



## Optimal Inequalities for Submanifolds in Statistical Manifolds of Quasi Constant Curvature

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**Abstract.** In this paper, we establish B.-Y. Chen's optimal inequalities for statistical submanifolds involving Casorati curvature and the normalized scalar curvature in a statistical manifold of quasi constant curvature. The equality cases of these inequalities are also considered. Further, we provide some applications of our results. Moreover, as a new example we construct minimal statistical surface (statistical submanifold) of a statistical manifold of quasi constant curvature.

### 1. Introduction and Motivation

A statistical structure can be considered as a generalization of Riemannian structure. The theory of abstract generalizations of statistical models as statistical manifolds is a fast growing area of research in differential geometry. In 1985, the notion of statistical manifolds (which was initiated from exploration of geometric structures on sets of certain probability distributions) was proposed by Amari [2] which brings a framework for the field of information geometry and it also associates a dual connection (known as conjugate connection). The applications of statistical manifolds draw the attention of distinguished geometers due to their applications in the field of science and engineering (for instance; see, [1, 3, 5, 14, 15, 18, 19, 21, 24, 27] and references therein). In recent years, Cihan Özgür et al. [17] studied statistical manifolds of quasi constant curvature in which they obtained Chen-Ricci inequality and generalized Wintgen inequality. In particular, a statistical space form is a particular case of statistical manifolds of quasi constant curvature.

On the other hand, Chen invariants conjecture yields the solutions to the problems which build the correlations concerning the main intrinsic and the extrinsic invariants [7]. B.-Y. Chen [7] initially developed some fundamental inequalities for submanifolds in real space form, eminently said to be Chen's inequalities. Later, he proposed the extended version of these optimal inequalities for different submanifolds of different manifolds (see [9] and the references therein). The Chen ideal submanifolds have also been investigated (see [4, 11, 12]).

Moreover, the extended version of the notion of the dominant curvatures of a hypersurface of a Riemannian manifold known as Casorati curvature introduced by F. Casorati [6]. Some extremities containing

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the congruous's essence and influence of the Casorati curvature have been examined by many researchers for different submanifolds of different ambient manifolds ([7, 14, 19]). So, it is both important and very interesting to obtain some extremities concerning such kind of algebraic curvatures of submanifolds in any Riemannian manifolds.

Motivated by the studies of above mentioned authors, in this paper we obtain optimal inequalities for the statistical submanifolds of a statistical manifold of quasi constant curvature involving the normalized scalar curvature and the extrinsic generalized normalized  $\delta$ -Casorati curvatures. The equality cases are also investigated. Also, we derive Chen's first inequality for a statistical submanifold of a statistical manifold of quasi constant curvature with applications. At the end we provide a non-trivial example of a statistical submanifold (a minimal translation surface) of a statistical manifold which support the obtained results.

Notice that for the simplicity, throughout this paper we denote the quasi constant curvature by QC.

## 2. Statistical manifolds of QC curvature and their submanifolds

A statistical manifold is a Riemannian manifold  $(N, \langle, \rangle, \tilde{\nabla})$  along a couple of torsionless affine connections  $\tilde{\nabla}$  and  $\tilde{\nabla}^*$  fascinating Codazzi equation and

$$X \langle Y, Z \rangle = \langle \tilde{\nabla}_X Y, Z \rangle + \langle Y, \tilde{\nabla}_X^* Z \rangle, \tag{1}$$

for  $X, Y, Z \in \Gamma(TN)$ . Then  $\tilde{\nabla}$  and  $\tilde{\nabla}^*$  are the dual (or conjugate) connections and the doublet  $(\tilde{\nabla}, \langle, \rangle)$  is statistical structure. Also, it appears  $(\tilde{\nabla}^*)^* = \tilde{\nabla}$ , where the dual connections are related by

$$2\tilde{\nabla}^\circ = \tilde{\nabla} + \tilde{\nabla}^*, \tag{2}$$

where  $\tilde{\nabla}^\circ$  is the Levi-Civita connection on  $N$ .

Denote  $\tilde{R}$  and  $\tilde{R}^*$  to be the curvature tensor fields on  $N$  associating with affine connection  $\tilde{\nabla}$  and  $\tilde{\nabla}^*$  respectively. Then we have following relation

$$\langle \tilde{R}(X, Y)Z, W \rangle = - \langle Z, \tilde{R}^*(X, Y)W \rangle$$

for any  $X, Y, Z, W \in \Gamma(TN)$ .

Let  $(N, \tilde{\nabla}, \langle, \rangle)$  be a statistical manifold,  $M$  be any Riemannian manifold and  $f : M \rightarrow N$  an immersion. We define  $g$  and  $\nabla$  on  $M$  by

$$g = f^* \langle, \rangle, \quad g(\nabla_X Y, Z) = \langle \tilde{\nabla}_X f^* Y, Z \rangle \tag{3}$$

for any  $X, Y, Z \in \Gamma(TM)$ . Then the pair  $(\nabla, g)$  is a statistical structure on  $M$ , which is said to be induced by  $f$  from  $(\tilde{\nabla}, \langle, \rangle)$  [16].

Let  $(M, \nabla, g)$  and  $(N, \tilde{\nabla}, \langle, \rangle)$  be two statistical manifolds. If  $(\nabla, g)$  coincides with the induced statistical structure then an immersion  $f : M \rightarrow N$  is called a statistical immersion and  $M$  is called a statistical submanifold of  $N$  [16].

Let  $M$  be a  $m$ -dimensional statistical submanifold of  $N^{m+d}$ . Let us denote  $\Gamma(TM)$  and  $\Gamma(T^\perp M)$  by set of all sections of tangent and normal bundle to  $M$ , respectively. Then for any  $X, Y \in \Gamma(TM)$ , the fundamental Gauss formulas for the connection  $\tilde{\nabla}$  and  $\tilde{\nabla}^*$  are outlined by [27]

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \tilde{\nabla}_X^* Y = \nabla_X^* Y + \sigma^*(X, Y). \tag{4}$$

respectively, whereas  $\sigma$  and  $\sigma^*$  are bilinear mapping from which the linear transformations  $A_V$  and  $A_V^*$  are outlined by [27]

$$\langle \sigma^*(X, Y), V \rangle = g(A_V X, Y), \quad \langle \sigma(X, Y), V \rangle = g(A_V^* X, Y), \tag{5}$$

respectively for  $V \in \Gamma(T^\perp M)$ . Furthermore, the Weingarten formulas for the connection  $\tilde{\nabla}$  and  $\tilde{\nabla}^*$  follows [27]

$$\tilde{\nabla}_X V = -A_V^* X + \nabla_X^\perp V, \quad \tilde{\nabla}_X^* V = -A_V X + \nabla_X^{\perp*} V, \tag{6}$$

respectively, for  $X \in \Gamma(TM)$  whereas the normal dual connections  $\nabla^\perp$  and  $\nabla^{*\perp}$  are the Riemannian dual connections on  $T^\perp M$ .

Let  $R$  and  $R^*$  be curvature tensor field of  $\nabla$  and  $\nabla^*$ . Then the fundamental Gauss equations are given by [27]

$$\langle \tilde{R}(X, Y)Z, W \rangle = g(R(X, Y)Z, W) + \langle \sigma(X, Z), \sigma^*(Y, W) \rangle - \langle \sigma^*(X, W), \sigma(Y, Z) \rangle, \tag{7}$$

$$\langle \tilde{R}^*(X, Y)Z, W \rangle = g(R^*(X, Y)Z, W) + \langle \sigma^*(X, Z), \sigma(Y, W) \rangle - \langle \sigma(X, W), \sigma^*(Y, Z) \rangle, \tag{8}$$

where  $X, Y, Z, W \in \Gamma(TM)$ .

A statistical manifold  $N$  with the statistical structure  $(\tilde{\nabla}, \langle, \rangle)$  is said to be of constant curvature  $c$  if the curvature tensor  $\tilde{R}$  of  $\tilde{\nabla}$  fascinates ([3, 22])

$$\tilde{R}(X, Y)Z = c\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\}. \tag{9}$$

Similar to the definition of a Riemannian manifold of quasi-constant curvature given in [13], we define a statistical manifold of quasi-constant curvature as follows:

**Definition 2.1.** A statistical manifold  $(N, \tilde{\nabla}, \langle, \rangle)$  is said to be of quasi-constant curvature if the curvature tensor  $\tilde{R}$  of  $\tilde{\nabla}$  satisfies

$$\tilde{R}(X, Y)Z = \alpha\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\} + \beta[\eta(Y)\eta(Z)X - \langle X, Z \rangle \eta(Y)\xi + \langle Y, Z \rangle \eta(X)\xi - \eta(X)\eta(Z)Y], \tag{10}$$

where  $\alpha, \beta$  are scalar functions and  $\eta$  is a 1-form defined by

$$g(X, \xi) = \eta(X)$$

and  $\xi$  is a unit vector field decomposed as  $\xi = \xi^T + \xi^\perp$ .

In particular, if  $\beta = 0$ , then  $N$  turns into a statistical manifold of constant curvature. Also, if  $(\tilde{\nabla}, \langle, \rangle)$  is a statistical structure of quasi-constant curvature, then so is  $(\tilde{\nabla}^*, \langle, \rangle)$ .

Let  $M$  be any  $m$ -dimensional statistical submanifold of a  $(m + d)$ -dimensional statistical manifold of QC curvature  $N^{m+d}$ . Let  $\{e_1, \dots, e_m\}$  and  $\{e_{m+1}, \dots, e_{m+d}\}$  be the standard orthonormal tangent and normal basis on  $M$ , respectively. Then, the mean curvature vector fields  $H$  and  $H^*$  of  $M$  have the following forms [19]

$$H = \frac{1}{m} \sum_{i=1}^m \sigma(e_i, e_i) = \frac{1}{m} \sum_{r=m+1}^{m+d} \left( \sum_{i=1}^m \sigma_{ii}^r \right) e_r, \quad H^* = \frac{1}{m} \sum_{i=1}^m \sigma^*(e_i, e_i) = \frac{1}{m} \sum_{r=m+1}^{m+d} \left( \sum_{i=1}^m \sigma_{ii}^{*r} \right) e_r, \tag{11}$$

where  $\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle$  and  $\sigma_{ij}^{*r} = \langle \sigma^*(e_i, e_j), e_r \rangle$ .

Next, the squared norm of the mean curvatures are stated by [19]

$$\|H\|^2 = \frac{1}{m^2} \sum_{r=m+1}^{m+d} \left( \sum_{i=1}^m \sigma_{ii}^r \right)^2, \quad \|H^*\|^2 = \frac{1}{m^2} \sum_{r=m+1}^{m+d} \left( \sum_{i=1}^m \sigma_{ii}^{*r} \right)^2.$$

We set

$$\|\sigma\|^2 = \sum_{r=m+1}^{m+d} \sum_{i,j=1}^m (\sigma_{ij}^r)^2, \quad \|\sigma^*\|^2 = \sum_{r=m+1}^{m+d} \sum_{i,j=1}^m (\sigma_{ij}^{*r})^2.$$

For any orthonormal vector fields  $X, Y \in \Gamma(TM)$ , the sectional curvature  $\kappa^{\tilde{\nabla}, \tilde{\nabla}^*}$  on  $M$  is defined by ([3], [25])

$$\kappa^{\tilde{\nabla}, \tilde{\nabla}^*}(X \wedge Y) = \frac{1}{2}[\langle \tilde{R}(X, Y)Y, X \rangle + \langle \tilde{R}^*(X, Y)Y, X \rangle].$$

The scalar curvature is determined by

$$\tau^{\nabla, \nabla^*} = \sum_{1 \leq i < j \leq m} \kappa^{\nabla, \nabla^*}(e_i \wedge e_j)$$

and the normalized scalar curvature  $\rho^{\nabla, \nabla^*}$  is calculated as [19]

$$\rho^{\nabla, \nabla^*} = \frac{2\tau^{\nabla, \nabla^*}}{m(m-1)}.$$

The Casorati curvatures  $C$  and  $C^*$  of the submanifold  $M$  are outlined as

$$C = \frac{1}{m} \sum_{r=m+1}^{m+d} \sum_{i,j=1}^m (\sigma_{ij}^r)^2 = \frac{\|\sigma\|^2}{m}, \quad C^* = \frac{1}{m} \sum_{r=m+1}^{m+d} \sum_{i,j=1}^m (\sigma_{ij}^{*r})^2 = \frac{\|\sigma^*\|^2}{m}.$$

Now, let us denote a  $k$ -dimensional subspace of  $T_pM$  by  $\mathcal{L}$ , where  $k > 2$  and  $\{e_i\}_1^k$  as an orthonormal basis of  $\mathcal{L}$ . Next,  $C(\mathcal{L})$  and  $C^*(\mathcal{L})$  of  $\mathcal{L}$  are given as follows [19]

$$C(\mathcal{L}) = \frac{1}{k} \sum_{r=m+1}^{m+d} \sum_{i,j=1}^k (\sigma_{ij}^r)^2, \quad C^*(\mathcal{L}) = \frac{1}{k} \sum_{r=m+1}^{m+d} \sum_{i,j=1}^k (\sigma_{ij}^{*r})^2.$$

We set

$$\mathcal{B} = \{C(\mathcal{L}) : \mathcal{L} \text{ a hyperplane of } T_pM\}, \quad \mathcal{B}^* = \{C^*(\mathcal{L}) : \mathcal{L} \text{ a hyperplane of } T_pM\}.$$

Then the normalized  $\delta$ -Casorati curvatures  $\delta_c(m-1)$  and  $\hat{\delta}_c(m-1)$  of  $M$  are given as [20]:

$$[\delta_c(m-1)](p) = \frac{1}{2}C(p) + \frac{m+1}{2m} \inf(\mathcal{B}), \quad [\hat{\delta}_c(m-1)](p) = 2C(p) + \frac{2m-1}{2m} \sup(\mathcal{B}).$$

The dual normalized  $\delta^*$ -Casorati curvatures  $\delta_c^*(m-1)$  and  $\hat{\delta}_c^*(m-1)$  of  $M$  are stated as

$$[\delta_c^*(m-1)](p) = \frac{1}{2}C^*(p) + \frac{m+1}{2m} \inf(\mathcal{B}^*), \quad [\hat{\delta}_c^*(m-1)](p) = 2C^*(p) + \frac{2m-1}{2m} \sup(\mathcal{B}^*).$$

Moreover, the generalized normalized  $\delta$ -Casorati curvatures  $\delta_c(t; m-1)$  and  $\hat{\delta}_c(t; m-1)$  of  $M$  for  $A(t, m-1) = \frac{(m-1)(m+t)[m^2-m-t]}{mt}$  are given by [20]:

$$[\delta_c(t; m-1)](p) = tC(p) + A(t, m-1) \inf(\mathcal{B}), \quad 0 < t < m(m-1),$$

$$[\hat{\delta}_c(t; m-1)](p) = tC(p) + A(t, m-1) \sup(\mathcal{B}), \quad t > m(m-1).$$

Furthermore, the dual generalized normalized  $\delta^*$ -Casorati curvatures  $\delta_c^*(t; m-1)$  and  $\hat{\delta}_c^*(t; m-1)$  of the submanifold  $M$  are stated as

$$[\delta_c^*(t; m-1)](p) = tC^*(p) + A(t, m-1) \inf(\mathcal{B}^*), \quad 0 < t < m(m-1)$$

$$[\hat{\delta}_c^*(t; m-1)](p) = tC^*(p) + A(t, m-1) \sup(\mathcal{B}^*), \quad t > m(m-1).$$

Next, we find some general extremities involving scalar curvature, normalized scalar curvature, Casorati curvature. From the definition of the scalar curvature, we find

$$2\tau^{\nabla, \nabla^*} = \alpha(m^2 - m) + 2\beta(m-1)\|\xi^T\|^2 + m^2g(H, H^*) - \sigma_{ij}^r \sigma_{ij}^{*r}$$

By the virtue of  $H^\circ = H + H^*$ , we have  $4\|H^\circ\|^2 = \|H\|^2 + \|H^*\|^2 + 2g(H, H^*)$  which yields

$$2\tau^{\nabla, \nabla^*} = \alpha(m^2 - m) + 2\beta(m-1)\|\xi^T\|^2 - \sigma_{ij}^r \sigma_{ij}^{*r} + \frac{m^2}{2}[4\|H^\circ\|^2 - \|H\|^2 - \|H^*\|^2], \tag{12}$$

or

$$2\tau^{\nabla, \nabla^*} = \alpha(m^2 - m) + 2\beta(m-1)\|\xi^T\|^2 - 2mC^\circ + \frac{m}{2}(C + C^*) + \frac{m^2}{2}[4\|H^\circ\|^2 - \|H\|^2 - \|H^*\|^2]. \tag{13}$$

Hence, we conclude that

**Theorem 2.2.** Let  $M^m$  be a statistical submanifold of a statistical manifold  $N^{m+d}$  of QC curvature. Then

$$2\tau^{\nabla, \nabla^*} \leq \alpha(m^2 - m) + 2\beta(m - 1)\|\xi^T\|^2 + \frac{m}{2}(C + C^*) + \frac{m^2}{2}[4\|H^\circ\|^2 - \|H\|^2 - \|H^*\|^2] \tag{14}$$

and equality holds in (14) if and only if  $\sigma = -\sigma^*$ .

We have the following consequences of above theorem.

**Corollary 2.3.** Let  $M^m$  be a totally umbilical statistical submanifold of a statistical manifold  $N^{m+d}$  of QC curvature. Then

$$2\tau^{\nabla, \nabla^*} = m(m - 1)(\alpha) + 2(m - 1)\beta\|\xi^T\|^2 + m^2 + g(H, H^*). \tag{15}$$

**Corollary 2.4.** The scalar curvature of a totally umbilical statistical submanifold  $M^m$  of a statistical manifold  $N^{m+d}$  of QC curvature satisfies

$$\tau^{\nabla, \nabla^*} = \left(\frac{m\alpha}{2} + \beta\|\xi^T\|^2\right)(m - 1) \tag{16}$$

if and only if one of the following holds

- (1)  $H = 0$
- (2)  $H^* = 0$
- (3)  $H$  and  $H^*$  are orthogonal.

### 3. Extremities involving Casorati curvature

Here, we shall prove a general inequality giving bounds for the normalized scalar curvature  $\rho^{\nabla, \nabla^*}$  involving the generalized normalized  $\delta$ -Casorati curvature and later some of its consequences are also discussed.

First, we need the following lemma, which plays an important role in the proof of our successive theorem.

**Lemma 3.1.** [26] Let  $S = \{(x_1, x_2, \dots, x_m) \in \mathbb{R}^m : x_1 + x_2 + \dots + x_m = k\}$  be a hyperplane of  $\mathbb{R}^m$  and  $\mathcal{F} : \mathbb{R}^m \rightarrow \mathbb{R}$  a quadratic form stated as

$$\mathcal{F}(x_1, x_2, \dots, x_m) = a \sum_{i=1}^{m-1} (x_i)^2 + b(x_m)^2 - 2 \sum_{1 \leq i < j \leq m} x_i x_j, \quad a > 0, b > 0.$$

Then by the constrained extremum problem,  $\mathcal{F}$  has a global solution given by

$$x_1 = x_2 = \dots = x_{m-1} = \frac{k}{a + 1}, \quad x_m = \frac{k}{b + 1} = (a - m + 2) \frac{k}{a + 1},$$

where  $b = \frac{m-1}{a-m+2}$ .

Now, we are going to present our result which shows that the normalized scalar curvature  $\rho^{\nabla, \nabla^*}$  is bounded above by generalized normalized  $\delta$ -Casorati curvature.

**Theorem 3.2.** Let  $M^m$  be a statistical submanifold of a statistical manifold  $N^{m+d}$  of QC curvature. Then, the generalized normalized  $\delta$ -Casorati curvature satisfies

$$\rho^{\nabla, \nabla^*} \leq \frac{2\delta_c^\circ(t; m - 1)}{m(m - 1)} - \frac{m}{2(m - 1)} (\|H\|^2 + \|H^*\|^2) + \frac{C + C^*}{2(m - 1)} + \alpha + \frac{2\beta}{m} \|\xi^T\|^2, \tag{17}$$

where  $2\delta_c^\circ(t; m - 1) = \delta_c(t; m - 1) + \delta_c^*(t; m - 1)$  for  $0 < t < m(m - 1)$ .

*Proof.* We consider the quadratic polynomial  $\mathcal{P}$  given by

$$\mathcal{P} = 2tC^\circ + 2A(t, m - 1)C^\circ(\mathcal{L}) - 2\tau^{\nabla_r, \nabla^*}(p) + \frac{m}{2}(C + C^*) + \alpha(m^2 - m) + 2\beta(m - 1)\|\xi^T\|^2 - \frac{m^2}{2}(\|H\|^2 + \|H^*\|^2).$$

Let us assume that  $\mathcal{L}$  is spanned by  $\{e_i\}_1^{m-1}$ . Then using (13) and writing the above relation in indices form, it gives

$$\begin{aligned} \mathcal{P} &= \sum_{r=m+1}^{m+d} \left[ \frac{t}{m} \sum_{i,j=1}^m (\sigma_{ij}^{or})^2 + \frac{2A(t, m - 1)}{m - 1} \sum_{i,j=1}^{m-1} (\sigma_{ij}^{or})^2 \right] + 2mC^\circ - 2m^2\|H^\circ\|^2 \\ &= \sum_{r=m+1}^{m+d} \left[ 2\left(\frac{(m - 1)(m + t)}{t} - 1\right) \sum_{i=1}^{m-1} (\sigma_{ii}^{or})^2 + \frac{4(m - 1)(m + t)}{t} \sum_{1 \leq i < j \leq m-1} (\sigma_{ij}^{or})^2 \right. \\ &\quad \left. + 4\left(\frac{t + m}{m}\right) \sum_{i=1}^{m-1} (\sigma_{im}^{or})^2 + \frac{2t}{m} (\sigma_{mm}^{or})^2 - 4 \sum_{1 \leq i < j \leq m} \sigma_{ii}^{or} \sigma_{jj}^{or} \right] \end{aligned}$$

or

$$\frac{\mathcal{P}}{2} = \sum_{r=m+1}^{m+d} \left[ \left(\frac{(m - 1)(m + t)}{t} - 1\right) \sum_{i=1}^{m-1} (\sigma_{ii}^{or})^2 + \frac{t}{m} (\sigma_{mm}^{or})^2 - 2 \sum_{1 \leq i < j \leq m} \sigma_{ii}^{or} \sigma_{jj}^{or} \right]. \tag{18}$$

Now, we consider a real valued function  $\mathcal{F}_r$  on  $\mathbb{R}^m$  given by

$$\mathcal{F}_r(\sigma_{11}^{or}, \dots, \sigma_{mm}^{or}) = \left(\frac{(m - 1)(m + t)}{t} - 1\right) \sum_{i=1}^{m-1} (\sigma_{ii}^{or})^2 + \frac{t}{m} (\sigma_{mm}^{or})^2 - 2 \sum_{1 \leq i < j \leq m} \sigma_{ii}^{or} \sigma_{jj}^{or}. \tag{19}$$

We contemplate with the optimization dilemma for invariant real constant  $c^r$

$$\begin{aligned} &\min \mathcal{F}_r \\ &\text{subjected to } \mathcal{F}_r : \sigma_{11}^{or} + \sigma_{22}^{or} + \dots + \sigma_{mm}^{or} = c^r. \end{aligned}$$

Next, using simple calculations the partial derivative of  $\mathcal{F}_r$  for  $i \in \{1, 2, \dots, m - 1\}$  are given as

$$\begin{cases} \frac{\partial \mathcal{F}_r}{\partial \sigma_{ii}^{or}} = \frac{2(m+t)(m-1)}{t} \sigma_{ii}^{or} - 2 \sum_{k=1}^m \sigma_{kk}^{or} \\ \frac{\partial \mathcal{F}_r}{\partial \sigma_{mm}^{or}} = \frac{2t}{m} \sigma_{mm}^{or} - 2 \sum_{k=1}^{m-1} \sigma_{kk}^{or}. \end{cases} \tag{20}$$

From Lemma 3.1, we have

$$a = \frac{(m - 1)(m + t)}{t} - 1, b = \frac{t}{m}.$$

Now, to get an extremum solution  $(\sigma_{11}^{or}, \sigma_{22}^{or}, \dots, \sigma_{mm}^{or})$  of the constraint  $\mathcal{F}_r$ , the vector  $\text{grad} \mathcal{F}_r \in T^\perp M$  at  $\mathcal{F}_r$ . From the system of equations of (20), the critical point of the optimized problem is outlined by

$$\begin{cases} \sigma_{ii}^{or} = \frac{tc^r}{(t+m)(m-1)} = \frac{c^r}{a+1}, \text{ for } 1 \leq i \leq m - 1 \\ \sigma_{mm}^{or} = \frac{mc^r}{m+t} = \frac{c^r}{b+1} \end{cases} \tag{21}$$

which is global minimum point. Then (19) and (21) yield

$$\mathcal{F}_r(\sigma_{11}^{or}, \dots, \sigma_{mm}^{or}) = 0$$

and  $\mathcal{P} \geq 0$ . Thus, we derive

$$2\tau^{\nabla, \nabla^*}(p) \leq 2tC^\circ + 2A(t, m - 1)C^\circ(\mathcal{L}) + \alpha(m^2 - m) + \frac{m}{2}(C + C^*) + 2\beta(m - 1)\|\xi^T\|^2 - \frac{m^2}{2}(\|H\|^2 + \|H^*\|^2)$$

equivalent to

$$\rho^{\nabla, \nabla^*} \leq \frac{2\delta_c^\circ(t; m - 1)}{m(m - 1)} - \frac{m}{2(m - 1)}(\|H\|^2 + \|H^*\|^2) + \frac{C + C^*}{2(m - 1)} + \alpha + \frac{2\beta}{m}\|\xi^T\|^2. \tag{22}$$

This completes the proof of theorem.  $\square$

**Theorem 3.3.** Let  $M^m$  be a statistical submanifold of a statistical manifold  $N^{m+d}$  of QC curvature. Then, for the equality case, the necessary and sufficient condition for (17) is

$$\sigma_{ij}^r = -\sigma_{ij}^{*r}, \text{ for } i \neq j \in \{1, 2, \dots, m\}, \quad \sigma_{mm}^{or} = \frac{m(m - 1)}{t}\sigma_{11}^{or} = \dots = \frac{m(m - 1)}{t}\sigma_{m-1\ m-1}^{or}.$$

**Remark 3.4.** Similarly, we have an inequality for a generalized normalized  $\delta$ -Casorati curvature  $\hat{\delta}_c^\circ(t; m - 1) (= (\delta_c(t; m - 1) + \delta_c^*(t; m - 1))/2)$  where  $t > m(m - 1)$ . Further, we can easily prove that the normalized scalar curvature is bounded above by the generalized normalized  $\delta$ -Casorati curvature  $\delta_c(t; m - 1)$  and its dual  $\delta_c^*(t; m - 1)$ .

**Corollary 3.5.** Let  $M^m$  be a statistical submanifold of a statistical manifold  $N^{m+d}$  of QC curvature. Then for  $0 < t < m(m - 1)$ , the generalized normalized  $\delta$ -Casorati curvature satisfies

(1) If  $\xi$  is tangent to  $M$

$$\rho^{\nabla, \nabla^*} \leq \frac{2\delta_c^\circ(t; m - 1)}{m(m - 1)} + \frac{C + C^*}{2(m - 1)} + \alpha + \frac{2\beta}{m} - \frac{m}{2(m - 1)}(\|H\|^2 + \|H^*\|^2).$$

(2) If  $\xi$  is normal to  $M$

$$\rho^{\nabla, \nabla^*} \leq \frac{2\delta_c^\circ(t; m - 1)}{m(m - 1)} - \frac{m}{2(m - 1)}(\|H\|^2 + \|H^*\|^2) + \frac{C + C^*}{2(m - 1)} + \alpha.$$

**Remark 3.6.** Similarly, we can obtain a result for the generalized normalized  $\delta$ -Casorati curvature  $\hat{\delta}_c^\circ(t; m - 1)$  where  $t > m(m - 1)$ .

**Remark 3.7.** Here, one can note that  $\delta_c(t; m - 1)$  and  $\hat{\delta}_c(t; m - 1)$  are the generalized versions of  $\delta_c(m - 1)$  and  $\hat{\delta}_c(m - 1)$  respectively by substituting  $t$  to  $\frac{m(m-1)}{2}$  as

$$\left[ \delta_c\left(\frac{m(m - 1)}{2}; m - 1\right) \right](p) = m(m - 1)[\delta_c(m - 1)](p); \quad \left[ \hat{\delta}_c\left(\frac{m(m - 1)}{2}; m - 1\right) \right](p) = m(m - 1)[\hat{\delta}_c(m - 1)](p),$$

for  $p \in M$ .

**Corollary 3.8.** Let  $M^m$  be a statistical submanifold of a statistical manifold  $N^{m+d}$  of QC curvature. Then, the normalized  $\delta$ -Casorati curvature satisfies

$$\rho^{\nabla, \nabla^*} \leq 2\delta_c^\circ(m - 1) - \frac{m}{2(m - 1)}(\|H\|^2 + \|H^*\|^2) + \frac{C + C^*}{2(m - 1)} + \alpha + \frac{2\beta}{m}\|\xi^T\|^2,$$

where  $2\delta_c^\circ(m - 1) = \delta_c(m - 1) + \delta_c^*(m - 1)$  for  $0 < t < m(m - 1)$ .

Moreover, the necessary and sufficient condition for above relation for equality is

$$\sigma_{ij}^r = -\sigma_{ij}^{*r}, \text{ for } i \neq j \in \{1, 2, \dots, m\}; \quad \sigma_{mm}^{or} = \frac{m(m - 1)}{t}\sigma_{11}^{or} = \dots = \frac{m(m - 1)}{t}\sigma_{m-1\ m-1}^{or}.$$

**Remark 3.9.** Similarly, one can have a result like Corollary 3.8 for the normalized  $\delta$ -Casorati curvature.

4. B.-Y. Chen’s first inequality for statistical submanifolds

This section is devoted to derive an inequality named as Chen’s first inequality for a statistical submanifold in a statistical manifold of QC curvature. Later, we give some consequences of our derived result.

Revoke that the Chen first inequality notion is given in [7, 8]

$$\delta_m(p) = \tau(p) - \inf\{\kappa(\pi)|\pi \subset T_pM^m, \dim \pi = 2\}, \tag{23}$$

where  $\kappa(\pi)$  is sectional curvature of  $M^m$  associated with 2-plane section  $\pi \subset T_pM$  and  $\tau(p)$  is scalar curvature at  $p$ .

Let  $\xi_\pi = pr_\pi\xi$ . Then, for a statistical submanifold of a statistical manifold of QC curvature, we state and prove Chen first inequality as follows.

**Theorem 4.1.** *Let  $M^m$  be a statistical submanifold of a statistical manifold  $N^{m+d}$  of QC curvature. Then we have*

$$\begin{aligned} \kappa^{\nabla, \nabla^*}(\pi) - 2\tau^{\nabla, \nabla^*} &\leq \alpha[1 - (m^2 - m)] + \beta[\|\xi_\pi\|^2 - 2(m - 1)\|\xi^T\|^2] + 2\|\sigma^\circ\|^2 \\ &\quad + \frac{3m^2}{4}[\|H\|^2 + \|H^*\|^2] - 2m^2\|H^\circ\|^2, \end{aligned} \tag{24}$$

provided  $2(\sigma_{11}^{*r}\sigma_{22}^r - \sigma_{12}^r\sigma_{12}^{*r}) = \sigma_{11}^r\sigma_{22}^r + \sigma_{11}^{*r}\sigma_{22}^{*r}$ .

*Proof.* From (12), we have

$$2\tau^{\nabla, \nabla^*} = \alpha(m^2 - m) + 2\beta(m - 1)\|\xi^T\|^2 + \frac{m^2}{2}[4\|H^\circ\|^2 - \|H\|^2 - \|H^*\|^2] - 2\|\sigma^\circ\|^2 + \frac{1}{2}(\|\sigma\|^2 + \|\sigma^*\|^2). \tag{25}$$

For a section  $\pi \subset T_pN$  generated by orthonormal vectors  $e_1, e_2$ , we have

$$\begin{aligned} \kappa^{\nabla, \nabla^*}(\pi) &= \alpha + \beta[g(e_1, \xi)^2 + g(e_2, \xi)^2] + \frac{1}{2}(\sigma_{22}\sigma_{11}^* - 2\sigma_{12}\sigma_{12}^* + \sigma_{22}^*\sigma_{11}) \\ &= \alpha + \beta\|\xi_\pi\|^2 + \frac{1}{2}(\sigma_{22}\sigma_{11}^* - 2\sigma_{12}\sigma_{12}^* + \sigma_{22}^*\sigma_{11}). \end{aligned} \tag{26}$$

From (25) and (26) one gets

$$\begin{aligned} \kappa^{\nabla, \nabla^*}(\pi) - 2\tau^{\nabla, \nabla^*} &= \alpha[1 - (m^2 - m)] + \beta[\|\xi_\pi\|^2 - 2(m - 1)\|\xi^T\|^2] + \frac{1}{2}(\sigma_{22}\sigma_{11}^* - 2\sigma_{12}\sigma_{12}^* + \sigma_{22}^*\sigma_{11}) \\ &\quad - \frac{1}{2}(\|\sigma\|^2 + \|\sigma^*\|^2) + 2\|\sigma^\circ\|^2 - \frac{m^2}{2}[4\|H^\circ\|^2 - \|H\|^2 - \|H^*\|^2]. \end{aligned} \tag{27}$$

Moreover, we can write

$$\begin{aligned} \|\sigma\|^2 &= \sum_{r=m+1}^{m+d} \left( (\sigma_{11}^r)^2 + (\sigma_{22}^r + \dots + \sigma_{mm}^r)^2 + \sum_{1 \leq i < j \leq m} (\sigma_{ij}^r)^2 - \sum_{2 \leq i \neq j \leq m} \sigma_{ii}^r \sigma_{jj}^r \right) \\ &= \frac{1}{2} \sum_{r=m+1}^{m+d} \left( (\sigma_{11}^r + \dots + \sigma_{mm}^r)^2 + (\sigma_{11}^r - \dots - \sigma_{mm}^r)^2 \right) + \sum_{r=m+1}^{m+d} \left( 2 \sum_{1 \leq i < j \leq m} (\sigma_{ij}^r)^2 - \sum_{2 \leq i \neq j \leq m} \sigma_{ii}^r \sigma_{jj}^r \right). \end{aligned}$$

Thus, we find

$$\|\sigma\|^2 \geq \frac{m^2}{2}\|H\|^2 - \sum_{r=m+1}^{m+d} \sum_{2 \leq i \neq j \leq m} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2). \tag{28}$$



Similarly, we have the same inequality for  $\nabla^*$  as follows

$$\|\sigma^*\|^2 \geq \frac{m^2}{2} \|H^*\|^2 - \sum_{r=m+1}^{m+d} \sum_{2 \leq i \neq j \leq m} \left( \sigma_{ii}^{*r} \sigma_{jj}^{*r} - (\sigma_{ij}^{*r})^2 \right). \tag{29}$$

Then from (28) and (29), we get

$$\begin{aligned} \|\sigma\|^2 + \|\sigma^*\|^2 &\geq \frac{m^2}{2} (\|H\|^2 + \|H^*\|^2) - \sum_{r=m+1}^{m+d} \sum_{2 \leq i \neq j \leq m} \left( \sigma_{ii}^r \sigma_{jj}^r + \sigma_{ii}^{*r} \sigma_{jj}^{*r} - (\sigma_{ij}^r)^2 - (\sigma_{ij}^{*r})^2 \right) \\ &= \frac{m^2}{2} (\|H\|^2 + \|H^*\|^2) - \sum_{r=m+1}^{m+d} \sum_{2 \leq i \neq j \leq m} \left( (\sigma_{ii}^r + \sigma_{ii}^{*r})(\sigma_{jj}^r + \sigma_{jj}^{*r}) - \sigma_{ii}^r \sigma_{jj}^{*r} - \sigma_{ii}^{*r} \sigma_{jj}^r \right) \\ &\quad + \sum_{r=m+1}^{m+d} \sum_{2 \leq i \neq j \leq m} \left( (\sigma_{ij}^r)^2 + (\sigma_{ij}^{*r})^2 \right) \end{aligned}$$

which yields

$$\|\sigma\|^2 + \|\sigma^*\|^2 \geq \frac{m^2}{2} (\|H\|^2 + \|H^*\|^2) - 2 \sum_{r=m+1}^{m+d} \sum_{2 \leq i \neq j \leq m} \left( 2\sigma_{ii}^{or} \sigma_{jj}^{or} - \sigma_{ii}^{*r} \sigma_{jj}^r \right) + \sum_{r=m+1}^{m+d} \sum_{2 \leq i \neq j \leq m} \left( (\sigma_{ij}^r)^2 + (\sigma_{ij}^{*r})^2 \right). \tag{30}$$

From (27) and (30), we obtain

$$\begin{aligned} \kappa^{\nabla, \nabla^*}(\tau) - 2\tau^{\nabla, \nabla^*} &\leq \alpha[1 - (m^2 - m)] + \beta[\|\xi_\pi\|^2 - 2(m - 1)\|\xi^T\|^2] + 2\|\sigma^\circ\|^2 + \frac{m^2}{4} [\|H\|^2 + \|H^*\|^2] - 2m^2\|H^\circ\|^2 \\ &\quad + \sum_{r=m+1}^{m+d} \left[ 2 \sum_{2 \leq i \neq j \leq m} \sigma_{ii}^{or} \sigma_{jj}^{or} - \sum_{1 \leq i \neq j \leq m} \sigma_{ii}^{*r} \sigma_{jj}^r \right] + \frac{1}{2} (3\sigma_{22}\sigma_{11}^* - 2\sigma_{12}\sigma_{12}^* + \sigma_{22}^*\sigma_{11}). \end{aligned} \tag{31}$$

Also, since

$$\sum_{r=m+1}^{m+d} \sum_{1 \leq i \neq j \leq m} \sigma_{ii}^{*r} \sigma_{jj}^r = \sum_{1 \leq i \neq j \leq m} g(\sigma(e_1, e_i), \sigma(e_j, e_j)) = m^2 g(H, H^*) = 2m^2\|H^\circ\|^2 - \frac{m^2}{2} [\|H\|^2 + \|H^*\|^2], \tag{32}$$

$$\sum_{r=m+1}^{m+d} \sum_{2 \leq i \neq j \leq m} \sigma_{ii}^{or} \sigma_{jj}^{or} = \sum_{2 \leq i \neq j \leq m} g(\sigma^\circ(e_1, e_i), \sigma^\circ(e_j, e_j)) = m^2 g(H^\circ, H^\circ) - \sigma_{11}^\circ \sigma_{22}^\circ = m^2\|H^\circ\|^2 - \sigma_{11}^\circ \sigma_{22}^\circ. \tag{33}$$

Using (32) and (33) in (31), we find

$$\begin{aligned} \kappa^{\nabla, \nabla^*}(\tau) - 2\tau^{\nabla, \nabla^*} &\leq \alpha[1 - (m^2 - m)] + \beta[\|\xi_\pi\|^2 - 2(m - 1)\|\xi^T\|^2] + 2\|\sigma^\circ\|^2 + \frac{m^2}{4} [\|H\|^2 + \|H^*\|^2] - 2m^2\|H^\circ\|^2 \\ &\quad + \frac{m^2}{2} [\|H\|^2 + \|H^*\|^2] - \sigma_{11}^{or} \sigma_{22}^{or} + \frac{1}{2} (3\sigma_{22}\sigma_{11}^* - 2\sigma_{12}\sigma_{12}^* + \sigma_{22}^*\sigma_{11}) \\ &= \alpha[1 - (m^2 - m)] + \beta[\|\xi_\pi\|^2 - 2(m - 1)\|\xi^T\|^2] + 2\|\sigma^\circ\|^2 + \frac{3m^2}{4} [\|H\|^2 + \|H^*\|^2] - 2m^2\|H^\circ\|^2 \\ &\quad + \frac{1}{2} (2\sigma_{11}^{*r} \sigma_{22}^r - 2\sigma_{12}^r \sigma_{12}^{*r} - \sigma_{11}^r \sigma_{22}^r - \sigma_{11}^{*r} \sigma_{22}^{*r}) \end{aligned}$$

which gives the desired result.  $\square$

**Remark 4.2.** For  $\beta = 0$  Theorem 4.1 states Chen first inequality for a statistical submanifold  $M^m$  of a statistical space form  $N^{m+d}(\alpha)$ .

As an application of Theorem 4.1, we have the the following result.

**Corollary 4.3.** *Let  $M^m$  be a statistical submanifold of a statistical manifold  $N^{m+d}$  of QC curvature. Then*

(1) *If  $\xi$  is tangent to  $M$*

$$\kappa^{\nabla, \nabla^*}(\pi) - 2\tau^{\nabla, \nabla^*} \leq \alpha[1 - (m^2 - m)] + \beta[\|\xi_\pi\|^2 - 2(m - 1)] + 2\|\sigma^\circ\|^2 + \frac{3m^2}{4}[\|H\|^2 + \|H^*\|^2] - 2m^2\|H^\circ\|^2,$$

(2) *If  $\xi$  is normal to  $M$*

$$\kappa^{\nabla, \nabla^*}(\pi) - 2\tau^{\nabla, \nabla^*} \leq \alpha[1 - (m^2 - m)] + 2\|\sigma^\circ\|^2 + \frac{3m^2}{4}[\|H\|^2 + \|H^*\|^2] - 2m^2\|H^\circ\|^2,$$

provided  $2(\sigma_{11}^{*r}\sigma_{22}^r - \sigma_{12}^r\sigma_{12}^{*r}) = \sigma_{11}^r\sigma_{22}^r + \sigma_{11}^{*r}\sigma_{22}^{*r}$ .

### 5. Example

Here, we construct a new example of a statistical submanifold (a minimal translation surface) of a statistical manifold of QC curvature using the following result.

**Proposition 5.1.** [23] *Let  $N = I \times_f M(c)$  be a statistical warped product manifold and  $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in \Gamma(TN)$ . Then the curvature tensor  $\tilde{R}$  of  $N$  is given by*

$$\begin{aligned} \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= \left[ \frac{c}{f^2} - \frac{(f')^2}{f^2} \right] \llbracket \langle \tilde{Y}, \tilde{Z} \rangle \langle \tilde{X}, \tilde{W} \rangle - \langle \tilde{X}, \tilde{Z} \rangle \langle \tilde{Y}, \tilde{W} \rangle \rrbracket \\ &+ \left[ \frac{c}{f^2} - \frac{(f')^2}{f^2} + \frac{f'\lambda + f''}{f} \right] \llbracket \langle \tilde{X}, \tilde{Z} \rangle \langle \tilde{Y}, \partial_t \rangle \langle \tilde{W}, \partial_t \rangle + \langle \tilde{Y}, \tilde{Z} \rangle \langle \tilde{X}, \partial_t \rangle \langle \tilde{W}, \partial_t \rangle \\ &+ \left[ \frac{c}{f^2} - \frac{(f')^2}{f^2} - \frac{f'\lambda - f''}{f} \right] \llbracket \langle \tilde{Y}, \tilde{W} \rangle \langle \tilde{X}, \partial_t \rangle \langle \tilde{Z}, \partial_t \rangle - \langle \tilde{X}, \tilde{W} \rangle \langle \tilde{Y}, \partial_t \rangle \langle \tilde{Z}, \partial_t \rangle \rrbracket, \end{aligned} \tag{34}$$

$$\begin{aligned} \tilde{R}^*(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= \left[ \frac{c}{f^2} - \frac{(f')^2}{f^2} \right] \llbracket \langle \tilde{Y}, \tilde{Z} \rangle \langle \tilde{X}, \tilde{W} \rangle - \langle \tilde{X}, \tilde{Z} \rangle \langle \tilde{Y}, \tilde{W} \rangle \rrbracket \\ &+ \left[ \frac{c}{f^2} - \frac{(f')^2}{f^2} - \frac{f'\lambda - f''}{f} \right] \llbracket \langle \tilde{X}, \tilde{Z} \rangle \langle \tilde{Y}, \partial_t \rangle \langle \tilde{W}, \partial_t \rangle - \langle \tilde{Y}, \tilde{Z} \rangle \langle \tilde{X}, \partial_t \rangle \langle \tilde{W}, \partial_t \rangle \\ &+ \left[ \frac{c}{f^2} - \frac{(f')^2}{f^2} + \frac{f'\lambda + f''}{f} \right] \llbracket \langle \tilde{Y}, \tilde{W} \rangle \langle \tilde{X}, \partial_t \rangle \langle \tilde{Z}, \partial_t \rangle - \langle \tilde{X}, \tilde{W} \rangle \langle \tilde{Y}, \partial_t \rangle \langle \tilde{Z}, \partial_t \rangle \rrbracket. \end{aligned} \tag{35}$$

If  $I \subset \mathbb{R}$  is a trivial statistical manifold and the warping  $f \equiv 1$ , a constant function; then by Proposition 5.1, we have

$$\begin{aligned} \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= c \llbracket \langle \tilde{Y}, \tilde{Z} \rangle \langle \tilde{X}, \tilde{W} \rangle - \langle \tilde{X}, \tilde{Z} \rangle \langle \tilde{Y}, \tilde{W} \rangle \rrbracket + c \llbracket \langle \tilde{X}, \tilde{Z} \rangle \langle \tilde{Y}, \partial_t \rangle \langle \tilde{W}, \partial_t \rangle \\ &+ \langle \tilde{Y}, \tilde{Z} \rangle \langle \tilde{X}, \partial_t \rangle \langle \tilde{W}, \partial_t \rangle + \langle \tilde{Y}, \tilde{W} \rangle \langle \tilde{X}, \partial_t \rangle \langle \tilde{Z}, \partial_t \rangle - \langle \tilde{X}, \tilde{W} \rangle \langle \tilde{Y}, \partial_t \rangle \langle \tilde{Z}, \partial_t \rangle \rrbracket. \end{aligned} \tag{36}$$

So, the statistical product manifold  $N = I \times M(c)$  has natural QC curvature by  $\alpha = \beta = c = \text{constant}$ .

On the other hand, D. W. Yoon [28] defined the following two translation surfaces in the Riemannian product manifold  $\mathbb{R} \times H^2(-1)$  as following

$$\varphi(s, t) = (sg(t), st, f(s)), \quad \psi(s, t) = (g(t), st, f(s)),$$

where  $f(s)$  and  $g(t)$  are smooth functions and  $s, t > 0$ . In his paper, he completely classified minimal translation surfaces in  $\mathbb{R} \times H^2(-1)$ .

Next, we consider one of the minimal translation surfaces [28]

$$M^2 : \varphi(s, t) = (as, st, bs + d), a, b, d \in \mathbb{R}. \tag{37}$$

Then in  $\mathbb{R} \times H^2(-1)$ , the product Riemannian metric is defined by

$$\tilde{g} = \frac{dx^2 + dy^2}{y^2} + dz^2.$$

With respect to the metric  $\tilde{g}$ , an orthonormal basis on  $\mathbb{R} \times H^2(-1)$  is defined by

$$e_1 = y \frac{\partial}{\partial x}, \quad e_2 = y \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

Now, we assume that  $\mathbb{R}$  has a trivial statistical structure. By using statistical structure on  $H^2(-1)$ , we can construct dualistic structure on a product of two statistical manifolds  $\mathbb{R} \times H^2(-1)$  as follows

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= 2e_2, & \tilde{\nabla}_{e_2} e_1 &= e_1, & \tilde{\nabla}_{e_3} e_1 &= 0, & \tilde{\nabla}_{e_1} e_2 &= 0, \\ \tilde{\nabla}_{e_2} e_2 &= 2e_2, & \tilde{\nabla}_{e_3} e_2 &= 0, & \tilde{\nabla}_{e_1} e_3 &= 0, & \tilde{\nabla}_{e_2} e_3 &= 0, & \tilde{\nabla}_{e_3} e_3 &= 0. \end{aligned}$$

Then with the help of (37), we obtain

$$\varphi_s = \frac{a}{st} e_1 + \frac{1}{s} e_2 + b e_3, \quad \varphi_t = \frac{1}{t} e_2.$$

So, we can find the unit normal vector field  $U$  of surface of  $M^2$

$$U = \frac{\varphi_s \times \varphi_t}{\|\varphi_s \times \varphi_t\|} = -\frac{b}{wt} e_1 + \frac{b}{wst^2} e_3, \quad w = \|\varphi_s \times \varphi_t\|. \tag{38}$$

On the other hand, we calculate

$$\tilde{\nabla}_{\varphi_s} \varphi_s = \left( 2 \frac{a^2}{s^2 t^2} + \frac{1}{s^2} \right) e_2, \quad \tilde{\nabla}_{\varphi_s} \varphi_t = \frac{2}{st} e_2, \quad \tilde{\nabla}_{\varphi_t} \varphi_t = \frac{2}{t^2} e_2. \tag{39}$$

By (38) and (39), we get coefficients of second fundamental form

$$L = \tilde{g}(\tilde{\nabla}_{\varphi_s} \varphi_s, U) = 0, \quad M = \tilde{g}(\tilde{\nabla}_{\varphi_s} \varphi_t, U) = 0, \quad N = \tilde{g}(\tilde{\nabla}_{\varphi_t} \varphi_t, U) = 0.$$

Hence,  $M^2$  is minimal statistical surface and also  $H = H^* = 0 = H^0$ .

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