SS-Discrete Modules

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Abstract. In this paper, we define (strongly) ss–discrete, semi-ss–discrete and quasi-ss–discrete modules as a strongly notion of (strongly) discrete, semi-discrete and quasi-discrete modules with the help of ss–supplements in [3]. We examined the basic properties of these modules and included characterization of strongly ss–discrete modules over semi-perfect rings.

1. Introduction

In this study, $R$ is used to show a ring which is associative and has an identity. All mentioned modules will be unital left $R$–modules. Let $M$ be an $R$–module. The notation $A \subseteq M$ means that $A$ is a submodule of $M$. Any submodule $A$ of an $R$–module $M$ is called small in $M$ and showed by $A \ll M$ whenever $A + C \neq M$ for all proper submodule $C$ of $M$. The Jacobson radical of $M$ denoted by $Rad(M)$. Dually, a submodule $A$ of a $R$–module $M$ is called to be essential in $M$ which is showed by $A \preceq M$ if $A \cap K \neq 0$ for each non-zero submodule $K$ of $M$. The socle of $M$ which is the sum of all simple submodules of $M$ is denoted by $Soc(M)$. A non-zero module $M$ is called hollow if every proper submodule of $M$ is small in $M$ and is called local providing that the sum of all proper submodules of $M$ is also a proper submodule of $M$. A submodule $N$ of $M$ is called coclosed in $M$ if whenever $\frac{N}{X} \ll \frac{M}{X}$ for a submodule $K$ of $M$ with $K \subseteq N, N = K$.

Let $A$ and $B$ be submodules of a module $M$. Then $A$ is called a supplement of $B$ in $M$ when $A$ is minimal with the property $M = A + B$; in other words, $M = A + B$ and $A \cap B \ll A$. $M$ is said to be supplemented if every submodule of $M$ has a supplement in $M$. Two submodules $A$ and $B$ of $M$ are called mutual supplements in $M$ if, $M = A + B, A \cap B \ll A$ and $A \cap B \ll B$ [1]. There are a lot of papers related with supplemented modules such as [7, 8]. If $M$ is supplemented and self-projective, then $M$ is called strongly discrete. The module $M$ is called amply supplemented if for any submodules $A$ and $B$ of $M$ with $M = A + B$, there exists a supplement $X$ of $A$ such that $X \subseteq B$.

In [7], a module $M$ is called lifting if for every submodule $A$ of $M$ lies over a direct summand, that is, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \ll A, A \cap M_2 \ll M_2$. By [8], $M$ is lifting iff $M$ is amply supplemented and every supplement submodule of $M$ is a direct summand of it.

Following [9], the sum of all simple submodules of $M$ which are small in $M$ is named with $Soc_s(M)$, that is, $Soc_s(M) = \sum [A \ll M | A$ is simple]. Note that $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$. In [3], a module $M$ is called strongly local providing that $M$ is local and $Rad(M) \subseteq Soc(M)$. In the same paper, a ring $R$ is called left strongly local ring if $R$ is a strongly local module.
According to [3], ss-supplemented modules was examined and founded as a strong notion of supplemented modules. Let \( M \) be a module and \( A, B \leq M \). If \( M = A + B \) and \( A \cap B \subseteq \text{soc}_c(B) \), then \( B \) is an ss-supplement of \( A \) in \( M \). Any module \( M \) is named ss-supplemented if each submodule \( A \) of \( M \) has a ss-supplement \( B \) in \( M \). As a result of this definition, any finitely generated module is ss-supplemented iff it is supplemented and \( \text{Rad}(M) \subseteq \text{soc}(M) \). In the same paper, amply ss-supplemented modules were defined. A submodule \( A \) of a module \( M \) has ample ss-supplements in \( M \) if \( A \) contains an ss-supplement of \( B \) in \( M \) with \( M = A + B \). \( M \) is called amply ss-supplemented if every submodule of \( M \) has ample ss-supplements in \( M \).

According to [2], a module \( M \) is called semisimple lifting or briefly ss-lifting if for every submodule \( A \) of \( M \), there is a decomposition \( M = M_1 \oplus M_2 \) such that \( M_1 \leq A, A \cap M_2 \ll M \) and \( A \cap M_2 \) is semisimple. Some new fundamental properties of ss-lifting modules will be examined in this paper.

Let \( c \) be a cardinal number. The module \( M \) is said to have the \( c \)-internal exchange property if every decomposition \( M = \bigoplus M_i \) with \( \text{card}(I) \leq c \) is exchangeable. A module \( M \) has the (finite) internal exchange property if it has the \( c \)-internal exchange property for every (finite) cardinal \( c \) [1, 11.34]. A lifting module with the finite internal exchange property is called a semi-discrete module. The module \( M \) is called discrete if \( M \) is lifting and satisfies the following condition:

\((D_2)\) : If \( N \subseteq M \) such that \( \frac{M}{N} \) is isomorphic to a direct summand of \( M \), then \( N \) is a direct summand of \( M \).

The module \( M \) is called quasi-discrete if \( M \) is lifting and satisfies the following condition;

\((D_3)\) : If \( N \) and \( K \) are direct summands of \( M \) such that \( M = N + K \), then \( N \cap K \) is a direct summand of \( M \) (See [7]).

By [7, Lemma 4.6], \((D_2)\) implies \((D_3)\). In [1, 4.29], the notion of \( \cap \)-direct projective modules is defined as an equivalent condition to the property \((D_3)\). By [1, 4.21], a module \( M \) is direct projective if and only if \( M \) has the property \((D_2)\).

In the first part of this study, we define semi-ss-discrete and quasi-ss-discrete modules based on the definition of ss-lifting module. We give examples of these modules. We show that every quasi-ss-discrete module is ss-lifting and amply ss-supplemented. The factor module of a quasi-ss-discrete module is showed to be quasi-ss-discrete again under special conditions. In addition, theorems related with the decomposition of quasi-ss-discrete modules are obtained. In the second part, we define (strongly) ss-discrete modules and determine their relationship with ss-supplemented modules.

2. Semi-SS-Discrete and Quasi-SS-Discrete Modules

In this section, semi-ss-discrete modules and quasi-ss-discrete modules are defined and some of the basic features of these modules are obtained.

**Definition 2.1.** If \( M \) is an ss-lifting module with finite internal exchange property, then \( M \) is called a semi-ss-discrete module. If \( M \) is both a \( \pi \)-projective and ss-supplemented module, then \( M \) is called a quasi-ss-discrete module. Let \( N \) be any submodule of \( M \). Any submodule \( K \) of \( M \) is called \( N \)-ss-lifting if every homomorphism \( M \rightarrow \frac{M}{N} \) where \( N \cap K \) is semisimple lifts to an endomorphism of \( M \). If \( K \) is a ss-supplement of \( N \) of \( M \), then \( K \) is called a \( N \)-lifting ss-supplement in \( M \).

Recall from [1] that a module \( K \) is said to be generalized \( M \)-projective if, for any epimorphism \( g : M \rightarrow X \) and homomorphism \( f : K \rightarrow X \), there exist decompositions \( K = K_1 \oplus K_2, M = M_1 \oplus M_2 \), a homomorphism \( h_1 : K_1 \rightarrow M_1 \) and an epimorphism \( h_2 : M_2 \rightarrow K_2 \), such that \( g \circ h_1 = f \), and \( f \circ h_2 = g \).

**Proposition 2.2.** The following statements are equivalent for \( M \):

1. \( M \) is semi-ss-discrete;
2. \( M \) is ss-supplemented, every ss-supplement in \( M \) is a direct summand and \( K \cap L \) are relatively generalized projective, for every decomposition \( M = K \oplus L \);
3. \( M \) is ss-lifting and \( K, L \) are relatively generalized projective, for every decomposition \( M = K \oplus L \).
Proof. (1) ⇒ (2) Since $M$ is ss-lifting, it is ss-supplemented and every ss-supplement is a direct summand by [2, Theorem 1]. Let $M = N + K$. Then $N$ contains an ss-supplement $N'$ of $K$ which is a direct summand of $M$. So, we have $M = N' \oplus L' \oplus K$ with $L' \subseteq L$ and $K' \subseteq K$ since $M$ has the finite internal exchange property. Thus $L$ is generalized $K$-projective by [1, 4.42]. Similarly, it is easy to see that $K$ is generalized $L$-projective.

(2) ⇒ (3) It is enough to prove that $M$ is ss-lifting. Let $N \subseteq M$. By the hypothesis, $N$ has an ss-supplement $K$ which is a direct summand of $M$, that is $M = L \oplus K$. Then $L$ is generalized $K$-projective and so $M = N' \oplus L' \oplus K' = N' + K$, where $N' \subseteq N$, $K' \subseteq K$ and $L' \subseteq L$ by [1, 4.42] since $M = N + K$. From here $N = N' + (N \cap K)$. Since $N \cap K \leq K$ and $N \cap K$ is semisimple, we have $M$ is an ss-lifting module.

(3) ⇒ (1) Suppose $M = K \oplus L$. It is obtained from [2, Theorem 3] that $K$ and $L$ are ss-lifting modules, and so $K$ and $L$ are relatively generalized projective. It follows from [1, 23.10] that $M$ has the 2-internal exchange property.

Recall from [5] that a module $M$ is called duo if for every submodule $U$ of $M$ is fully invariant, i.e. $f(U) \subseteq U$ for every $f \in \text{End}(M)$ and $U \subseteq M$.

Proposition 2.3. Let $M = M_1 \oplus \ldots \oplus M_n$ be a duo module where each $M_i$ is semi-ss-discrete. Then the following statements are equivalent:

1. $M$ is semi-ss-discrete;
2. $M$ is ss-lifting and $M = M_1 \oplus \ldots \oplus M_n$ is an exchange decomposition;
3. For any direct summand $K$ of $\bigoplus M_i$, and any direct summand $L$ of $\bigoplus M_j$, $K$ and $L$ are relatively generalized projective where $I, J$ non-empty disjoint subsets of $\{1, 2, \ldots, n\}$;
4. If $M'_i$ is any direct summand of $M_i$ and $T$ is any direct summand of $\bigoplus M_j$, then $M'_i$ and $T$ are relatively generalized projective for any $1 \leq i \leq n$;

Proof. is clear by [1, 23.14] and [2, Theorem 10].

As an immediate consequence of Proposition 2.3, we have the following corollary.

Corollary 2.4. Let $M = M_1 \oplus \ldots \oplus M_n$ be a duo module where each $M_i$ is a semi-ss-discrete module. If $M_i$ and $M_j$ are relatively generalized projective for each $i \neq j$, then $M$ is semi-ss-discrete.

Recall from [1, 12.1] that an $R$-module $M$ is said to be an $LE$-module if its endomorphism ring $\text{End}(M)$ is local.

Theorem 2.5. Let $M$ be an ss-lifting module with an indecomposable decomposition $M = \bigoplus M_i$ is a duo module. Then $M$ is a semi-ss-discrete module if one of the following statements is satisfied:

1. $M_i$ is an LE-module for all $i \in I$;
2. every non-zero direct summand of $M$ contains a non-zero indecomposable direct summand and the decomposition $M = \bigoplus_{i \in I} M_i$ complements maximal direct summands.

Proof. A module $M$ with an indecomposable exchange decomposition has the internal exchange property. Hence we can apply [1, 24.13, 24.10] to [3, Theorem 30].

We can compare quasi-ss-discrete modules, ss-supplemented modules and ss-lifting modules in following lemmas.

Lemma 2.6. If $M$ is a quasi-ss-discrete module, then $M$ is ss-lifting.
Proof. Since $M$ is $\pi$-projective, it is clear by [1, 20.9] and [2, Theorem 1] that ss-supplements are direct summands in $M$. So it is enough to prove that $M$ is amply ss-supplemented. Suppose that $M = U + V$ and $X$ is an ss-supplement of $U$ in $M$. Then for any $f \in \text{End}(M)$ with $\text{Im}(f) \subseteq V$ and $\text{Im}(1 - f) \subseteq U$, we have $M = U + f(X)$ and $U \cap f(X) = f(U \cap X) \gg f(X)$. Since $U \cap X$ is semisimple, $U \cap f(X)$ is semisimple by [8, 20.3]. Thus $f(X)$ is an ss-supplement of $U$ contained in $V$. □

By the help of [8, 41.15], it can be seen that if the intersection of any pair of mutual ss-supplements is zero in an ss-supplemented module, then ss-supplement submodules of $M$ are direct summands.

Lemma 2.7. If $M$ is an ss-lifting and $\pi$-projective module, then $M$ is amply ss-supplemented and the intersection of any pair of mutual ss-supplements in $M$ is zero.

Proof. Follows from [2, Theorem 1] and [1, 20.9]. □

Corollary 2.8. If $M$ is a quasi-ss-discrete module, then $M$ is amply ss-supplemented and the intersection of any pair of mutual ss-supplements in $M$ is zero.

Proof. Clear by Lemmas 2.6 and 2.7. □

It is clear that every quasi-ss-discrete module is quasi-discrete by Definition 2.1. The following example shows that the converse is not need to be true. So the notion of quasi-ss-discrete module is a stronger than that of quasi-discrete module.

Example 2.9. For any prime integer $p$, consider the left $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$. $M$ is supplemented but not ss-supplemented by [3, Example 17]. Since $M$ has the property $(D_3)$, $M$ is quasi-discrete but not quasi-ss-discrete.

The following corollary is obtained by automatically by Lemma 2.7.

Corollary 2.10. If $M$ is an ss-lifting module and has the property $(D_3)$, then $M$ is a quasi-ss-discrete module.

Lemma 2.11. Let $M$ be a quasi-ss-discrete module, $K$ be a submodule of $M$ and $L$ be an ss-supplement of $K$. If $N$ is an ss-supplement submodule of $M$ contained in $K$, then $N \cap L = 0$ and $N \oplus L$ is a direct summand of $M$.

Proof. Since $M$ is a quasi-ss-discrete module, $M$ is ss-lifting by Lemma 2.6. If we use [2, Theorem 1], it can be concluded that $L$ and $N$ are direct summand of $M$. Therefore there exists a submodule $N_1$ of $M$ such that $M = N \oplus N_1$. It is clear that $K = (K \cap N_1) \oplus N$ and so $M = N + L + (K \cap N_1)$. By [2, Theorem 1], $K \cap N_1$ contains an ss-supplement $X$ of $N + L$, where $X$ is a direct summand of $M$. Thus $X \oplus N$ is a direct summand of $M$ due to $X \subseteq N$. However, we have that $(X \oplus N) \cap L$ is a direct summand of $M$ by [4.14 (4)]. From here $(X \oplus N) \cap L \subseteq K \cap L \subseteq \text{Soc}_s(L)$. Finally we can get $(X \oplus N) \cap L = 0$ and so $M = X \oplus N \oplus L$. □

Proposition 2.12. If $K, L$ are direct summand of a quasi-ss-discrete module $M$ and $L$ is hollow, then

(i) $K \cap L = 0$ and $K \oplus L$ is a direct summand of $M$ or
(ii) $K + L = K \oplus S$ with $S \subseteq \text{Soc}_s(M)$ and $L$ is isomorphic to a summand of $K$.

Proof. Suppose that $T$ is an ss-supplement of $K + L$. Then we have $M = T + (K + L)$ and $T \cap (K + L) \subseteq \text{Soc}_s(T)$. By Lemma 2.11, $K \cap T = 0$. Let’s complete the proof by evaluating the following two situations.

(1) If $L \not\subseteq K \oplus T$, then $L \cap (K + T) = 0$ and so $L$ is an ss-supplement of $K + T$. It follows that $K \cap L = 0$ and $K \oplus L$ is a direct summand of $M$ by Lemma 2.11.

(2) Assume that $L \subseteq K \oplus T$. Since $M = K + T + L = K + T$ and $K \cap T = 0$, we have $M = K \oplus T$. If we intersect the equality $M = K + T$ with $K + L$, then we can write $K + L = K \oplus S$ where $S = (K + L) \cap T$. Moreover $S \subseteq \text{Soc}_s(M)$ by [2, Theorem 1]. Since $L$ is a direct summand of $M$, there exists a submodule $L_1$ of $M$ such that $M = L \oplus L_1$. It follows that $M = K + L + L_1 = K \oplus (K + L) \cap T + L_1 = K + L_1$ because $(K + L) \cap T \ll M$. Let $N_1$ be an ss-supplement of $L_1$ contained in $K$. Then, we get $M = [N_1 \oplus (K \cap L_1)] + L_1 = N_1 \oplus L_1$ and $L \not\subseteq N_1$. □

Theorem 2.13. If $M$ is a quasi-ss-discrete module, then $M$ is ss-lifting and for every decomposition $M = K \oplus L$, $K$ and $L$ are relatively projective.
Proof. We obtain by Lemmas 2.6 and 2.7 that \( M \) is amply \( ss \)-supplemented and the intersection of any pair of mutual \( ss \)-supplements in \( M \) is zero. Since \( M \) is \( ss \)-supplemented, \( ss \)-supplements are direct summands and so \( M \) is \( ss \)-lifting by [2, Theorem 1]. Suppose that \( M = U + V \) where \( U \) and \( V \) are direct summands of \( M \). Let \( X \) be an \( ss \)-supplement of \( V \) such that \( X \subseteq U \). Then \( M = X \oplus V \). As \( U = X \oplus (U \cap V) \), we get \( U \cap V \) is a direct summand of \( M \). Therefore \( M \) is \( \cap \)-direct projective. The rest follows from [1, 4.14(2)].  

By the definition, every quasi \( ss \)-discrete module is semi-\( ss \)-discrete. But the converse is not always true as in the following example.

Example 2.14. Consider the \( \mathbb{Z} \)-module \( U = \mathbb{Z}/p\mathbb{Z} \) and \( V = \mathbb{Z}/q\mathbb{Z} \) where \( p \) is prime. Then \( U \) and \( V \) are relatively generalised projective but \( U \) is not \( V \)-projective. So \( M \) is not a quasi \( ss \)-discrete module although \( M \) is an \( ss \)-lifting module.

Now we can obtain properties of quasi \( ss \)-discrete modules.

Proposition 2.15. Let \( M \) be a quasi-\( ss \)-discrete module. Then every direct summand of \( M \) is quasi-\( ss \)-discrete and every \( ss \)-supplement submodule of it is a direct summand.

Proof. Let \( N \) be a direct summand of \( M \). Since \( M \) is \( ss \)-lifting and \( \pi \)-projective, every \( ss \)-supplement submodule of \( M \) is a direct summand by [2, Theorem 1]. Since every direct summand of a \( \pi \)-projective module is again \( \pi \)-projective, \( N \) is \( ss \)-supplemented by [3, Corollary 38]. Therefore \( N \) is quasi-\( ss \)-discrete module.  

Since \( ss \)-supplemented modules are supplemented, proofs of the following facts are clear by [8, 41.16-(2,3)].

Lemma 2.16. Let \( M \) be a quasi-\( ss \)-discrete module and \( S = \text{End}(M) \). Let \( e \in S \) be an idempotent and \( N \) be a direct summand of \( M \). If \((1 - e)(N) \ll (1 - e)(M)\), then \( N \cap (1 - e)(M) = 0 \) and \( N \oplus (1 - e)(M) \) is a direct summand in \( M \).

Proposition 2.17. Let \( M \) be a quasi-\( ss \)-discrete module. If \( \{N_i \}_{i \in I} \) is a directed family of direct summands of \( M \) with respect to inclusion, then \( \bigcup_{i \in I} N_i \) is also a direct summand in \( M \).

Recall from [3, Proposition 16] that an \( ss \)-supplemented hollow module is strongly local.

Lemma 2.18. Let \( M \) be a quasi-\( ss \)-discrete module. Then for every \( 0 \neq m \in M \), there is a decomposition \( M = M_1 \oplus M_2 \) such that \( m \notin M_1 \) and \( M_2 \) is strongly local.

Proof. Given \( 0 \neq m \in M \). Let’s define the set \( S = \{ T \subset M \mid T \text{ is direct summand and } m \notin T \} \). This set is non-empty and inductive with respect to inclusion by Proposition 2.17 and has a maximal element \( M_1 \) by Zorn’s Lemma. Since \( M_1 \) is a direct summand, there exists a submodule \( M_2 \) of \( M \) such that \( M = M_1 \oplus M_2 \). By Proposition 2.15 and Lemma 2.6, \( M_2 \) is a quasi-\( ss \)-discrete module and \( M_2 \) is \( ss \)-lifting. Therefore \( M_2 \) must be strongly local. If \( M_2 \) is not hollow, then there is a proper non-supercuous submodule in \( M_2 \), say \( U \). It follows that there exists an nontrivial decomposition \( M_2 = V \oplus V_1 \) with \( V \subset U \) and \( U \cap V_1 \subseteq \text{Soc}(V_1) \) for some submodule \( V, V_1 \) of \( M_2 \). Then we can write \( M = M_1 \oplus M_2 = M_1 \oplus V \oplus V_1 \). By the maximality of \( M_1 \), we get \( m \in M_1 \oplus V \) and \( m \in M_1 \oplus V_1 \). But this means \( m \in M_1 \) contradicting the choice of \( M_1 \). Therefore all proper submodules in \( M_2 \) are superfluous, i.e. \( M_2 \) is hollow. By [3, Proposition 16], we deduce that \( M_2 \) is strongly local.  

Observe from [3, Lemma 13] that an \( ss \)-supplemented and radical module is zero. Using this fact we prove that the following fact:

Theorem 2.19. Let \( M \) be a quasi-\( ss \)-discrete module. Then \( M \) has a decomposition \( M = \bigoplus_{i \in I} H_i \), where each \( H_i \) is strongly local. In particular, if \( N \) is a direct summand of \( M \), there exists a subset \( J \subset I \) such that \( M = \left( \bigoplus_{i \in J} H_i \right) \oplus N \).
Suppose that \( N \) is a direct summand of \( M \). Let’s define \( S = \{ \lambda \subset I \} N \cap \bigoplus_{\lambda} H_{\lambda} = \{ 0 \} \) and \( N \cap \bigoplus_{\lambda} H_{\lambda} \) is a direct summand in \( M \). By using Proposition 2.17 and Zorn’s Lemma, we can say that \( S \) has a maximal element \( I \). Assume that \( L = N \cap \bigoplus_{\lambda} H_{\lambda} \). We must prove that \( M = L \). Assume that \( L \neq M \). Therefore there exists an element \( a \in M \setminus L \). Then by Lemma 2.18, we have a decomposition \( M = K \oplus H \) with \( L \subset K \) and \( H \) is strongly local. If we show that \( H = \{ 0 \} \), then the proof is completed. Suppose that \( H \neq \{ 0 \} \). We consider the canonical projection \( p : M \to H \). It is clear that if \( p(H_{i}) = H \) holds for some \( j \in I \), then \( M = K + H_{j} \). If \( K \cap H_{j} = H_{j} \), then \( M = K \) and so \( H = \{ 0 \} \). Because of \( K \cap H_{j} \neq H_{j} \), we get that \( K \cap H_{j} \ll H_{j} \). Since \( M \) is \( \pi \)-projective, we have \( K \cap H_{j} = \{ 0 \} \), i.e. \( M = K \oplus H_{j} \). \( L \oplus H_{j} \) is a direct summand of \( M \) because \( L \) is a direct summand of \( M \). Since \( j \neq j' \), this is a contradiction to the maximality of \( I \). It follows from \( p(H_{i}) \neq H \) for every \( i \in I \). From here, if we say \( T = H_{i_{1}} \oplus H_{i_{2}} \oplus \ldots \oplus H_{i_{n}} \) for every finite \( i_{1}, i_{2}, \ldots, i_{n} \in I \), then \( p(T) = p(H_{i_{1}}) \oplus p(H_{i_{2}}) \oplus \ldots \oplus p(H_{i_{n}}) \ll H \). Moreover, for the canonical projection \( e : M \to K \), we get that \( p = l_{M} - e \) and \( p(T) = (l_{M} - e)(T) \ll H = (l_{M} - e)(M) \). Then we have \( T \cap H = \{ 0 \} \) by Lemma 2.16. This situation is valid for every finite \( i_{1}, i_{2}, \ldots, i_{n} \) we obtain \( \bigoplus_{j} H_{j} \cap N = \{ 0 \} \) and so \( H = M \cap H = \{ 0 \} \). It is a contradiction to the \( H \neq \{ 0 \} \). Hence \( H = \{ 0 \} \), this means \( M = L \). \( \square \)

Recall that a module \( M \) is called coatomic if every proper submodule of \( M \) is contained in a maximal submodule of \( M \). A ring \( R \) is called left max if every non-zero \( R \)-module has a maximal submodule. Note that if \( R \) is a left max ring, then every \( R \)-module is coatomic.

**Corollary 2.20.** Let \( M \) be a quasi-ss-discrete. Then \( M \) is coatomic and \( \text{Rad}(M) \) is semisimple.

**Proof.** It follows from Theorem 2.19 and [3, Theorem 27]. \( \square \)

**Proposition 2.21.** The following statements are equivalent for an amply ss-supplemented module \( M \).

1. \( M \) is quasi-ss-discrete;
2. \( M \) is \( \pi \)-projective.

**Proof.** Clear by [8, 41.15] and [3, Proposition 26]. \( \square \)

Recall from [1, 4.13] that any factor module \( \frac{M}{N} \) of a \( \pi \)-projective module \( M \) by a fully invariant submodule \( N \) is \( \pi \)-projective.

The following proposition can be proven by [3, Proposition 26].

**Proposition 2.22.** Let \( M \) be a quasi-ss-discrete module and \( N \) be a fully invariant submodule of \( M \). Then \( \frac{M}{N} \) is quasi-ss-discrete.

**Proposition 2.23.** The following statements are equivalent for any module \( M \).

Proof. We indicate by \( \Omega \) the set of all strongly local submodules in \( M \) and take into account \( \Phi = \{ \varphi \subset \Omega \mid \sum_{H \in \Phi} H \) is a direct sum and a direct summand in \( M \} \). Then, since \( M \) is a quasi-ss-discrete module, \( M \) has a strongly local submodule that is a direct summand of its by [3, Lemma 13] and Lemma 2.6. So this set is non-empty and inductive with respect to inclusion by Proposition 2.17 has a maximal element \( \varphi \) by Zorn’s Lemma. By indexing the elements in \( \varphi \) with \( i \), let \( L = \bigoplus_{i} H_{i} \). Since \( L \) is a direct summand, there exists a submodule \( K \) of \( M \) such that \( M = L \oplus K \). If we prove that \( K = \{ 0 \} \), then the proof will be completed. Suppose that \( K \neq \{ 0 \} \). Then, there is an element \( a \) of \( K \) with \( a \neq 0 \). Moreover, \( K \) is a quasi-ss-discrete module by Proposition 2.15. We get that a decomposition \( K = K_{1} \oplus K_{2} \) such that \( a \notin K_{1} \) and \( K_{2} \) is strongly local by Lemma 2.18. Then we have \( M = L \oplus K = L \oplus K_{1} \oplus K_{2} = (L \oplus K_{2}) \oplus K_{1} \) and so \( K_{2} \neq \{ 0 \} \) because of \( a \notin K_{1} \). Therefore, the direct summand \( L \oplus K_{2} \) of \( M \) is properly larger than \( L \). This contradicts the maximality of \( L \). Consequently, \( K = \{ 0 \} \) and we deduce that \( M = \bigoplus_{i} H_{i} \).
1. $M$ is quasi-ss-discrete;
2. $M$ is amply ss-supplemented and all ss-supplements of any coclosed submodule $N$ of $M$ are $K$-ss-lifting.

Proof. (1) $\Rightarrow$ (2) It is clear that $M$ is amply ss-supplemented by [3, Proposition 37]. Let $N$ be a coclosed submodule of $M$ and $K$ be an ss-supplement of $N$ in $M$. Then $N$ and $K$ are ss-supplements of each other and so $K \cap N = 0$ by [7, Proposition 4.11].

(2) $\Rightarrow$ (1) It is enough to prove that $M$ is $\pi$-projective. Let $N$ and $K$ be submodules of $M$ with $M = N + K$. Since $M$ is amply ss-supplemented, there exists a submodule $K'$ of $M$ such that $M = N + K'$, $N \cap K' \ll K'$, $N \cap K$ is semisimple, $K \subseteq K'$ and a submodule $N'$ of $M$ such that $M = K' + N'$, $K \cap N' \ll N'$, $K \cap N'$ is semisimple and $N' \subseteq N$. Therefore $K'$ and $N'$ are ss-supplements of each other. Define $\varphi : M \to K_{N'}$ by $\varphi(k + n') = k' + (K' \cap N')$ ($k' \in K'$, $n' \in N'$). By the hypothesis, there exists a homomorphism $\theta : M \to M$ where $\theta(M) \subseteq K'$ and $(1 - \theta)(M) \subseteq N'$. Hence $M$ is $\pi$-projective. $\square$

Lemma 2.24. Let $N$ be a submodule of $M$ such that $\frac{M}{N} \cong \frac{M}{K}$ with $N'$ is a coclosed submodule of $M$. If $K$ is a $N$-lifting ss-supplement, then $M = N \oplus K$.

Proof. Suppose that $K$ is an ss-supplement of $N$ in $M$. Then we have $M = N + K$, $N \cap K \ll K$ and $N \cap K$ is semisimple, and every homomorphism $\psi : M \to M_{N'}$ lifts to a homomorphism of $M$. Since $\frac{M}{N} \cong \frac{M}{N'}$, then an isomorphism $\xi : \frac{M}{N} \to \frac{M}{N'}$. We can similarly obtain rest of the proof follows from [4, Lemma 2.2]. $\square$

Corollary 2.25. Let $N$ be a coclosed submodule of $M$. If $K$ is a $N$-lifting ss-supplement in $M$, then $M = N \oplus K$.

Proof. Clear by Lemma 2.24. $\square$

In the following theorem, we give a characterization of ss-lifting modules via coclosed submodule from renaissance of [4, Theorem 2.4].

Theorem 2.26. Let $M$ be an amply ss-supplemented module. $M$ is ss-lifting if and only if every coclosed submodule $N$ of $M$ has a $N$-lifting ss-supplement.

Proof. Follows from Corollary 2.25 and [2, Theorem 1]. $\square$

3. SS-Discrete Modules and Strongly SS-Discrete Modules

In this section, we define notions of ss-discrete modules and strongly ss-discrete modules, and we obtain some elementary characterizations of these notions.

Definition 3.1. Let $M$ be a ss-supplemented module which is $\pi$-projective and direct projective, then $M$ is called a ss-discrete module. If $M$ is a ss-supplemented module which is self-projective, then $M$ is called a strongly ss-discrete module.

By this definition, we can obtain that if a module $M$ is ss-lifting and has the property (D2), then $M$ is a ss-discrete module.

Lemma 3.2. Let $N$ be an ss-supplement in $M$. $N$ is a direct summand of $M$ if and only if there exists an ss-supplement $K$ of $N$ in $M$ such that $K$ is a direct summand of $M$ and every homomorphism $f : M \to \frac{M}{N_{K}}$ can be lifted to a homomorphism $\varphi : M \to M$.

Proof. (⇒) Clear.

(⇐) Let $K$ be an ss-supplement of $N$ in $M$ with the stated property and $f : M \to \frac{M}{N_{K}}$ be the homomorphism defined by $f(a + b) = a + (N \cap K)$ for every $a \in N$ and $b \in K$. By the hypothesis, there exists a homomorphism $\varphi : M \to M$ such that $f$ can be lifted to the homomorphism $\varphi$. We have $M = K \oplus K'$ for some submodule $K'$ of $M$ and $K \cap N \ll N$ and $K \cap N$ is semisimple. By [6, Lemma 2.1], we have $M = \varphi(K') \oplus K$. Since $\varphi(K') \leq N$, then $N = \varphi(K') \oplus (N \cap K)$. This implies that $N \cap K = 0$. Thus $N$ is a direct summand of $M$. $\square$
Now we can characterize ss-lifting modules via the above lemma.

**Corollary 3.3.** Let $M$ be an amply ss-supplemented module. $M$ is ss-lifting if and only if for every ss-supplement $N$ in $M$ there is a direct summand ss-supplement $K$ of $N$ in $M$ such that every homomorphism $f : M \rightarrow \frac{M}{N \cap K}$ can be lifted to a homomorphism $\phi : M \rightarrow M$.

**Proposition 3.4.** Let $M$ be a module with $\text{Rad}(M) \subseteq \text{Soc}(M)$. If $M$ is a (quasi-)discrete module, then $M$ is a (quasi-)ss-discrete module.

*Proof.* Clear by [3, Theorem 20]. □

**Proposition 3.5.** Let $M$ be an ss-discrete module. Then every direct summand of $M$ is an ss-discrete module.

*Proof.* Let $N$ be a direct summand of $M$. Since $M$ is direct projective by [1, 4.22], we have $N$ is direct projective, i.e. $N$ has the property $(D_2)$. Since $M$ is ss-supplemented and $\pi$-projective, $M$ is ss-lifting by [2, Theorem 2]. Thus $N$ is ss-lifting by [2, Theorem 3] and so $N$ is an ss-discrete module. □

**Example 3.6.** Consider the self-projective $\mathbb{Z}$-module $M = \mathbb{Z}_4 \mathbb{Z}_4$. Since $M$ is ss-supplemented, $M$ is strongly ss-discrete.

**Proposition 3.7.** Let $M$ be a projective module. $M$ is a strongly ss-discrete module if and only if $M$ is a strongly discrete module and $\text{Rad}(M) \subseteq \text{Soc}(M)$.

*Proof.* Since $M$ be a projective module, $M$ is self-projective. The proof is obvious by [3, Theorem 20]. □

**Proposition 3.8.** Let $M$ be a strongly ss-discrete module. Then every direct summand of $M$ is a strongly ss-discrete module.

*Proof.* As self-projective modules are closed under direct summands, the proof clear by [2, Theorem 3]. □

**Theorem 3.9.** Let $\{M_i\}_{i \in I}$ be any finite family of $R$-modules and let $M = \bigoplus_{i \in I} M_i$. Suppose that for every $i \in I$, $\text{Rad}(M_i) \subseteq \text{Soc}(M_i)$. Then the following statements are equivalent.

1. $M$ is strongly ss-discrete;
2. (a) each $M_i$ is strongly discrete;
   (b) for each $i \in I$, $M_i$ is $M_j$-projective for $j \neq i$.

*Proof.* The proof similar to these of [1, 27.16] and [3, Theorem 20]. □

In the following corollary, we prove that strongly ss-discrete rings thanks to semiperfect ring.

**Corollary 3.10.** The following statements are equivalent for a ring $R$:

1. $R$ is ss-supplemented;
2. $R$ is semiperfect and $\text{Rad}(R) \subseteq \text{Soc}(R)$;
3. for any finite set $I$ and for each $i \in I$, every left $R$-module $M = \bigoplus_{i \in I} M_i$ where $M_i$ is a strongly local $M$-projective module;
4. $R$ is strongly ss-discrete.

*Proof.* Follows from [3, Theorem 41]. □

Finally we give the following hierarchy for any module:

strongly ss-discrete $\Rightarrow$ ss-discrete $\Rightarrow$ quasi-ss-discrete $\Rightarrow$ semi-ss-discrete $\Rightarrow$ ss-lifting
References