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Pricing American Put Option under Fractional Model

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Abstract. In this research work, our chief target is to elaborate an analytical solution of the fractional linear complement problem related to the evaluation of American put option generated by the fractional Black and Scholes model using the Adomian decomposition method, a numerical study is set forward to perform the theoretical result. Compared to the existent fractional model we prove that our result has a prompt convergence to the solution.

1. Introduction

American options are very popular in the worldwide financial markets. Their evaluation is a challenge. Indeed, it is one of the most thorny problems in option pricing literature. Compared to the European options, the American ones are more common. They allow more flexibility since they can be exercised at any time, between the current time and maturity. This issue has whetted the interest of both academics and traders.

Over the last few decades, several papers investigated the problem of the American pricing options generated by different models using many methods for instance [2], [3], [5], [9], [14], [15], [19],[23], [24], [26] and [27]. The early exercise feature inherent in American options was related to a free boundary condition problem in mathematics see [7], [12], and [16] , which was very complicated. For this reason, American options have no closed form solutions. The most famous one is the Black and Scholes model [6], which rests upon the concept that the stock price of the underlying asset is log-normally distributed conditional on the current stock price with a constant volatility.

The fractional calculus is invested in several research axes [4], [10], [25], [29]. For example, fractional derivation models have shown an ability to describe shape memory materials better than full derivation models. When a material is purely elastic, it is described by an integer derivation of order zero while when it is purely viscous it is described by an integer derivation of order one. Immediately, we can imagine describing a viscous-elastic material by a derivation between zero and one. This justifies the use of fractional derivation for this kind of material. We can refer the reader the Podlubny's book [25] where he introduces various physical models generated by differential equations with non-integer derivatives order. So out of mathematical curiosity and to get closer to the reality of the financial market we find ourselves obliged to use models based on fractional derivatives. Lately, it has been integrated in the Mathematical finance

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field [13],[17], [18],[20], and especially designed to resolve the pricing option problem. For instance [17], [18], [21], [22],[28] and [29] which are devoted for the evaluation of the European option. Refer back to [23], [30] and [31] for the American option. Numerous methods are elaborated so as to resolve linear and nonlinear fractional differential equations. Lastly, Chen et al. [8] have shown that the splitting method is a promising method and especially is more efficiency than the projected LU method. They have shown also that the optimal exercise price increases as the derivative order increases. In this research work, the Adomian decomposition method [1], [10]and [11] are used. This method is a powerful tool to compute analytical solutions in the linear or non-linear equations.

For an American option, under the hypotheses of Black and Scholes, with the exception of that of the payoff, the option can be exercised at any time between the date of purchase and the maturity date. By choosing the geometrical Brownian motion as dynamics of the underlying asset price:

$$dS_t = rS_t dt + \sigma S_t dW_t^S \tag{1}$$

where S_t is the underlying asset price at time t, σ is the volatility and r is the interest rate, both are supposed to be constants.

Using the Ito formula, we obtain the following differential equation:

$$\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 P}{\partial S^2} - rP = 0 \tag{2}$$

where P is the American put price. The Boundary conditions regarding time can be written as follows :

$$P(S_t, t) = max(K - S_t, 0) \qquad \textit{in the exercice case} \tag{3}$$

and

$$P(S_t, t) > max(K - S_t, 0) \qquad \textit{in the other case} \tag{4}$$

Therefore, the problem of pricing American put option comes down to a linear complementarity problem under the following system :

$$\begin{gathered}
\left(\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 P}{\partial S^2} - rP\right)(P - (K - S_t)) = 0 \\
\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 P}{\partial S^2} - rP \leq 0 \\
P - (K - S_t) \geq 0 \quad \forall t.
\end{gathered}$$

The outline of this work is as follows. In section 2, we present some preliminaries which are the basics of fractional calculus. In section 3, we establish our main results based on the Adomian decompositions. Section 4 includes the simulations and the numerical results. In the last section, we conclude.

2. Preliminaries

In what follows, we set forward some definitions related to the fractional calculus constituting the cornerstone of our work. For an organic presentation of the fractional theory, we can refer readers to Podlubny's book [25].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as,*

$$I_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau) d\tau$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-t} t^{\alpha-1} dt.$

Definition 2.2. *The Caputo fractional derivative is defined as*

$$D^{\alpha}_{t_0,t}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}\frac{d^m}{d\tau^m}x(\tau)d\tau,\ (m-1<\alpha<m).$$

When $0<\alpha<1$*, then the Caputo fractional derivative of order* α *of* f *reduces to*

$$D^{\alpha}_{t_0,t}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\frac{d}{d\tau}x(\tau)d\tau. \tag{5}$$

Note that the relation between Riemann-Liouville operator and Caputo fractional differential operator is expressed by the following equality:

$$I^{\alpha}_{t_0}D^{\alpha}_{t_0,t}f(t) = D^{-\alpha}_{t_0,t}D^{\alpha}_{t_0,t}f(t) = f(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}f^k(0), \qquad m-1<\alpha\le m. \tag{6}$$

Similar to the exponential function used in the solutions of integer-order differential systems, Mittag-Leffler function is frequently used in the solutions of fractional-order differential systems.

Definition 2.3. *The Mittag-Leffler function with two parameters is defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha>0$, $\beta>0$, $z\in C$.
When $\beta=1$*, we have* $E_{\alpha}(z)=E_{\alpha,1}(z)$*. Furthermore,* $E_{1,1}(z)=e^z$.

Adomian Method
The Adomian decomposition method consists to represented a solution $Z(x)$ of linear or nonlinear differential equations as a decomposition form named Adomian series:

$$Z(x) = \sum_{k=0}^{+\infty}Z_k(x)$$

where the components Z_k $k\ge 0$ are converging series and can be computed in a recursive manner.

3. Main results

In order to price American put option, we need to resolve the following fractional linear complementarity problem:

$$\left(\frac{\partial^{\alpha}P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma S^2\frac{\partial^2 P}{\partial S^2} - rP\right)(P-(K-S_t)) = 0$$
$$\left(\frac{\partial^{\alpha}P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma S^2\frac{\partial^2 P}{\partial S^2} - rP\right) \le 0$$
$$P-(K-S_t)\ge 0 \quad \forall t.$$

where $0 < \alpha \le 1$.

Under a constant volatility, to compute the value of American put price $P(S_t,V_t)$, we have to resolve the following nonlinear fractional differential equation:

$$D^{\alpha}_t P(S_t,V_t) + A[P](S_t,V_t) = 0 \qquad 0 < \alpha \le 1 \tag{7}$$

in the unbounded domain $\{(S_t, V_t) | S_t \geq 0, V_t \geq 0 \text{ and } t \in [0, T]\}$ with the initial value

$$P(S_0, V_0). \tag{8}$$

For boundary conditions, in the case of a put option, at maturity T with an exercise price K, the payoff function is

$$max(K - S_T, 0) \tag{9}$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t}$ and

$$A[P] = rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 P}{\partial S^2} - rP.$$

Theorem 3.1. *Let $(P_t)_{t \geq 0}$ be the American option price at time t. Under the hypotheses of the Black and Scholes model, at time l with $l < t$, the American put option price, which is the solution of the previous fractional linear complementarity problem, is equal to:*

$$P(S_l, V_l) = max(max(K - S_l, 0); e^{-r(t-l)} E_\alpha(-(t-l)^\alpha A[P(S_t, V_t)]))$$

where $0 < \alpha \leq 1$, E_α is the Mittag-Leffler function and $A[P] = rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma S^2 \frac{\partial^2 P}{\partial S^2} - rP$.

Proof

Multiplying equation (8)by the operator $D_t^{-\alpha}$ and taking into account of (7), we get

$$P(S_t, V_t) = P(S_l, V_l) + D_t^{-\alpha}(-A[P](S_t, V_t)). \tag{10}$$

Therefore, using the Adomian decomposition method in the domain $[l, t]$, the solution has the following form

$$P(S_t, V_t) = P(S_l, V_l) + \sum_{k=1}^{\infty} P_k(S_t, V_t). \tag{11}$$

By substituting (11) into (7), we have

$$\begin{aligned} P_{n+1}(S_t, V_t) &= D_t^{-\alpha}(-A[P_n](S_t, V_t)) \\ &= -A[P(S_l, V_l)]^n D_t^{-\alpha}(\frac{(t-l)^{n\alpha}}{\Gamma(1+n\alpha)}) \end{aligned} \tag{12}$$

Thus, we get

$$\begin{aligned} P(S_l, V_l) &= \sum_{k=0}^{\infty} (-1)^k \frac{(t-l)^{k\alpha}}{\Gamma(1+k\alpha)} A[P(S_t, V_t)]^k \\ &= E_\alpha(-(t-l)^\alpha A[P(S_t, V_t)]). \end{aligned} \tag{13}$$

The convergence of the power series of the fractional Black and Scholes model is guaranteed for a real and positive α. □

4. Simulations and Numerical Results

In this section, we perform the found results by presenting a numerical study of the pricing American put option under our proposed model for different values of the fractional order (see Tables 1, 3 and 5, Figures 1,2 and 3).

Table 1: Pricing American put option for different values of the fractional model compared to the classical binomial model (1000 time-steps) and splitting model $Spl(\alpha = 0.9)$, as a function of moneyness, (K=100, $\sigma = 0.2$, r=0.05, T=1/12)

S/K	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
BIN1000	20	15	10	4.627	2.131	1.093	0.094	0.051	0.001
$Spl(\alpha = 0.9)$	20	15	10	5.088	2.379	1.283	0.123	0.061	0.002
$P(\alpha = 1)$	20	15	10	4.598	2.077	1.125	0.117	0.043	0.0017
$P(\alpha = 0.9)$	20	15	10	5.127	2.354	1.274	0.129	0.056	0.003
$P(\alpha = 0.7)$	20	15	10.05	5.202	2.253	1.301	0.154	0.062	0.0034
$P(\alpha = 0.5)$	20	15	10.23	5.291	2.469	1.347	0.286	0.095	0.0068

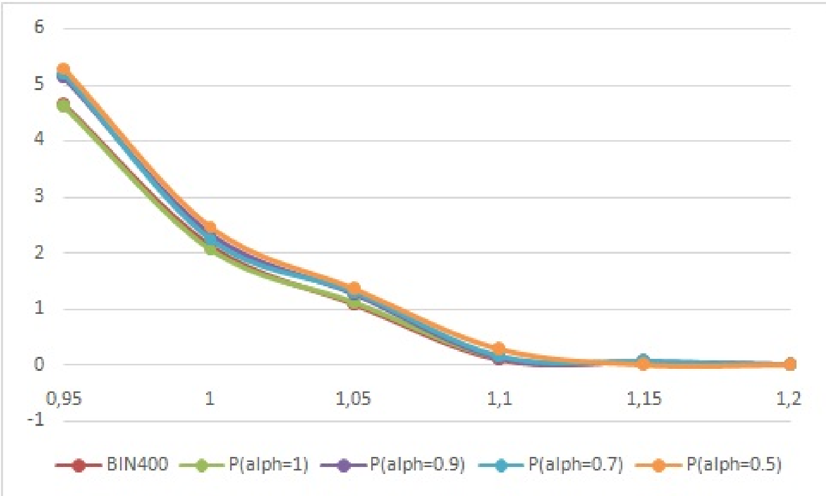

Figure 1: *Fig.1 Pricing American put option for different values of the fractional model compared to the classical binomial model (1000 time-steps), as a function of moneyness, (K=100, $\sigma = 0.2$, r=0.05, T=1/12).*

We have investigated the American put price as a function of moneyness. As data, we have considered K=100, $\sigma = 0.2$, r=0.05. For the maturity time, we considered three cases: the first one is equal to 1/12, the second equals 1/4 and finally the third equals 1/2.

We take as "true" reference price, the one issued of the Binomial model [3] with 1000 steps. From the obtained results, all curves have the same profiles as the one related to the binomial model, which goes in good accordance with the option's theory.

We notice that, when moneyness is located near to one, there is slight difference between the obtained results and the binomial model. In the otherwise, the difference between the premium of the American put option is almost useless, which proves in terms of accuracy, for every value of the fractional order, all results are almost the same(see Tables 2, 4 and 6).

In their work [8], Chen et al. prove that the splitting method is promising and especially is more efficient than the projected LU method. In correlation with our method, Chen et al. prove that the optimal exercise price increases as the derivative order increases. So, to compare with our solution we implement the results of Chen et al. for $\alpha = 0.9$ noted by $Spl(\alpha = 0.9)$.

Now we make a comparison between our method based on the Adomian decomposition and the method presented in the work of Chen et al. [8] based on the splitting method. In the first we must indicate that the two methods are easy to implement nevertheless the Adomian decomposition has a rapid convergence (CPU time) to the solution than the splitting method (see Table 7), and this is logical since the solution of the first method is written as a Mittag-Leffler function contrariwise to the second method which based on iterative functions.

Table 2: The error between the value of American put option under the fractional model for different values of α and the classical binomial model (1000 time-steps), as a function of moneyness, (K=100, σ = 0.2, r=0.05, T=1/12)

S/K	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
$\lvert BIN1000 - Spl(\alpha = 0.9)\rvert$	0	0	0	0.461	0.248	0.19	0.029	0.01	0.001
$\lvert BIN1000 - P(\alpha = 1)\rvert$	0	0	0	0.029	0.054	0.032	0.023	0.008	0.0006
$\lvert BIN1000 - P(\alpha = 0.9)\rvert$	0	0	0	0.5	0.223	0.181	0.035	0.005	0.0019
$\lvert BIN1000 - P(\alpha = 0.7)\rvert$	0	0	0.05	0.575	0.122	0.208	0.06	0.011	0.0023
$\lvert BIN1000 - P(\alpha = 0.5)\rvert$	0	0	0.23	0.664	0.338	0.254	0.192	0.044	0.0057

Table 3: Pricing American put option for different values of the fractional model compared to the classical binomial model (1000 time-steps) and splitting model $Spl(\alpha = 0.9)$, as a function of moneyness, (K=100, σ = 0.2, r=0.05, T=1/4)

S/K	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
BIN1000	20	15	10.092	5.015	3.429	1.912	0.722	0.181	0.041
$Spl(\alpha = 0.9)$	20	15.01	10.203	5.021	3.470	1.986	0.745	0.213	0.075
$P(\alpha = 1)$	20	15	10.156	5.003	3.451	1.857	0.736	0.186	0.047
$P(\alpha = 0.9)$	20	15.012	10.184	5.038	3.492	2.094	0.759	0.197	0.069
$P(\alpha = 0.7)$	20	15.057	10.197	5.109	3.617	2.251	0.883	0.244	0.089
$P(\alpha = 0.5)$	20	15.113	10.175	5.483	3.983	2.475	1.088	0.319	0.095

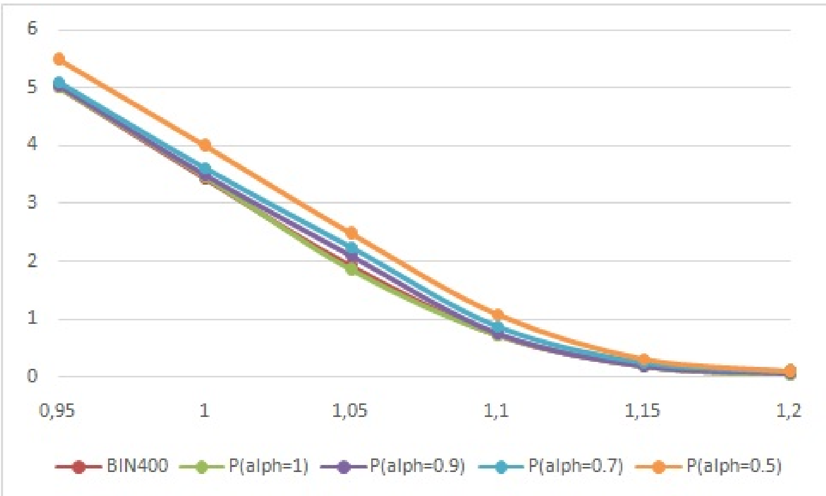

Figure 2: *Fig.5 Pricing American put option for different values of the fractional model compared to the classical binomial model (1000 time-steps), as a function of moneyness, (K=100, σ = 0.2, r=0.05, T=1/4).*

5. Conclusion

Investing the Adomian decomposition, we provide and show the rapid convergence (CPU time) of the power series related to the pricing American put option problem under our proposed model. In order to perform the theoretical results, we set forward numerical solutions for different values of the fractional order. All results are correlated with the American option's theory.

Table 4: The error between the value of American put option under the fractional model for different values of α and the classical binomial model (1000 time-steps), as a function of moneyness, (K=100, σ = 0.2, r=0.05, T=1/4)

S/K	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
$\lvert BIN1000 - Spl(\alpha = 0.9)\rvert$	0	0.01	0.111	0.006	0.041	0.074	0.023	0.032	0.034
$\lvert BIN1000 - P(\alpha = 1)\rvert$	0	0	0.064	0.012	0.022	0.055	0.014	0.005	0.006
$\lvert BIN1000 - P(\alpha = 0.9)\rvert$	0	0.012	0.092	0.023	0.063	0.182	0.037	0.016	0.028
$\lvert BIN1000 - P(\alpha = 0.7)\rvert$	0	0.057	0.105	0.094	0.188	0.339	0.161	0.063	0.048
$\lvert BIN1000 - P(\alpha = 0.5)\rvert$	0	0.113	0.083	0.468	0.554	0.563	0.366	0.138	0.054

Table 5: Pricing American put option for different values of the fractional model compared to the classical binomial model (1000 time-steps) and splitting model $Spl(\alpha = 0.9)$, as a function of moneyness, (K=100, σ = 0.2, r=0.05, T=1/2)

S/K	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
BIN1000	20	15	10.155	5.252	3.482	2.009	0.769	0.194	0.0688
$Spl(\alpha = 0.9)$	20	15.054	10.206	5.294	3.596	2.149	0.813	0.229	0.125
$P(\alpha = 1)$	20	15.003	10.171	5.277	3.475	2.059	0.788	0.209	0.0696
$P(\alpha = 0.9)$	20	15.081	10.193	5.316	3.554	2.177	0.791	0.237	0.109
$P(\alpha = 0.7)$	20	15.099	10.217	5.441	3.702	2.301	0.933	0.264	0.157
$P(\alpha = 0.5)$	20	15.131	10.379	5.509	3.896	2.593	1.025	0.351	0,232

Table 6: The error between the value of American put option under the fractional model for different values of α and the classical binomial model (1000 time-steps), as a function of moneyness, (K=100, σ = 0.2, r=0.05, T=1/2)

S/K	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
$\lvert BIN1000 - Spl(\alpha = 0.9)\rvert$	0	0.054	0.051	0.042	0.114	0.14	0.044	0.035	0.0562
$\lvert BIN1000 - P(\alpha = 1)\rvert$	0	0.003	0.016	0.025	0.007	0.05	0.019	0.015	0.0008
$\lvert BIN1000 - P(\alpha = 0.9)\rvert$	0	0.081	0.038	0.064	0.072	0.168	0.022	0.043	0.0402
$\lvert BIN1000 - P(\alpha = 0.7)\rvert$	0	0.099	0.062	0.189	0.22	0.292	0.164	0.07	0.0882
$\lvert BIN1000 - P(\alpha = 0.5)\rvert$	0	0.131	0.224	0.257	0.414	0.584	0.256	0.157	0.1632

Table 7: Calculation time for the various methods, (K=100, σ = 0.2, r=0.05, T=1/2)

.	$BIN1000$	$Spl(\alpha = 0.9)\rvert$	$P(\alpha = 1)$
$CPUtime$	55s	247s	62s

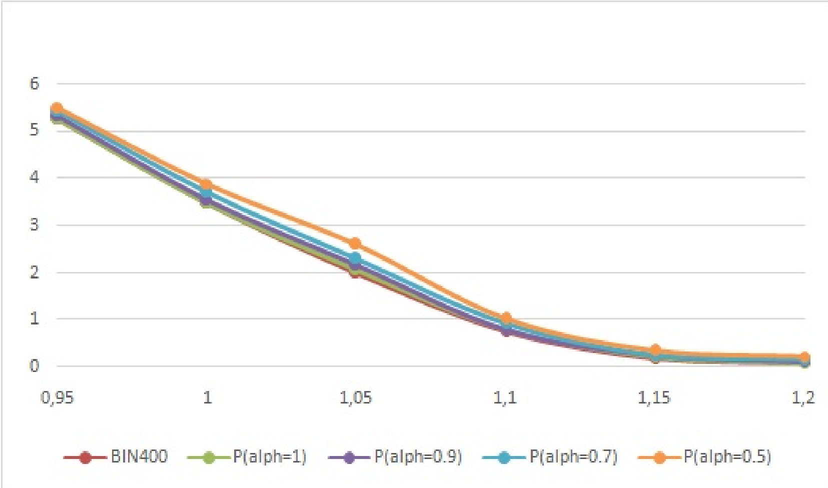

Figure 3: *Fig.5 Pricing American put option for different values of the fractional model compared to the classical binomial model (1000 time-steps), as a function of moneyness, (K=100,* σ = 0.2*, r=0.05, T=1/2).*

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