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Fuzzy Points Based Betweenness Relations in L-Convex Spaces

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Abstract. As topology-like mathematical structures, convex structures can be characterized by betweenness relations via (restricted) hull operators in convex spaces. In a topological approach, the aim of this paper is to present the fuzzy counterpart of betweenness relations based on fuzzy points in fuzzy convex spaces. Concretely, the notion of L-betweenness relations via restricted L-hull operators is introduced. Firstly, it is proved that L-betweenness relations are categorically isomorphic to restricted L-hull operators and L-remotehood systems, respectively. Secondly, it is shown that L-betweenness relations from two perspectives of restricted L-hull operators and L-remotehood systems are unified. Finally, a new type of restricted L-hull operators in accordance with L-betweenness relations is proposed and the relationship between two types of restricted L-hull operators is displayed.

1. Introduction

Convexity, which is intriguing the extremum problems in area of applied mathematics, has been showing its great importance. In 1993, M. van de Vel collected the theory of convexity systematically in his famous book [19]. A convex structure on a set X is defined to be a subset $\mathcal{E}$ of 2^X which contains both the empty set $\emptyset$ and X itself and which is closed under arbitrary intersections and directed unions. A convex structure can be completely determined by its hull operator or even by its effect on finite sets (restricted hull operator). In fact, a point which is in the hull of a finite set can be regarded as being between this set. That is, restricted hull operators and betweenness relations can be determined by each other.

With the development of fuzzy set theory, the notion of convex structures has been extended to the fuzzy case. Up to now, there have been three typical kinds of fuzzy convex structures, including L-convex structures [5, 13], M-fuzzifying convex structures [17] and (L, M)-fuzzy convex structures [4, 18]. Many researchers studied fuzzy convex structures from different aspects, such as fuzzy hull operators [6, 10, 16], fuzzy (fuzzifying) interval operators [20, 22, 33, 34], categorical properties [9, 23, 26, 31], convergence properties [7, 30], bases and subbases [8, 11, 32], degree presentations [3, 24, 29], topological convexity [21, 28] and geometric properties [25, 27]. In particular, Shi and Li [16] extended the concept of restricted hull operators to the M-fuzzifying case, namely, restricted M-fuzzifying hull operators, to characterize

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M-fuzzifying convex structures. More recently, Shen and Shi [15] proposed the concept of restricted L-hull operators to characterize L-convex structures from a categorical aspect.

Considering the fuzzy counterpart of betweenness relations, Shi and Li [16] first introduced the notion of M-fuzzifying betweenness relations to describe the fuzzy relations between classical points and finite subsets and then investigated its categorical relationship with restricted M-fuzzifying hull operators. By this motivation, we will consider fuzzy betweenness relations in the framework of L-convex spaces. In this situation, we will introduce fuzzy points based betweenness relations to describe the relations between fuzzy points and fuzzy finite L-subsets, which will be called L-betweenness relations in this paper. Also, we will induce L-betweenness relations by means of restricted L-hull operators. Moreover, we will induce L-betweenness relations from the aspect of L-remotehood systems, which can be used to characterize L-convex structures. Finally, we will discuss the unities of L-betweenness relations induced by restricted L-hull operators and L-remotehood systems.

The paper is organized as follows. In Section 2, we recall some necessary concepts and results. In Sections 3 and 4, we first propose the concept of L-betweenness relations and then establish its categorical relationship with restricted L-hull operators and L-remotehood systems, respectively. In Section 5, we prove that both of the approaches of restricted L-hull operators and L-remotehood systems to L-betweenness relations are unified. Correspondingly, a new type of restricted L-hull operators is proposed and the relationship between the two types of restricted L-hull operators is displayed.

2. Preliminaries

Let L be a complete lattice. The largest element and the smallest element in L are denoted by $\top$ and $\bot$, respectively. A nonempty subset $D \subseteq L$ is called directed (in symbols $D \subseteq^{dir} L$) if for each $a, b \in D$, there exists $c \in D$ such that $a, b \leqslant c$. In particular, we use the notation $x = \bigvee^{\uparrow} D$ to express that the set D is directed and x is its least upper bound. For $x, y \in L$, x is way below y (in symbols $x \ll y$) if for any $D \subseteq^{dir} L$ such that $\bigvee^{\uparrow} D$ exists, $y \leqslant \bigvee^{\uparrow} D$ always implies the existence of some $d \in D$ with $x \leqslant d$. A complete lattice L is called continuous if it satisfies the axiom of approximation: $(\forall x \in L)\ \ x = \bigvee^{\uparrow} \Downarrow x$, where $\Downarrow x = \{u \in L \mid u \ll x\}$ (See [2]).

Throughout this article, L is always assumed to be a continuous lattice.

For a nonempty set X, we write 2^X and $2^{(X)}$ for the powerset of X and for the collection of all finite subsets of X, respectively. Each mapping $A : X \longrightarrow L$ is called an L-subset on X, and the collection of all L-subsets is denoted by L^X. L^X is also a continuous lattice by defining $\leqslant$ on L^X in a pointwise way. The way below relation on L^X is also denoted by $\ll$, if no confusion will rise. Further, for each $A \in L^X$, $\Downarrow A = \{F \in L^X \mid F \ll A\}$ is directed and $A = \bigvee^{\uparrow} \Downarrow A$. The largest element and the smallest element in L^X are denoted by $\underline{\top}$ and $\underline{\bot}$, respectively. We call an L-subset A on X finite if its support set $\{x \in X \mid A(x) \neq \bot\}$ is finite. Let $L^{(X)}$ denote the collection of all finite L-subsets on X. The set of all fuzzy points x_λ (i.e., an L-subset $A \in L^X$ such that $A(x) = \lambda \neq \bot$ and $A(y) = \bot$ for $y \neq x$) is denoted by $J(L^X)$.

Given a mapping $f : X \longrightarrow Y$, define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $f_L^{\leftarrow}(B) = B \circ f$ for $B \in L^Y$, respectively.

We give some useful properties of the way below relation $\ll$ between L-subsets on X (refer to [15]).

Proposition 2.1. *The following statements hold for any $A, B \in L^X$ and any $\{D_i \mid i \in I\} \subseteq^{dir} L^X$:*

(1) *if $A \leqslant B$, then $\Downarrow A \subseteq \Downarrow B$;*

(2) $\Downarrow \bigvee^{\uparrow}_{i\in I} D_i = \bigcup_{i\in I} \Downarrow D_i$.

Proposition 2.2. *Let $f : X \longrightarrow Y$ be a mapping and let $A \in L^X$.*

(L1) *$F \ll A$ implies $f_L^{\rightarrow}(F) \ll f_L^{\rightarrow}(A)$;*

(L2) *$F \ll f_L^{\leftarrow}(H)$ if and only if $f_L^{\rightarrow}(F) \ll H$.*

Next, we recall briefly some basic definitions and results on L-convex spaces.

Definition 2.3. (Maruyama [5] and Rose [13]) A subset $\mathcal{C}$ of L^X is called an L-convex structure on X if it satisfies the following conditions:

(LC1) $\underline{\perp}, \underline{\top} \in \mathcal{C}$;
(LC2) if $\{C_i \mid i \in I\} \subseteq \mathcal{C}$ is nonempty, then $\bigwedge_{i\in I} C_i \in \mathcal{C}$;
(LC3) if $\{C_j \mid j \in J\} \subseteq^{dir} \mathcal{C}$, then $\bigvee^{\uparrow}_{j\in J} C_j \in \mathcal{C}$.

For an L-convex structure $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is called an L-convex space.

Definition 2.4. (Pang and Shi [9]) A mapping $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ between L-convex spaces is called L-convexity-preserving (L-CP, in short) provided that for any $B \in \mathcal{C}_Y$, $f_L^{\leftarrow}(B) \in \mathcal{C}_X$.

The category whose objects are L-convex spaces and whose morphisms are L-CP mappings will be denoted by L-**CS**.

Definition 2.5. (Shen and Shi [15]) A mapping $\mathfrak{h} : L^{(X)} \longrightarrow L^X$ is called a restricted L-hull operator on X if it satisfies the following conditions:

(LRH1) $\mathfrak{h}(\underline{\perp}) = \underline{\perp}$;
(LRH2) for any $F \in L^{(X)}$, $F \leqslant \mathfrak{h}(F)$;
(LRH3) for any $F \in L^{(X)}$, $G \ll \mathfrak{h}(F)$ implies $\mathfrak{h}(G) \leqslant \mathfrak{h}(F)$;
(LRH4) for any $F \in L^{(X)}$, $\mathfrak{h}(F) = \bigvee^{\uparrow}_{G \ll F} \mathfrak{h}(G)$.

For a restricted L-hull operator on X, the pair $(X, \mathfrak{h})$ is called a restricted L-hull space.

Definition 2.6. (Shen and Shi [15]) A mapping $f : (X, \mathfrak{h}_X) \longrightarrow (Y, \mathfrak{h}_Y)$ between restricted L-hull spaces is called L-hull-preserving (L-HP, in short) if for any $F \in L^{(X)}$, $f_L^{\rightarrow}(\mathfrak{h}_X(F)) \leqslant \mathfrak{h}_Y(f_L^{\rightarrow}(F))$.

The category whose objects are restricted L-hull spaces and whose morphisms are L-HP mappings will be denoted by L-**RHS**.

For notions on category theory, we refer to [1, 12].

3. *L*-Betweenness Relations from Restricted *L*-Hull Operators

In the classical case, a betweenness relation is a subset $\mathcal{B} \subseteq 2^{(X)} \times X$ and a restricted hull operator is a mapping $h : 2^{(X)} \longrightarrow 2^X$ which satisfies certain axiomatic conditions, respectively. Further, a restricted hull operator h can induce a betweenness relation $\mathcal{B}^h$ in a natural way [19]:

$$(F, x) \in \mathcal{B}^h \Longleftrightarrow x \in h(F).$$

In the theory of L-convex structures, we usually replace the points by fuzzy points and replace (finite) subsets by (finite) L-subsets. This results in the definition of restricted L-hull operators $\mathfrak{h} : L^{(X)} \longrightarrow L^X$ in [15]. By the above-mentioned analysis, what is the fuzzy counterpart of a betweenness relation induced by a restricted L-hull operator? It should be a subset $\mathfrak{B}^{\mathfrak{h}} \subseteq L^{(X)} \times J(L^X)$ and

$$(F, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}} \Longleftrightarrow \text{“} x_\lambda \in \mathfrak{h}(F) \text{”}.$$

Here, x_λ and $\mathfrak{h}(F)$ are both L-subsets and thus there is no belonging relation between them. In order to deal with “$x_\lambda \in \mathfrak{h}(F)$”, we usually adopt “$x_\lambda \leqslant \mathfrak{h}(F)$”. Hence we obtain

$$(F, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}} \Longleftrightarrow x_\lambda \leqslant \mathfrak{h}(F).$$

However, what kind of conditions should $\mathfrak{B}^{\mathfrak{h}}$ satisfy? To this end, we first present the following proposition.

Proposition 3.1. *Let $(X, \mathfrak{h})$ be a restricted L-hull space and define $\mathfrak{B}^{\mathfrak{h}} \subseteq L^{(X)} \times J(L^X)$ as follows:*

$$\mathfrak{B}^{\mathfrak{h}} = \{(F, x_\lambda) \in L^{(X)} \times J(L^X) \mid x_\lambda \leqslant \mathfrak{h}(F)\}.$$

Then $\mathfrak{B}^{\mathfrak{h}}$ satisfies the following conditions:

(LB1) $(\underline{\perp}, x_\lambda) \notin \mathfrak{B}^{\mathfrak{h}}$;
(LB2) $\forall x_\lambda \leqslant F,\ (F, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}}$;
(LB3) *if* $(G, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}}$ *and* $(F, y_\mu) \in \mathfrak{B}^{\mathfrak{h}}$ *for all* $y_\mu \leqslant G$*, then* $(F, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}}$;
(LB4) $(F, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}}$ *if and only if* $\forall \mu \ll \lambda, \exists G \ll F$ *s.t.* $(G, x_\mu) \in \mathfrak{B}^{\mathfrak{h}}$;
(LB5) $(F, x_{\bigvee_{i\in I}\lambda_i}) \in \mathfrak{B}^{\mathfrak{h}}$ *if and only if* $\forall i \in I, (F, x_{\lambda_i}) \in \mathfrak{B}^{\mathfrak{h}}$.

Proof. (LB1) and (LB2) are straightforward.

(LB3) Suppose that $(G, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}}$ and $(F, y_\mu) \in \mathfrak{B}^{\mathfrak{h}}$ for all $y_\mu \leqslant G$. Then $x_\lambda \leqslant \mathfrak{h}(G)$ and $y_\mu \leqslant \mathfrak{h}(F)$ for all $y_\mu \leqslant G$. This implies $G \leqslant \mathfrak{h}(F)$. Then

$$\begin{aligned} x_\lambda &\leqslant \mathfrak{h}(G) = \textstyle\bigvee^{\uparrow}_{H\ll G} \mathfrak{h}(H) \ \text{ (by (LRH4))} \\ &\leqslant \textstyle\bigvee^{\uparrow}_{H\ll \mathfrak{h}(F)} \mathfrak{h}(H) \\ &\leqslant \mathfrak{h}(F), \ \text{ (by (LRH3))} \end{aligned}$$

which means $(F, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}}$.

(LB4) By (LRH4), it follows that

$$\begin{aligned} (F, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}} &\iff x_\lambda \leqslant \mathfrak{h}(F) = \textstyle\bigvee^{\uparrow}_{G\ll F} \mathfrak{h}(G) \\ &\iff \forall \mu \ll \lambda, \exists G \ll F \ s.t.\ x_\mu \leqslant \mathfrak{h}(G) \\ &\iff \forall \mu \ll \lambda, \exists G \ll F \ s.t.\ (G, x_\mu) \in \mathfrak{B}^{\mathfrak{h}}. \end{aligned}$$

(LB5) Suppose that $F \in L^{(X)}$ and $\{x_{\lambda_i} \mid i \in I\} \subseteq J(L^X)$. Then we have

$$\begin{aligned} (F, x_{\bigvee_{i\in I}\lambda_i}) \in \mathfrak{B}^{\mathfrak{h}} &\iff x_{\bigvee_{i\in I}\lambda_i} \leqslant \mathfrak{h}(F) \\ &\iff \forall i \in I, x_{\lambda_i} \leqslant \mathfrak{h}(F) \\ &\iff \forall i \in I, (F, x_{\lambda_i}) \in \mathfrak{B}^{\mathfrak{h}}. \end{aligned}$$

This completes the proof. □

By means of (LB1)–(LB5), we will introduce the fuzzy counterpart of betweenness relations, which will be called L-betweenness relations. Now we give the axiomatic definition.

Definition 3.2. An L-betweenness relation on X is a subset $\mathfrak{B} \subseteq L^{(X)} \times J(L^X)$ which satisfies (LB1)–(LB5). For an L-betweenness relation $\mathfrak{B}$ on X, the pair $(X, \mathfrak{B})$ is called an L-betweenness space.

Next, we give some examples of L-betweenness relations on X.

Example 3.3. (1) Let X be any nonempty set. Define $\mathfrak{B} = \{(F, x_\lambda) \in L^{(X)} \times J(L^X) \mid x_\lambda \leqslant F\}$. It is trivial that $\mathfrak{B}$ is an L-betweenness relation on X.

(2) Let X be a poset. Define $\mathfrak{B} = \{(F, x_\lambda) \in L^{(X)} \times J(L^X) \mid \lambda \leqslant \bigvee(F(x_1) \wedge F(x_2)), x_1, x_2 \in X, x_1 \leqslant x \leqslant x_2\}$. Then $\mathfrak{B}$ is an L-betweenness relation on X.

(3) Let X be a vector space over a totally ordered filed $\mathbb{K}$. Define $\mathfrak{B} = \{(F, x_\lambda) \in L^{(X)} \times J(L^X) \mid \lambda \leqslant \bigvee(F(x_1) \wedge \cdots \wedge F(x_n)), x_i \in X, x = \sum_{i=1}^n t_i x_i, \sum_{i=1}^n t_i = 1, n \in \mathbb{Z}^+, t_i \in \mathbb{K}, t_i \geqslant 0\ (i = 1, 2, \cdots n)\}$. Then $\mathfrak{B}$ is an L-betweenness relation on X.

In order to construct the category of L-betweenness spaces, we further introduce the following definition.

Definition 3.4. A mapping $f : (X, \mathfrak{B}_X) \longrightarrow (Y, \mathfrak{B}_Y)$ between L-betweenness spaces is called L-betweenness-preserving (L-BP, in short) provided that

$$\forall F \in L^{(X)}, \forall x_\lambda \in J(L^X),\ (F, x_\lambda) \in \mathfrak{B}_X \text{ implies } (f_L^{\rightarrow}(F), f(x)_\lambda) \in \mathfrak{B}_Y.$$

It is easy to check that all L-betweenness spaces as objects and all L-BP mappings as morphisms form a category, denoted by L-**Bet**.

Considering L-HP mappings between restricted L-hull spaces, we have

Proposition 3.5. *If $f : (X, \mathfrak{h}_X) \longrightarrow (Y, \mathfrak{h}_Y)$ is L-HP, then $f : (X, \mathfrak{B}^{\mathfrak{h}_X}) \longrightarrow (Y, \mathfrak{B}^{\mathfrak{h}_Y})$ is L-BP.*

Proof. Since $f : (X, \mathfrak{h}_X) \longrightarrow (Y, \mathfrak{h}_Y)$ is L-HP, it follows that $\mathfrak{h}_X(F) \leqslant f_L^{\leftarrow}(\mathfrak{h}_Y(f_L^{\rightarrow}(F)))$ for any $F \in L^{(X)}$. Then for each $x_\lambda \in J(L^X)$, we have

$$\begin{aligned}
(F, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}_X} &\iff x_\lambda \leqslant \mathfrak{h}_X(F)\\
&\implies x_\lambda \leqslant f_L^{\leftarrow}(\mathfrak{h}_Y(f_L^{\rightarrow}(F)))\\
&\iff f(x)_\lambda \leqslant \mathfrak{h}_Y(f_L^{\rightarrow}(F))\\
&\iff (f_L^{\rightarrow}(F), f(x)_\lambda) \in \mathcal{B}^{\mathfrak{h}_Y},
\end{aligned}$$

as desired. □

By Propositions 3.1 and 3.5, we construct a functor $\mathbb{F} : L\text{-}\mathbf{RHS} \longrightarrow L\text{-}\mathbf{Bet}$ defined by

$$\mathbb{F}(X, \mathfrak{h}) = (X, \mathfrak{B}^{\mathfrak{h}}) \text{ and } \mathbb{F}(f) = f.$$

Conversely, we will construct restricted L-hull operators via L-betweenness relations.

Given an L-betweenness relation $\mathfrak{B}$ on X, define $\mathfrak{h}^{\mathfrak{B}} : L^{(X)} \longrightarrow L^X$ as follows:

$$\forall F \in L^{(X)},\ \mathfrak{h}^{\mathfrak{B}}(F) = \bigvee \{x_\lambda \in J(L^X) \mid (F, x_\lambda) \in \mathfrak{B}\}.$$

In order to show that $\mathfrak{h}^{\mathfrak{B}}$ is a restricted L-hull operator, we first give the following lemma.

Lemma 3.6. *Let $(X, \mathfrak{B})$ be an L-betweenness space. Then:*

(1) *$\mu \leqslant \lambda$ and $(F, x_\lambda) \in \mathfrak{B}$ imply $(F, x_\mu) \in \mathfrak{B}$;*

(2) *$x_\lambda \leqslant \mathfrak{h}^{\mathfrak{B}}(F)$ if and only if $(F, x_\lambda) \in \mathfrak{B}$.*

Proof. (1) It follows immediately from (LB5).

(2) It suffices to show the necessity. Suppose that $x_\lambda \leqslant \mathfrak{h}^{\mathfrak{B}}(F)$, i.e.,

$$\lambda \leqslant \mathfrak{h}^{\mathfrak{B}}(F)(x) = \bigvee \{\mu \in L \mid (F, x_\mu) \in \mathfrak{B}\}.$$

Denote $U = \{\mu \in L \mid (F, x_\mu) \in \mathfrak{B}\}$. By (LB5), we have $(F, x_{\bigvee U}) \in \mathfrak{B}$. Since $\lambda \leqslant \bigvee U$, it follows from (1) that $(F, x_\lambda) \in \mathfrak{B}$. □

Proposition 3.7. *Let $(X, \mathfrak{B})$ be an L-betweenness space. Then $\mathfrak{h}^{\mathfrak{B}}$ is a restricted L-hull operator on X.*

Proof. It suffices to show that $\mathfrak{h}^{\mathfrak{B}}$ satisfies (LRH1)–(LRH4).

(LRH1) $\mathfrak{h}^{\mathfrak{B}}(\underline{\perp}) = \bigvee \{x_\lambda \in J(L^X) \mid (\underline{\perp}, x_\lambda) \in \mathfrak{B}\} = \bigvee \emptyset = \underline{\perp}$.

(LRH2) For each $F \in L^{(X)}$, take each $x_\lambda \in J(L^X)$ with $x_\lambda \leqslant F$. By (LB2), we have $(F, x_\lambda) \in \mathfrak{B}$. Then it follows from Lemma 3.6 (2) that $x_\lambda \leqslant \mathfrak{h}^{\mathfrak{B}}(F)$. By the arbitrariness of x_λ, we have $F \leqslant \mathfrak{h}^{\mathfrak{B}}(F)$.

(LRH3) Suppose that $F, G \in L^{(X)}$ with $G \ll \mathfrak{h}^{\mathfrak{B}}(F)$. Take each $x_\lambda \in J(L^X)$ such that $x_\lambda \leqslant \mathfrak{h}^{\mathfrak{B}}(G)$. By Lemma 3.6 (2), we have $(G, x_\lambda) \in \mathfrak{B}$. Then for each $y_\mu \in J(L^X)$ such that $y_\mu \leqslant G$, it follows that $y_\mu \leqslant \mathfrak{h}^{\mathfrak{B}}(F)$ i.e., $(F, y_\mu) \in \mathfrak{B}$. This shows $(F, y_\mu) \in \mathfrak{B}$ for all $y_\mu \leqslant G$. By (LB3), we obtain $(F, x_\lambda) \in \mathfrak{B}$, i.e., $x_\lambda \leqslant \mathfrak{h}^{\mathfrak{B}}(F)$. By the arbitrariness of x_λ, we have $\mathfrak{h}^{\mathfrak{B}}(G) \leqslant \mathfrak{h}^{\mathfrak{B}}(F)$.

(LRH4) Take each $F \in L^{(X)}$ and $x_\lambda \in J(L^X)$. It follows from Lemma 3.6 (2) and (LB4) that

$$\begin{array}{rcl} x_\lambda \leqslant \mathfrak{h}^{\mathfrak{B}}(F) & \Longleftrightarrow & (F, x_\lambda) \in \mathfrak{B} \\ & \Longleftrightarrow & \forall \mu \ll \lambda, \exists G \ll F \ s.t.\ (G, x_\mu) \in \mathfrak{B} \\ & \Longleftrightarrow & \forall \mu \ll \lambda, \exists G \ll F \ s.t.\ x_\mu \leqslant \mathfrak{h}^{\mathfrak{B}}(G) \\ & \Longleftrightarrow & x_\lambda \leqslant \bigvee^{\uparrow}_{G \ll F} \mathfrak{h}^{\mathfrak{B}}(G). \end{array}$$

Therefore, $\mathfrak{h}^{\mathfrak{B}}(F) = \bigvee^{\uparrow}_{G \ll F} \mathfrak{h}^{\mathfrak{B}}(G)$. $\square$

Proposition 3.8. *If $f : (X, \mathfrak{B}_X) \longrightarrow (Y, \mathfrak{B}_Y)$ is L-BP, then $f : (X, \mathfrak{h}^{\mathfrak{B}_X}) \longrightarrow (Y, \mathfrak{h}^{\mathfrak{B}_Y})$ is L-HP.*

Proof. Since $f : (X, \mathfrak{B}_X) \longrightarrow (Y, \mathfrak{B}_Y)$ is L-BP, it follows that $(F, x_\lambda) \in \mathfrak{B}_X$ implies $(f_L^{\rightarrow}(F), f(x)_\lambda) \in \mathfrak{B}_Y$. Then, for any $F \in L^{(X)}$, we have

$$\begin{array}{rcl} f_L^{\rightarrow}(\mathfrak{h}^{\mathfrak{B}_X}(F)) & = & f_L^{\rightarrow}(\bigvee\{x_\lambda \in J(L^X) \mid (F, x_\lambda) \in \mathfrak{B}_X\}) \\ & = & \bigvee\{f(x)_\lambda \in J(L^Y) \mid (F, x_\lambda) \in \mathfrak{B}_X\} \\ & \leqslant & \bigvee\{f(x)_\lambda \in J(L^Y) \mid (f_L^{\rightarrow}(F), f(x)_\lambda) \in \mathfrak{B}_Y\} \\ & \leqslant & \bigvee\{y_\mu \in J(L^Y) \mid (f_L^{\rightarrow}(F), y_\mu) \in \mathfrak{B}_Y\} \\ & = & \mathfrak{h}^{\mathfrak{B}_Y}(f_L^{\rightarrow}(F)), \end{array}$$

as desired. $\square$

By Propositions 3.7 and 3.8, we construct a functor $\mathbb{G} : L$-**Bet**$\longrightarrow L$-**RHS** defined by

$$\mathbb{G}(X, \mathfrak{B}) = (X, \mathfrak{h}^{\mathfrak{B}}) \text{ and } \mathbb{G}(f) = f.$$

Theorem 3.9. *L-**Bet** and L-**RHS** are isomorphic.*

Proof. It suffices to verify that (1) $\mathfrak{h}^{\mathcal{B}^{\mathfrak{h}}} = \mathfrak{h}$ and (2) $\mathfrak{B}^{\mathfrak{h}^{\mathfrak{B}}} = \mathfrak{B}$ for any restricted L-hull space $(X, \mathfrak{h})$ and any L-betweenness space $(X, \mathfrak{B})$.

(1) For any $F \in L^{(X)}$, we have

$$\mathfrak{h}^{\mathcal{B}^{\mathfrak{h}}}(F) = \bigvee\{x_\lambda \in J(L^X) \mid (F, x_\lambda) \in \mathcal{B}^{\mathfrak{h}}\} = \bigvee\{x_\lambda \in J(L^X) \mid x_\lambda \leqslant \mathfrak{h}(F)\} = \mathfrak{h}(F).$$

(2) For any $F \in L^{(X)}$ and $x_\lambda \in J(L^X)$, we have

$$\begin{array}{rcl} (F, x_\lambda) \in \mathfrak{B}^{\mathfrak{h}^{\mathfrak{B}}} & \Longleftrightarrow & x_\lambda \leqslant \mathfrak{h}^{\mathfrak{B}}(F) \\ & \Longleftrightarrow & (F, x_\lambda) \in \mathfrak{B}, \ \text{(by Lemma 3.6 (2))} \end{array}$$

as desired. $\square$

4. *L*-Betweenness Relations from *L*-Remotehood Systems

In [34], Yang and Li introduced the concept of L-remotehood systems, which can be used to characterize L-convex structures. In this section, we will study L-betweenness relations from the perspective of L-remotehood systems. Firstly, let us recall the definition of L-remotehood systems.

Definition 4.1. (Yang and Li [34]) An L-remotehood system on X is a set $\mathcal{R} = \{\mathcal{R}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$, where $\mathcal{R}_{x_\lambda} \subseteq L^X$ satisfies the following conditions:

(LR1) $\underline{\perp} \in \mathcal{R}_{x_\lambda}$;
(LR2) $\forall A \in \mathcal{R}_{x_\lambda}, x_\lambda \nleqslant A$;
(LR3) $\forall A \in L^X, A \in \mathcal{R}_{x_\lambda}$ if and only if $\exists B \in L^X$ *s.t.* $x_\lambda \nleqslant B \geqslant A$ and $\forall y_\mu \nleqslant B, B \in \mathcal{R}_{y_\mu}$;
(LR4) $\forall \{A_j\}_{j \in J} \subseteq^{dir} L^X, \bigvee^{\uparrow}_{j \in J} A_j \in \mathcal{R}_{x_\lambda}$ if and only if $\exists \mu \ll \lambda$ such that $A_j \in \mathcal{R}_{x_\mu}$ for each $j \in J$.

For an L-remotehood system $\mathcal{R}$ on X, the pair $(X,\mathcal{R})$ is called an L-remotehood space and $\mathcal{R}_{x_\lambda}$ is called an L-remotehood of x_λ.

Proposition 4.2. (Yang and Li [34]) *Let $(X,\mathcal{R})$ be an L-remotehood space. If $A \in \mathcal{R}_{x_\lambda}$ and $B \leqslant A$, then $B \in \mathcal{R}_{x_\lambda}$.*

Definition 4.3. A mapping $f : (X,\mathcal{R}^X) \longrightarrow (Y,\mathcal{R}^Y)$ between L-remotehood spaces is called L-CP provided that

$$\forall B \in L^Y, x_\lambda \in J(L^X),\ B \in \mathcal{R}^Y_{f(x)_\lambda} \text{ implies } f_L^{\leftarrow}(B) \in \mathcal{R}^X_{x_\lambda}.$$

It is easy to check that all L-remotehood spaces as objects and all L-CP mappings as morphisms form a category, denoted by L-**REH**.

In [34], Yang and Li provided the transformation formulas between L-convex space $(X,\mathcal{C})$ and L-remotehood space $(X,\mathcal{R})$ as follows:

$$\mathcal{R} \longmapsto \mathcal{C}^{\mathcal{R}} = \{A \in L^X \mid \forall x_\lambda \nleqslant A,\ A \in \mathcal{R}_{x_\lambda}\};$$

$$\mathcal{C} \longmapsto \mathcal{R}^{\mathcal{C}} = \{\mathcal{R}^{\mathcal{C}}_{x_\lambda} \mid x_\lambda \in J(L^X)\},$$

where $\mathcal{R}^{\mathcal{C}}_{x_\lambda} = \{A \in L^X \mid \exists B \in \mathcal{C}\ s.t.\ x_\lambda \nleqslant B \geqslant A\}$. Moreover, $\mathcal{C}^{\mathcal{R}^{\mathcal{C}}} = \mathcal{C}$ and $\mathcal{R}^{\mathcal{C}^{\mathcal{R}}} = \mathcal{R}$ (i.e., $\mathcal{R}^{\mathcal{C}^{\mathcal{R}}}_{x_\lambda} = \mathcal{R}_{x_\lambda}$ for all $x_\lambda \in J(L^X)$).

Now, let us show the relationships between L-**REH** and L-**CS**.

Proposition 4.4. (1) *If $f : (X,\mathcal{C}_X) \longrightarrow (Y,\mathcal{C}_Y)$ is L-CP, then so is $f : (X,\mathcal{R}^{\mathcal{C}_X}) \longrightarrow (Y,\mathcal{R}^{\mathcal{C}_Y})$.*

(2) *If $(X,\mathcal{R}^X) \longrightarrow (Y,\mathcal{R}^Y)$ is L-CP, then so is $f : (X,\mathcal{C}^{\mathcal{R}^X}) \longrightarrow (Y,\mathcal{C}^{\mathcal{R}^Y})$.*

Proof. (1) Since $f : (X,\mathcal{C}_X) \longrightarrow (Y,\mathcal{C}_Y)$ is L-CP, it follows that $f_L^{\leftarrow}(C) \in \mathcal{C}_X$ for every $C \in \mathcal{C}_Y$. Then for each $x_\lambda \in J(L^X)$ and $B \in L^Y$, we have

$$\begin{aligned} B \in \mathcal{R}^{\mathcal{C}_Y}_{f(x)_\lambda} &\iff \exists C \in \mathcal{C}_Y\ s.t.\ f(x)_\lambda \nleqslant C \geqslant B \\ &\Longrightarrow \exists f_L^{\leftarrow}(C) \in \mathcal{C}_X\ s.t.\ x_\lambda \nleqslant f_L^{\leftarrow}(C) \geqslant f_L^{\leftarrow}(B) \\ &\iff f_L^{\leftarrow}(B) \in \mathcal{R}^{\mathcal{C}_X}_{x_\lambda}. \end{aligned}$$

(2) Since $(X,\mathcal{R}^X) \longrightarrow (Y,\mathcal{R}^Y)$ is L-CP, it follows that $B \in \mathcal{R}^Y_{f(x)_\lambda}$ implies $f_L^{\leftarrow}(B) \in \mathcal{R}^X_{x_\lambda}$ for any $x_\lambda \in J(L^X)$ and $B \in L^Y$. Then we have

$$\begin{aligned} B \in \mathcal{C}^{\mathcal{R}^Y} &\iff \forall y_\mu \nleqslant B, B \in \mathcal{R}^Y_{y_\mu} \\ &\Longrightarrow \forall f(x)_\lambda \nleqslant B, B \in \mathcal{R}^Y_{f(x)_\lambda} \\ &\Longrightarrow \forall x_\lambda \nleqslant f_L^{\leftarrow}(B), f_L^{\leftarrow}(B) \in \mathcal{R}^X_{x_\lambda} \\ &\iff f_L^{\leftarrow}(B) \in \mathcal{C}^{\mathcal{R}^X}. \end{aligned}$$

This completes the proof. □

Proposition 4.5. (Yang and Li [34]) *L-remotehood systems and L-convex structures are one-to-one corresponding.*

By Propositions 4.4 and 4.5, we have

Theorem 4.6. *L-**REH** and L-**CS** are isomorphic.*

Next, we will induce L-betweenness relations via L-remotehood systems.

Proposition 4.7. *Let $(X,\mathcal{R})$ be an L-remotehood space and define $\mathfrak{B}^{\mathcal{R}} \subseteq L^{(X)} \times J(L^X)$ as follows:*

$$\mathfrak{B}^{\mathcal{R}} = \{(F, x_\lambda) \in L^{(X)} \times J(L^X) \mid F \notin \mathcal{R}_{x_\lambda}\}.$$

Then $\mathfrak{B}^{\mathcal{R}}$ is an L-betweenness relation on X.

Proof. It suffices to show that $\mathfrak{B}^{\mathcal{R}}$ satisfies (LB1)–(LB5).

(LB1) and (LB2) follow immediately from (LR1) and (LR2).

(LB3) Suppose that $(G, x_\lambda) \in \mathfrak{B}^{\mathcal{R}}$ and $(F, y_\mu) \in \mathfrak{B}^{\mathcal{R}}$ for any $y_\mu \leqslant G$, i.e., $G \notin \mathcal{R}_{x_\lambda}$ and $F \notin \mathcal{R}_{y_\mu}$ for all $y_\mu \leqslant G$. Then $F \notin \mathcal{R}_{x_\lambda}$. Otherwise, $F \in \mathcal{R}_{x_\lambda}$. By (LR3), there exists $A \in L^X$ such that $x_\lambda \nleqslant A \geqslant F$ and $A \in \mathcal{R}_{z_\nu}$ for each $z_\nu \nleqslant A$. Let $B = \bigvee\{z_\nu \in J(L^X) \mid F \notin \mathcal{R}_{z_\nu}\}$ and $C = \bigvee\{z_\nu \in J(L^X) \mid A \notin \mathcal{R}_{z_\nu}\}$. Then $G \leqslant B \leqslant C \leqslant A$. Since $G \notin \mathcal{R}_{x_\lambda}$, it follows from Proposition 4.2 that $A \notin \mathcal{R}_{x_\lambda}$. This implies $x_\lambda \leqslant A$, which is a contradiction.

(LB4) Take each $F \in L^{(X)}$ and $x_\lambda \in J(L^X)$. Then

$$\begin{aligned}(F, x_\lambda) \in \mathfrak{B}^{\mathcal{R}} &\iff F \notin \mathcal{R}_{x_\lambda}\\ &\iff \textstyle\bigvee^{\uparrow}_{G \ll F} G \notin \mathcal{R}_{x_\lambda}\\ &\iff \forall \mu \ll \lambda, \exists G \ll F\ s.t.\ G \notin \mathcal{R}_{x_\mu}\ \text{(by (LR4))}\\ &\iff \forall \mu \ll \lambda, \exists G \ll F\ s.t.\ (G, x_\mu) \in \mathfrak{B}^{\mathcal{R}}.\end{aligned}$$

(LB5) Take each $F \in L^{(X)}$ and $\{x_{\lambda_i} \mid i \in I\} \subseteq J(L^X)$. Then

$$\begin{aligned}&(F, x_{\vee_{i\in I}\lambda_i}) \notin \mathfrak{B}^{\mathcal{R}}\\ \iff\ & F \in \mathcal{R}_{x_{\vee_{i\in I}\lambda_i}}\\ \iff\ & \exists A \in L^X\ s.t.\ x_{\vee_{i\in I}\lambda_i} \nleqslant A \geqslant F\ \text{ and }\ \forall y_\mu \nleqslant A, A \in \mathcal{R}_{y_\mu}\ \text{(by (LR3))}\\ \iff\ & \exists A \in L^X, \exists i_0 \in I, x_{\lambda_{i_0}} \nleqslant A \geqslant F\ \text{ and }\ \forall y_\mu \nleqslant A, A \in \mathcal{R}_{y_\mu}\\ \iff\ & \exists i_0 \in I, F \in \mathcal{R}_{x_{\lambda_{i_0}}}\ \text{(by (LR4))}\\ \iff\ & \exists i_0 \in I, (F, x_{\lambda_{i_0}}) \notin \mathfrak{B}^{\mathcal{R}}.\end{aligned}$$

This implies

$$(F, x_{\vee_{i\in I}\lambda_i}) \in \mathfrak{B}^{\mathcal{R}} \iff \forall i \in I, (F, x_{\lambda_{i_0}}) \in \mathfrak{B}^{\mathcal{R}}.$$

As a consequence, we obtain that $\mathfrak{B}^{\mathcal{R}}$ is an L-betweenness relation on X. □

Proposition 4.8. *If $f : (X, \mathcal{R}^X) \longrightarrow (Y, \mathcal{R}^Y)$ is L-CP, then $f : (X, \mathfrak{B}^{\mathcal{R}^X}) \longrightarrow (Y, \mathfrak{B}^{\mathcal{R}^Y})$ is L-BP.*

Proof. Since $f : (X, \mathcal{R}^X) \longrightarrow (Y, \mathcal{R}^Y)$ is L-CP, it follows that

$$\forall B \in L^Y,\ \forall x_\lambda \in J(L^X),\ B \in \mathcal{R}^Y_{f(x)_\lambda}\ \text{ implies }\ f_L^{\leftarrow}(B) \in \mathcal{R}^X_{x_\lambda}.$$

Now take each $F \in L^{(X)}$ such that $(F, x_\lambda) \in \mathfrak{B}^{\mathcal{R}^X}$, i.e., $F \notin \mathcal{R}^X_{x_\lambda}$. By Proposition 4.2, it follows that $f_L^{\leftarrow}(f_L^{\rightarrow}(F)) \notin \mathcal{R}^X_{x_\lambda}$. This implies $f_L^{\rightarrow}(F) \notin \mathcal{R}^Y_{f(x)_\lambda}$, whence $(f_L^{\rightarrow}(F), f(x)_\lambda) \in \mathfrak{B}^{\mathcal{R}^Y}$. □

By Propositions 4.7 and 4.8, we obtain a functor $\mathbb{H} : L$-**REH**$\longrightarrow L$-**Bet** defined by

$$\mathbb{H}(X, \mathcal{R}) = (X, \mathfrak{B}^{\mathcal{R}}) \text{ and } \mathbb{H}(f) = f.$$

Conversely, we induce L-remotehood systems via L-betweenness relations. For this purpose, we first give the following lemmas.

Lemma 4.9. *Let $(X, \mathfrak{B})$ be an L-betweenness space and define $\mathcal{C}^{\mathfrak{B}} \subseteq L^X$ as follows:*

$$\mathcal{C}^{\mathfrak{B}} = \{C \in L^X \mid \forall F \ll C,\ \forall x_\lambda \in J(L^X),\ (F, x_\lambda) \in \mathfrak{B}\ \text{ implies }\ x_\lambda \leqslant C\}.$$

Then $\mathcal{C}^{\mathfrak{B}}$ is an L-convex structure on X.

Proof. It suffices to verify that $\mathcal{C}^{\mathfrak{B}}$ satisfies (LC1)–(LC3).

(LC1) It is trivial.

(LC2) Suppose $\{C_i \mid i \in I\} \subseteq \mathcal{C}^{\mathfrak{B}}$. If $F \ll \bigwedge_{i\in I} C_i$, then it follows that $F \ll C_i$ for all $i \in I$. Since $\{C_i \mid i \in I\} \subseteq \mathcal{C}^{\mathfrak{B}}$, we have $(F, x_\lambda) \in \mathfrak{B}$ implies $x_\lambda \leqslant C_i$ for all $i \in I$. That is, $(F, x_\lambda) \in \mathfrak{B}$ implies $x_\lambda \leqslant \bigwedge_{i\in I} C_i$. Hence $\bigwedge_{i\in I} C_i \in \mathcal{C}^{\mathfrak{B}}$.

(LC3) Suppose $\{C_j \mid j \in J\} \subseteq^{dir} \mathcal{C}^{\mathfrak{B}}$. If $F \ll \bigvee^{\uparrow}_{j\in J} C_j$, then there exists $k \in J$ such that $F \ll C_k$. Note that $C_k \in \mathcal{C}^{\mathfrak{B}}$, we have $(F, x_\lambda) \in \mathfrak{B}$ implies $x_\lambda \leqslant C_k \leqslant \bigvee^{\uparrow}_{j\in J} C_j$. This shows $\bigvee^{\uparrow}_{j\in J} C_j \in \mathcal{C}^{\mathfrak{B}}$. □

Lemma 4.10. *If $f:(X,\mathfrak{B}_X)\longrightarrow(Y,\mathfrak{B}_Y)$ is L-BP, then $f:(X,\mathcal{C}^{\mathfrak{B}_X})\longrightarrow(Y,\mathcal{C}^{\mathfrak{B}_Y})$ is L-CP.*

Proof. Since $f:(X,\mathfrak{B}_X)\longrightarrow(Y,\mathfrak{B}_Y)$ is L-BP, it follows that for each $F\in L^{(X)}$ and $x_\lambda\in J(L^X)$, $(F,x_\lambda)\in\mathfrak{B}_X$ implies $(f_L^{\rightarrow}(F),f(x)_\lambda)\in\mathfrak{B}_Y$. Then for each $B\in\mathcal{C}^{\mathfrak{B}_Y}$, we have

$$\begin{array}{rl} & F\ll f_L^{\leftarrow}(B)\ \textit{and}\ (F,x_\lambda)\in\mathfrak{B}_X\\ \Longrightarrow & f_L^{\rightarrow}(F)\ll B\ \text{and}\ (f_L^{\rightarrow}(F),f(x)_\lambda)\in\mathfrak{B}_Y\\ \Longrightarrow & f(x)_\lambda\leqslant B\\ \Longleftrightarrow & x_\lambda\leqslant f_L^{\leftarrow}(B). \end{array}$$

This means that for each $F\ll f_L^{\leftarrow}(B)$, $(F,x_\lambda)\in\mathfrak{B}_X$ implies $x_\lambda\leqslant f_L^{\leftarrow}(B)$. Thus we obtain $f_L^{\leftarrow}(B)\in\mathcal{C}^{\mathfrak{B}_X}$. □

Now, we show how to generate an L-remotehood system via an L-betweenness relation.

Proposition 4.11. *Let $(X,\mathfrak{B})$ be an L-betweenness space and define $\mathcal{R}^{\mathfrak{B}}=\{\mathcal{R}^{\mathfrak{B}}_{x_\lambda}\mid x_\lambda\in J(L^X)\}$ as follows:*

$$\forall x_\lambda\in J(L^X),\ \ \mathcal{R}^{\mathfrak{B}}_{x_\lambda}=\mathcal{R}^{\mathcal{C}^{\mathfrak{B}}}_{x_\lambda}.$$

Then $\mathcal{R}^{\mathfrak{B}}$ is an L-remotehood system on X.

Proof. By Lemma 4.9, it is straightforward. □

Proposition 4.12. *If $f:(X,\mathfrak{B}_X)\longrightarrow(Y,\mathfrak{B}_Y)$ is L-BP, then $f:(X,\mathcal{R}^{\mathfrak{B}_X})\longrightarrow(Y,\mathcal{R}^{\mathfrak{B}_Y})$ is L-CP.*

Proof. Since $f:(X,\mathfrak{B}_X)\longrightarrow(Y,\mathfrak{B}_Y)$ is L-BP, it follows that for each $F\in L^{(X)}$ and $x_\lambda\in J(L^X)$, $(F,x_\lambda)\in\mathfrak{B}_X$ implies $(f_L^{\rightarrow}(F),f(x)_\lambda)\in\mathfrak{B}_Y$. Then for each $B\in L^Y$ and $x_\lambda\in J(L^X)$, we have

$$\begin{array}{rl} & B\in\mathcal{R}^{\mathfrak{B}_Y}_{f(x)_\lambda}=\mathcal{R}^{\mathcal{C}^{\mathfrak{B}_Y}}_{f(x)_\lambda}\\ \Longleftrightarrow & \exists C\in\mathcal{C}^{\mathfrak{B}_Y}\ s.t.\ f(x)_\lambda\nleqslant C\geqslant B\\ \Longrightarrow & \exists C\in\mathcal{C}^{\mathfrak{B}_Y}\ s.t.\ x_\lambda\nleqslant f_L^{\leftarrow}(C)\geqslant f_L^{\leftarrow}(B)\\ \Longrightarrow & \exists f_L^{\leftarrow}(C)\in\mathcal{C}^{\mathfrak{B}_X}\ s.t.\ x_\lambda\nleqslant f_L^{\leftarrow}(C)\geqslant f_L^{\leftarrow}(B)\ \ (\text{by}\ \ L\text{–CP})\\ \Longrightarrow & f_L^{\leftarrow}(B)\in\mathcal{R}^{\mathcal{C}^{\mathfrak{B}_X}}_{x_\lambda}=\mathcal{R}^{\mathfrak{B}_X}_{x_\lambda}. \end{array}$$

This shows that $B\in\mathcal{R}^{\mathfrak{B}_Y}_{f(x)_\lambda}$ implies $f_L^{\leftarrow}(B)\in\mathcal{R}^{\mathfrak{B}_X}_{x_\lambda}$, as desired. □

By Propositions 4.11 and 4.12, we obtain a functor $\mathbb{K}:L$-**Bet**$\longrightarrow L$-**REH** defined by

$$\mathbb{K}(X,\mathfrak{B})=(X,\mathcal{R}^{\mathfrak{B}})\ \text{and}\ \mathbb{K}(f)=f.$$

Theorem 4.13. *L-**Bet** and L-**REH** are isomorphic.*

Proof. It suffices to verify (1) $\mathcal{R}^{\mathfrak{B}^{\mathcal{R}}}=\mathcal{R}$ and (2) $\mathfrak{B}^{\mathcal{R}^{\mathfrak{B}}}=\mathfrak{B}$ for any L-betweenness space $(X,\mathfrak{B})$ and any L-remotehood space $(X,\mathcal{R})$.

For (1), we first show $\mathcal{C}^{\mathfrak{B}^{\mathcal{R}}}=\mathcal{C}^{\mathcal{R}}$. Take each $B\in\mathcal{C}^{\mathfrak{B}^{\mathcal{R}}}$. Then for each $F\ll B,(F,z_\nu)\in\mathfrak{B}^{\mathcal{R}}$ implies $z_\nu\leqslant B$. In order to show $B\in\mathcal{C}^{\mathcal{R}}$, take each $y_\mu\in J(L^X)$ such that $y_\mu\nleqslant B$. Then there exists $\nu\ll\mu$ such that $y_\nu\nleqslant B$. Thus, for each $F\ll B$, it follows that $y_\nu\nleqslant B$. This implies $(F,y_\nu)\notin\mathfrak{B}^{\mathcal{R}}$. That is, $F\in\mathcal{R}_{y_\nu}$. Then we obtain that there exists $\nu\ll\mu$ such that $F\in\mathcal{R}_{y_\nu}$ for all $F\ll B$. By (LR4), we have $B=\bigvee^{\uparrow}_{F\ll B}F\in\mathcal{R}_{y_\mu}$. Thus, $B\in\mathcal{R}_{y_\mu}$ for all $y_\mu\nleqslant B$, whence $B\in\mathcal{C}^{\mathcal{R}}$. By the arbitrariness of B, we have $\mathcal{C}^{\mathfrak{B}^{\mathcal{R}}}\subseteq\mathcal{C}^{\mathcal{R}}$.

Conversely, take each $B\in\mathcal{C}^{\mathcal{R}}$. Then for each $F\ll B$ and $y_\mu\nleqslant B$, it follows that $F\ll B$ and $B\in\mathcal{R}_{y_\mu}$. By Proposition 4.2, we have $F\in\mathcal{R}_{y_\mu}$, i.e., $(F,y_\mu)\notin\mathfrak{B}^{\mathcal{R}}$. This shows that for each $F\ll B,(F,y_\mu)\in\mathfrak{B}^{\mathcal{R}}$ implies $y_\mu\leqslant B$. That is, $B\in\mathcal{C}^{\mathfrak{B}^{\mathcal{R}}}$. Hence, $\mathcal{C}^{\mathcal{R}}\subseteq\mathcal{C}^{\mathfrak{B}^{\mathcal{R}}}$.

Now we show $\mathcal{R}^{\mathfrak{B}^{\mathcal{R}}} = \mathcal{R}$. Take each $x_\lambda \in J(L^X)$. Then

$$\mathcal{R}^{\mathfrak{B}^{\mathcal{R}}}_{x_\lambda} = \mathcal{R}^{\mathcal{C}^{\mathfrak{B}^{\mathcal{R}}}}_{x_\lambda} = \mathcal{R}^{\mathcal{C}^{\mathcal{R}}}_{x_\lambda} = \mathcal{R}_{x_\lambda},$$

which implies $\mathcal{R}^{\mathfrak{B}^{\mathcal{R}}} = \mathcal{R}$.

For (2), we first show $\mathfrak{B} \subseteq \mathfrak{B}^{\mathcal{R}^{\mathfrak{B}}}$. Take each $F \in L^{(X)}$ and $x_\lambda \in J(L^X)$ such that $(F, x_\lambda) \notin \mathfrak{B}^{\mathcal{R}^{\mathfrak{B}}}$. It follows that $F \in \mathcal{R}^{\mathfrak{B}}_{x_\lambda} = \mathcal{R}^{\mathcal{C}^{\mathfrak{B}}}_{x_\lambda}$. Then there exists $A \in L^X$ such that $x_\lambda \nleqslant A \geqslant F$ and for each $G \ll A, (G, y_\mu) \in \mathfrak{B}$ implies $y_\mu \leqslant A$. Since $x_\lambda \nleqslant A \geqslant$ F, there exists $\mu \ll \lambda$ such that $x_\mu \nleqslant A \geqslant F$. This implies that there exists $\mu \ll \lambda$ and for each $G \ll F, (G, x_\mu) \notin \mathfrak{B}$. By (LB4), we have $(F, x_\lambda) \notin \mathfrak{B}$. Hence $\mathfrak{B} \subseteq \mathfrak{B}^{\mathcal{R}^{\mathfrak{B}}}$.

Conversely, take each $F \in L^{(X)}$ and $x_\lambda \in J(L^X)$ such that $(F, x_\lambda) \notin \mathfrak{B}$. Let $A = \bigvee\{z_\nu \in J(L^X) \mid (F, z_\nu) \in \mathfrak{B}\}$. By (LB2), we have $z_\nu \leqslant F$ implies $(F, z_\nu) \in \mathfrak{B}$. This means $F \leqslant A$. Further, $x_\lambda \nleqslant A$. Otherwise, $\lambda \leqslant A(x) = \bigvee\{\mu \in L \mid (F, x_\mu) \in \mathfrak{B}\}$. Denote $U = \{\nu \in L \mid (F, x_\nu) \in \mathfrak{B}\}$. Then it follows from (LB5) that $(F, x_{\bigvee U}) \in \mathfrak{B}$. By Lemma 3.6 (1), we have $(F, x_\lambda) \in \mathfrak{B}$, which is a contradiction. This shows $x_\lambda \nleqslant A$. Then we show $A \in \mathcal{C}^{\mathfrak{B}}$. For each $G \ll A$ and $(G, y_\mu) \in \mathfrak{B}$, take each $z_\nu \leqslant G$. Then $\nu \leqslant G(z) \leqslant A(z) = \bigvee\{\omega \in L \mid (F, z_\omega) \in \mathfrak{B}\}$. It follows that $(F, z_\nu) \in \mathfrak{B}$. This shows that $z_\nu \leqslant G$ implies $(F, z_\nu) \in \mathfrak{B}$. Since $(G, y_\mu) \in \mathfrak{B}$, it follows from (LB3) that $(F, y_\mu) \in \mathfrak{B}$. By the construction of A, we have $y_\mu \leqslant A$. Thus, for each $G \ll A, (G, y_\mu) \in \mathfrak{B}$ implies $y_\mu \leqslant A$. This shows that $A \in \mathcal{C}^{\mathfrak{B}}$. Now we have shown that $A \in \mathcal{C}^{\mathfrak{B}}$ and $x_\lambda \nleqslant A \geqslant F$, which means $F \in \mathcal{R}^{\mathcal{C}^{\mathfrak{B}}}_{x_\lambda} = \mathcal{R}^{\mathfrak{B}}_{x_\lambda}$, i.e., $(F, x_\lambda) \notin \mathfrak{B}^{\mathcal{R}^{\mathfrak{B}}}$. By the arbitrariness of (F, x_λ), we have $\mathfrak{B}^{\mathcal{R}^{\mathfrak{B}}} \subseteq \mathfrak{B}$.

As a consequence, we obtain $\mathfrak{B}^{\mathcal{R}^{\mathfrak{B}}} = \mathfrak{B}$. □

5. Unities of *L*-Betweenness Relations from two Perspectives

In this section, we will study the relationship between *L*-betweenness relations from two perspectives. Moreover, we will propose a new type of restricted *L*-hull operators from the aspect of *L*-betweenness relations.

Given an *L*-convex structure, we can construct *L*-betweenness relation via its restricted *L*-hull operator and *L*-remotehood system, respectively. We will show *L*-betweenness relations induced by these two perspectives are unified.

Theorem 5.1. *Let $(X, \mathcal{C})$ be an L-convex space. Then $\mathfrak{B}^{\mathfrak{h}^{\mathcal{C}}} = \mathfrak{B}^{\mathcal{R}^{\mathcal{C}}}$.*

Proof. Take each $F \in L^{(X)}$ and $x_\lambda \in J(L^X)$. Then

$$\begin{aligned}(F, x_\lambda) \notin \mathfrak{B}^{\mathfrak{h}^{\mathcal{C}}} &\iff x_\lambda \nleqslant \mathfrak{h}^{\mathcal{C}}(F) = \bigwedge\{A \in \mathcal{C} \mid F \leqslant A\}\\ &\iff \exists A \in \mathcal{C}\ s.t.\ x_\lambda \nleqslant A \geqslant F\\ &\iff F \in \mathcal{R}^{\mathcal{C}}_{x_\lambda}\\ &\iff (F, x_\lambda) \notin \mathfrak{B}^{\mathcal{R}^{\mathcal{C}}}.\end{aligned}$$

This shows $\mathfrak{B}^{\mathfrak{h}^{\mathcal{C}}} = \mathfrak{B}^{\mathcal{R}^{\mathcal{C}}}$. □

As mentioned at the beginning of Section 3, there exists a natural way to induce a betweenness relation by a restricted hull operator, and vice versa. That is,

$$(F, x) \in \mathcal{B} \iff x \in h(F).$$

In the fuzzy case, it should be

$$(F, x_\lambda) \in \mathfrak{B} \iff \text{“}x_\lambda \in \mathfrak{h}(F)\text{”}.$$

But the restricted *L*-hull operator $\mathfrak{h}$ is a mapping from $L^{(X)}$ to L^X. This means that $\mathfrak{h}(F)$ is an *L*-subset of *X*. As we all know, x_λ is also an *L*-subset of *X*. So there is no " $\in$ " relation between x_λ and $\mathfrak{h}(F)$. Then we characterize "$x_\lambda \in \mathfrak{h}(F)$" by "$x_\lambda \leqslant \mathfrak{h}(F)$" and fortunately, *L*-betweenness relations and restricted *L*-hull operators are coincident by means of this transformation formula. However, there is still a question:

Is there a kind of restricted L-hull operator h which can characterize "$x_\lambda \in \mathsf{h}(F)$"?

In order to answer this question, we propose a new type of restricted L-hull operators which can be connected with L-betweenness relations in a natural way.

For convenience, we denote $\downarrow F = \{x_\lambda \in J(L^X) \mid x_\lambda \leqslant F\}$ for any $F \in L^{(X)}$ and $P(J(L^X))$ for the powerset of $J(L^X)$. Then we propose a new type of restricted L-hull operators.

Definition 5.2. A restricted L-hull operator on X is a mapping $\mathsf{h} : L^{(X)} \longrightarrow P(J(L^X))$ which satisfies:

(LRH1) $\mathsf{h}(\underline{\perp}) = \emptyset$;
(LRH2) $\downarrow F \subseteq \mathsf{h}(F)$;
(LRH3) $\downarrow G \subseteq \mathsf{h}(F)$ implies $\mathsf{h}(G) \subseteq \mathsf{h}(F)$;
(LRH4) $x_\lambda \in \mathsf{h}(F)$ if and only if $\forall \mu \ll \lambda, x_\mu \in \bigcup_{G \ll F} \mathsf{h}(G)$;
(LRH5) $x_{\bigvee_{i\in I}\lambda_i} \in \mathsf{h}(F)$ if and only if $\forall i \in I, x_{\lambda_i} \in \mathsf{h}(F)$.

For a restricted L-hull operator h on X, the pair (X, h) is called a restricted L-hull space.

Definition 5.3. A mapping $f : (X, \mathsf{h}_X) \longrightarrow (Y, \mathsf{h}_Y)$ between restricted L-hull spaces is called L-hull-preserving provided that

$$\forall F \in L^{(X)}, \forall x_\lambda \in J(L^X), x_\lambda \in \mathsf{h}(F) \text{ implies } f(x)_\lambda \in \mathsf{h}_Y(f_L^{\rightarrow}(F)).$$

It is easy to check that all restricted L-hull spaces as objects and all L-hull-preserving mappings as morphisms form a category, denoted by L-**RHSS**.

As mentioned in the motivation of this concept, we can show that this kind of restricted L-hull operators and L-betweenness relations can be induced by each other in a natural way, which is presented as follows:

$$(F, x_\lambda) \in \mathfrak{B} \Longleftrightarrow x_\lambda \in \mathsf{h}(F).$$

Concretely, we can obtain the following result.

Theorem 5.4. *L-**RHSS** and L-**Bet** are isomorphic.*

Proof. It is straightforward and is omitted. □

By Theorems 5.4 and 3.9, L-**RHSS** and L-**RHS** are isomorphic, in a theoretical sense. Here we only provide the transformation formulas between these two kinds of restricted L-hull operators.

$$(X, \mathfrak{h}) \longmapsto (X, \mathsf{h}^{\mathfrak{h}}) : \mathsf{h}^{\mathfrak{h}}(F) = \{x_\lambda \in J(L^X) \mid x_\lambda \leqslant \mathfrak{h}(F)\}.$$

$$(X, \mathsf{h}) \longmapsto (X, \mathfrak{h}^{\mathsf{h}}) : \mathfrak{h}^{\mathsf{h}}(F) = \bigvee \{x_\lambda \in J(L^X) \mid x_\lambda \in \mathsf{h}(F)\}.$$

6. Conclusions

In this paper, we proposed the definition of L-betweenness relations by means of restricted L-hull operators, and showed its equivalence to restricted L-hull operators and L-remotehood systems, respectively. Further, we proved that both of the approach of restricted L-hull operators and the approach of L-remotehood systems to L-betweenness relations were unified. Finally, we gave a new type of restricted L-hull operators and established the relationship between these two kinds of restricted L-hull operators. As the future work, we will consider the following problems:

• As a generalization of L-convex structures and M-fuzzifying convex structures, the notion of (L, M)-fuzzy convex structures was introduced in [18]. Also, some further research has been done to (L, M)-fuzzy convex structures [6, 8, 26]. Thus, it will be interesting to consider fuzzy counterpart of betweenness relations in the framework of (L, M)-fuzzy convex structures.

• In the theory of convex structures, there is a result that convex systems are correspondent to partial restricted hull operators, where the difference between convex systems and convex structures is that the universal set does not need to be convex in convex systems (see [19], 2.21]). Shen and Shi [14] have already given the definition of L-convex systems. This motivates us to consider how to generalize partial restricted hull operators to fuzzy case and subsequently construct the relationship among L-betweenness relations, L-convex systems and generalized fuzzy partial restricted hull operators.

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