicit midpoint iterative schemes:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \quad (18)$$

where $\{\alpha_n\}$ is the sequence in $(0, 1)$ and satisfying the conditions (i)-(iii) of Theorem 3.1. Then the sequence $\{x_n\}$ converge strongly to $z \in \text{Fix}(S)$.

Remark 3.4. Theorem 3.1 extends and generalize the viscosity implicit midpoint rule of Xu *et al.* [19] and the implicit midpoint rule of Alghamdi *et al.* [18] to a general viscosity implicit midpoint rule for a nonexpansive mappings, which also includes the results of [1, 9] as special cases.

4. Applications

4.1. Application to Variational Inequalities

Consider the following classical variational inequality problem (In short, VIP): Find $x^* \in K$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in K, \quad (19)$$

where A is a single-valued monotone mapping on H and K is a nonempty, closed and convex subset of H . We assume $K \subset \text{dom}(A)$. An example of VIP (19) is the constrained minimization problem: Find $x^* \in K$ such that

$$\min_{x \in K} \phi(x^*) \quad (20)$$

where $\phi : H \rightarrow \mathbb{R}$ is a lower-semicontinuous convex function. If ϕ is Fréchet differentiable, then the minimization problem (20) is equivalently reformulated as VIP (19) with $A = \nabla\phi$. Notice that the VIP (19) is equivalent to the following fixed point problem, for any $\lambda > 0$,

$$Sx^* = x^*, \quad Sx := P_K(I - \lambda A)x. \quad (21)$$

If A is Lipschitz continuous and strongly monotone, then, for $\lambda > 0$ small enough, S is a contraction mapping has a unique fixed point is also the unique solution of the VIP (19). However, if A is not strongly monotone, S is no longer a contraction, in general. In this case we must deal with nonexpansive mappings for solving the VIP (19). More precisely, we assume

(i) A is θ -Lipschitz continuous for some $\theta > 0$, i.e.,

$$\|Ax - Ay\| \leq \theta \|x - y\|, \quad \forall x, y \in H.$$

(ii) A is μ -inverse strongly monotone (μ -ism) for some $\mu > 0$, namely,

$$\langle Ax - Ay, x - y \rangle \geq \mu \|Ax - Ay\|^2, \quad \forall x, y \in H.$$

It is well known that by using the conditions (i) and (ii), the operator $S = P_K(I - \lambda A)$ is nonexpansive provided that $0 < \lambda < 2\mu$. It turns out that for this range of values of λ , fixed point algorithms can be applied to solve the VIP (19). Applying Theorem 3.1, we get the following result.

Theorem 4.1. *Assume that VIP (19) is solvable in which A satisfies the conditions (i) and (ii) with $0 < \lambda < 2\mu$. Let $B : H \rightarrow H$ be a strongly positive bounded linear operator with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$ and $Q : H \rightarrow H$ be a contraction mapping with constant $\alpha \in (0, 1)$. Let the iterative sequence $\{x_n\}$ be generated by the following general viscosity implicit midpoint iterative schemes:*

$$x_{n+1} = \alpha_n \gamma Q(x_n) + (1 - \alpha_n B) P_K(I - \lambda A) \left(\frac{x_n + x_{n+1}}{2} \right), \quad n \geq 0, \quad (22)$$

where $\{\alpha_n\}$ is the sequence in $(0, 1)$ and satisfying the conditions (i)-(iii) of Theorem 3.1. Then the sequence $\{x_n\}$ converge strongly to a solution z of VIP (19), which is also unique solution of variational inequality (4).

4.2. Fredholm Integral Equation

Consider a Fredholm integral equation of the following form

$$x(t) = g(t) + \int_0^t F(t, s, x(s)) ds, \quad t \in [0, 1], \quad (23)$$

where g is a continuous function on $[0, 1]$ and $F : [0, 1] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Note that if F satisfies the Lipschitz continuity condition, i.e.,

$$|F(t, s, x) - F(t, s, y)| \leq |x - y|, \quad \forall t, s \in [0, 1], \quad x, y \in \mathbb{R},$$

then equation (23) has at least one solution in $L^2[0, 1]$ (see [24]). Define a mapping $S : L^2[0, 1] \rightarrow L^2[0, 1]$ by

$$(Sx)(t) = g(t) + \int_0^t F(t, s, x(s)) ds, \quad t \in [0, 1]. \quad (24)$$

It is easy to observe that S is nonexpansive. In fact, for $x, y \in L^2[0, 1]$, we have

$$\begin{aligned} \|Sx - Sy\|^2 &= \int_0^1 |(Sx)(t) - (Sy)(t)|^2 dt \\ &= \int_0^1 \left| \int_0^t (F(t, s, x(s)) - F(t, s, y(s))) ds \right|^2 dt \\ &\leq \int_0^1 \left| \int_0^t |x(s) - y(s)| ds \right|^2 dt \\ &= \int_0^1 |x(s) - y(s)|^2 ds \\ &= \|x - y\|^2. \end{aligned}$$

This means that to find the solution of integral equation (23) is reduced to finding a fixed point of the nonexpansive mapping S in the Hilbert space $L^2[0, 1]$. Initiating with any function $x_0 \in L^2[0, 1]$. The sequence of functions $\{x_n\}$ in $L^2[0, 1]$ generated by the general viscosity implicit midpoint iterative scheme:

$$x_{n+1} = \alpha_n \gamma Q(x_n) + (1 - \alpha_n B) S \left(\frac{x_n + x_{n+1}}{2} \right), \quad n \geq 0, \quad (25)$$

where $\{\alpha_n\}$ is the sequence in $(0, 1)$ and satisfying the conditions (i)-(iii) of Theorem 3.1. Then the sequence $\{x_n\}$ converge strongly in $L^2[0, 1]$ to the solution of integral equation (23).

4.3. Periodic solution of a nonlinear evolution equation

Consider the following time-dependent nonlinear evolution equation in a Hilbert space H ,

$$\frac{du}{dt} + A(t)u = f(t, u), \quad t > 0, \quad (26)$$

where $A(t)$ is a family of closed linear operators in H and $f : \mathbb{R} \times H \rightarrow H$. The following result is the existence of periodic solutions of nonlinear evolution equation (26) due to Browder [25].

Theorem 4.2. [25] *Suppose that $A(t)$ and $f(t, u)$ are periodic in t of period $\omega > 0$ and satisfy the following assumptions:*

(i) *For each t and each pair $u, v \in H$,*

$$\operatorname{Re}\langle f(t, u) - f(t, v), u - v \rangle \leq 0.$$

(ii) *For each t and each $u \in D(A(t))$, $\operatorname{Re}\langle A(t)u, u \rangle \geq 0$.*

(iii) *There exists a mild solution u of equation (26) on \mathbb{R}^+ for each initial value $v \in H$. Recall that u is a mild solution of (26) with the initial value $u(0) = v$ if, for each $t > 0$,*

$$u(t) = \mathcal{U}(t, 0)v + \int_0^t \mathcal{U}(t, s)f(s, u(s))ds,$$

where $\{\mathcal{U}(t, s)\}_{t \geq s \geq 0}$ is the evolution system for the homogeneous linear system

$$\frac{du}{dt} + A(t)u = 0, \quad (t > s). \quad (27)$$

(iv) *There exists some $R > 0$ such that*

$$\operatorname{Re}\langle f(t, u), u \rangle < 0,$$

for $\|u\| = R$ and all $t \in [0, \omega]$.

Then there exists an element v of H with $\|v\| < R$ such that the mild solution of equation (26) with the initial condition $u(0) = v$ is periodic of period ω .

Next, we apply the general viscosity implicit midpoint rule for nonexpansive mappings to obtain an implicit iterative scheme for finding a periodic solution of (26). As a matter of fact, define a mapping $S : H \rightarrow H$ by assigning to each $v \in H$ the value $u(\omega)$, where u is the solution of (26) satisfying the initial condition $u(0) = v$. Namely, we define S by $Sv = u(\omega)$, where u solves (26) with $u(0) = v$.

We then find that S is nonexpansive. Moreover, condition (iv) of Theorem 4.2 forces S to map the closed ball $\mathcal{B} := \{v \in H : \|v\| \leq R\}$ into itself. Consequently, S has a fixed point which we denote by v , and the corresponding solution u of (26) with the initial condition $u(0) = v$ is a desired periodic solution of (26) with period ω . In other words, to find a periodic solution u of (26) is equivalent to finding a fixed point of S . Therefore the general viscosity implicit midpoint rule is applicable to solve (26), in which $\{x_n\}$ is generated by the general viscosity implicit midpoint iterative scheme:

$$x_{n+1} = \alpha_n \gamma Q(x_n) + (1 - \alpha_n)BS\left(\frac{x_n + x_{n+1}}{2}\right), \quad (28)$$

where $\{\alpha_n\}$ is the sequence in $(0, 1)$ and satisfying the conditions (i)-(iii) of Theorem 3.1. Then the sequence $\{x_n\}$ converges weakly to a fixed point v of S , and the solution of (26) with the initial value $u(0) = v$ is a periodic solution of (26).

5. Numerical Experiments

In this section, we provide a numerical example to illustrate the operational reliability and strong convergence of the presented iterative scheme in Theorem 3.1.

Example 5.1. Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y \rangle = xy, \forall x, y \in \mathbb{R}$, and induced usual norm $|\cdot|$. Let the mappings $B, Q, S : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $B(x) = 2x, Q(x) = \frac{1}{2}x, S(x) = \frac{1}{3}x$. Then there exist unique sequences $\{x_n\}$, be generated by the iterative schemes;

$$x_0 = x \in \mathbb{R},$$

$$x_{n+1} = \frac{1}{n^2 + 1} (2) \frac{x_n}{2} + \left(1 - \frac{1}{n^2 + 1} B\right) S \left(\frac{x_n + x_{n+1}}{2}\right), \tag{29}$$

where $\alpha_n = \frac{1}{n^2+1}$, and $\gamma = 2$. Then $\{x_n\}$ converge strongly to $\{0\}$.

Proof. It is easy to observe that Q is contraction mapping with constant $\alpha = \frac{1}{2}$ and B is a strongly positive self-adjoint bounded linear operator with constant $\bar{\gamma} = 1$ on \mathbb{R} . Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$.

Furthermore it is easy to observe that S is a nonexpansive mapping with $\text{Fix}(S) = \{0\}$. After simplification, iterative scheme (29) reduce to

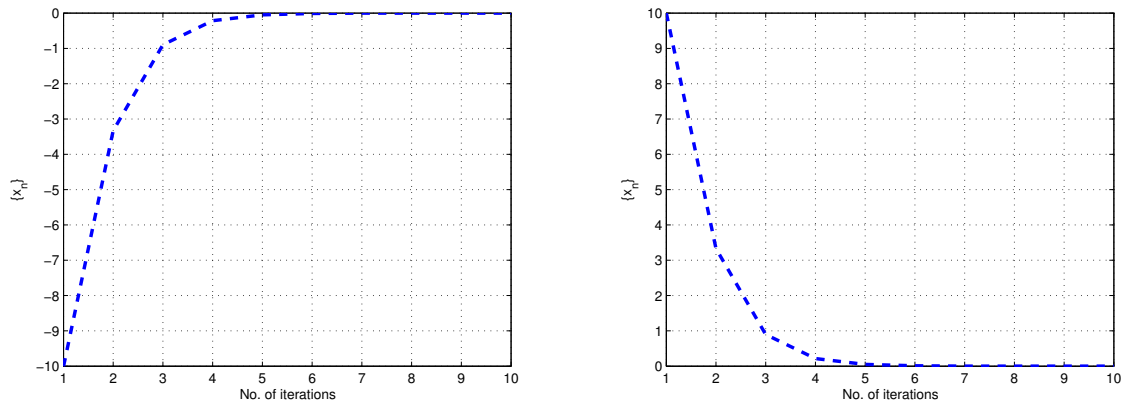
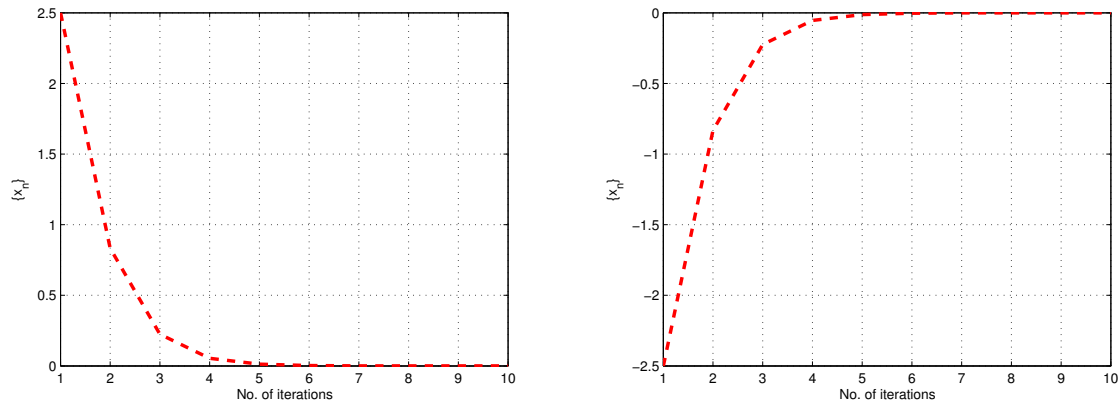
$$x_{n+1} = \frac{n^2 + 5}{5n^2 + 7} x_n. \tag{30}$$

Setting $\|x_n - p\| < 10^{-4}$ as stop criterion, then we obtain the numerical results of the iterative scheme (29) with different initial points x_0 . in Table 1. The computations are performed by Matlab R2007a running on a PC Desktop Intel(R) Core(TM)i3-2330M, CPU @2.20 GHz, 790 MHz, 2 GB RAM.

Table 1: Numerical results with different initial values $x(0) = -20, 20, 5, -5$

Iter. (n)	$x(0) = -20$	Iter. (n)	$x(0) = 20$	Iter. (n)	$x(0) = 5$	Iter. (n)	$x(0) = -5$
1.	-10.0000	1.	10.0000	1.	2.5000	1.	-2.5000
2.	-3.3333	2.	3.3333	2.	0.8333	2.	-0.8333
3.	-0.8974	3.	0.8974	3.	0.2244	3.	-0.2244
4.	-0.2166	4.	0.2166	4.	0.0542	4.	-0.0542
5.	-0.0492	5.	0.0492	5.	0.0123	5.	-0.0123
6.	-0.0108	6.	0.0108	6.	0.0027	6.	-0.0027
7.	-0.0023	7.	0.0023	7.	0.0006	7.	-0.0006
8.	-0.0005	8.	0.0005	8.	0.0001	8.	-0.0001
9.	-0.0001	9.	0.0001	9.	0.0000	9.	-0.0000
10.	-0.0000	10.	0.0000	10.	0.0000	10.	-0.0000

Next, by using the software Matlab 7.0, we study the convergence behavior of $\{x_n\}$, for different initial values and obtained the following figures, which shows that $\{x_n\}$ converge strongly to $\{0\}$.

Figure 1: Convergence behavior of the sequence $\{x_n\}$ Figure 2: Convergence behavior of the sequence $\{x_n\}$

This completes the proof. \square

Conclusion: The present work has been aimed to study the general viscosity implicit midpoint rule for nonexpansive mapping and proved the strong convergence theorem for solving fixed point for a nonexpansive mapping. Theorem 3.1 extends and generalize the viscosity implicit midpoint rule of Xu *et al.* [19] and the implicit midpoint rule of Alghamdi *et al.* [18] to a general viscosity implicit midpoint rule for a nonexpansive mappings, which also includes the results of [1, 9] as special cases.

References

- [1] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, **241**(1) (2000) 46–55.
- [2] F.E. Browder, Convergence of approximants to fixed points of non-expansive maps in Banach spaces, *Arch. Rational Mech. Anal.*, **24** (1967) 82–90.
- [3] B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.*, **73** (1967) 591–597.
- [4] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004) 279–291.
- [5] V. Berinde, *Iterative Approximation of Fixed Points*, (Lecture Notes in Mathematics, No. 1912), Springer, Berlin, 2007.
- [6] C.E. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, (Lecture Notes in Mathematics, No. 1965), Springer-Verlag, London, 2009.

- [7] C.E. Chidume and M.O. Nnakwe, A strong convergence theorem for an inertial algorithm for a countable family of generalized nonexpansive maps, *Fixed Point Theory*, **21** (2) (2020) 441–452, DOI:10.24193/fpt-ro.2020.2.31.
- [8] I. Yamada, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004) 279–291.
- [9] G. Marino and H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.*, **318** (2006) 43–52.
- [10] E. Hofer, A partially implicit method for large stiff systems of ODEs with only few equations introducing small time-constants, *SIAM J. Numer. Anal.*, **13** (1976) 645–663.
- [11] G. Bader and P. Deufhard, A semi-implicit mid-point rule for stiff systems of ordinary differential equations, *Numer. Math.*, **41** (1983) 373–398.
- [12] P. Deufhard, Recent progress in extrapolation methods for ordinary differential equations, *SIAM Rev.*, **27**(4) (1985) 505–535.
- [13] W. Auzinger and R. Frank, Asymptotic error expansions for stiff equations: an analysis for the implicit midpoint and trapezoidal rules in the strongly stiff case, *Numer. Math.*, **56** (1989) 469–499.
- [14] A. Bayreuth, The implicit midpoint rule applied to discontinuous differential equations, *Computing*, **49** (1992) 45–62.
- [15] C. Schneider, Analysis of the linearly implicit mid-point rule for differential-algebra equations, *Electron. Trans. Numer. Anal.*, **1** (1993) 1–10.
- [16] S. Somalia and S. Davulcua, Implicit midpoint rule and extrapolation to singularly perturbed boundary value problems, *Int. J. Comput. Math.*, **75**(1) (2000) 117–127.
- [17] S. Somalia, Implicit midpoint rule to the nonlinear degenerate boundary value problems, *Int. J. Comput. Math.*, **79**(3) (2002) 327–332.
- [18] M.A. Alghamdi, N. Shahzad and H.K. Xu, The implicit midpoint rule for nonexpansive mappings. *Fixed Point Theory Appl.*, **96** (2014), 9 pages.
- [19] H.K. Xu, M.A. Alghamdi and N. Shahzad, The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces, *Fixed Point Theory Appl.*, **41** (2015), 12 pages.
- [20] Y. Yao, N. Shahzad and Y.C. Liou, Modified semi-implicit midpoint rule for nonexpansive mappings, *Fixed Point Theory Appl.*, **15** (2015), 12 pages.
- [21] L.C. Zhao, S.S. Chang and C.F. Wen, Viscosity approximation methods for the implicit midpoint rule of asymptotically nonexpansive mappings in Hilbert spaces, *J. Nonlinear Sci. Appl.*, **9**(1) (2016) 4478–4488.
- [22] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, London, 2011.
- [23] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.
- [24] P.L. Lions, Approximation de points fixes de contractions, *C.R. Acad. Sci., Ser. A-B Paris* **284** (1977) 1357–1359.
- [25] F.E. Browder, Existence of periodic solutions for nonlinear equations of evolution, *Proc. Natl. Acad. Sci., USA*, **53** (1965) 1100–1103.