Almost Riemann Solitons and Gradient Almost Riemann Solitons on LP-Sasakian Manifolds

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Abstract. The upcoming article aims to investigate almost Riemann solitons and gradient almost Riemann solitons in a LP-Sasakian manifold $M^3$. At first, it is proved that if $(g, Z, \lambda)$ be an almost Riemann soliton on a LP-Sasakian manifold $M^3$, then it reduces to a Riemann soliton, provided the soliton vector $Z$ has constant divergence. Also, we show that if $Z$ is pointwise collinear with the characteristic vector field $\xi$, then $Z$ is a constant multiple of $\xi$, and the ARS reduces to a Riemann soliton. Furthermore, it is proved that if a LP-Sasakian manifold $M^3$ admits gradient almost Riemann soliton, then the manifold is a space form. Also, we consider a non-trivial example and validate a result of our paper.

1. Introduction

The idea of Ricci flow was introduced by Hamilton [5] and defined by $\frac{d}{dt}g(t) = -2S(t)$, where $S$ denotes the Ricci tensor.

As a natural generalization, the concept of Riemann flow ([14],[15]) is defined by $\frac{d}{dt}G(t) = -2Rg(t)$, $G = \frac{1}{2}g \otimes g$, where $R$ is the Riemann curvature tensor and $\otimes$ is Kulkarni-Nomizu product (executed as (see Besse [2], p. 47),

$$(P \otimes Q)(X, Y, Z, W) = P(X, W)Q(Y, U) + P(Y, U)Q(X, W) - P(X, U)Q(Y, W) - P(Y, W)Q(X, U)).$$

Similar to Ricci soliton, the interesting idea of Riemann soliton was introduced by Hirica and Udriste [6]. Analogous to Hirica and Udriste [6], a Lorentzian metric $g$ on a Lorentzian manifold $M$ is called a Riemann solitons if there exists a $C^\infty$ vector field $Z$ and a real scalar $\lambda$ such that

$$2R + \lambda g \otimes g + g \otimes \mathcal{L}_Z g = 0.$$  \hspace{1cm} (1)

On this occasion, we should mention that the space of constant sectional curvature is generalized by the Riemann soliton. If the vector field $Z$ is the gradient of the potential function $\gamma$, then the manifold is called gradient Riemann soliton. Then the foregoing equation can be written as

$$2R + \lambda g \otimes g + g \otimes \nabla^2 \gamma = 0,$$  \hspace{1cm} (2)
where $\nabla^2 f$ denotes the Hessian of $\gamma$. If we modified the equation (1) and (2) by fixing the condition on the parameter $\lambda$ to be a variable function, then it reduces to $ARS$ and gradient $ARS$ respectively. Here the terminology “almost Riemann solitons” is written as $ARS$ which will be applied throughout the article.

A general idea of Lorentzian para-Sasakian (briefly $LP$-Sasakian) manifold has been introduced by K. Matsumoto [7], in 1989 and several geometors in different context ([1], [8], [9], [10]) have studied $LP$-Sasakian manifolds. Riemann solitons and gradient Riemann solitons on Sasakian manifolds have been discussed in detail by Hirica and Udriste (see, [6]). Moreover, Riemann’s soliton concerning infinitesimal harmonic transformation was investigated in [13]. Here it is appropriate to notice that Sharma in [11] investigated almost Ricci soliton in $K$-contact geometry and in [12], with divergence-free soliton vector field. Very recently in [4], the authors studied Riemann solitons within the framework of a contact manifold and proved various fascinating results.

The above studies motivate us to investigate an $ARS$ and the gradient $ARS$ in a 3-dimensional $LP$-Sasakian manifold.

The upcoming article is structured as follows: In section 2, we recall some fundamental facts and formulas of $LP$-Sasakian manifolds, which will be needed in later sections. Beginning from Section 3, after providing the proof, we will write our prime theorems. This article terminates with a concise bibliography which has been used during the formulation of the upcoming article.

2. $LP$-Sasakian manifolds

Let $\eta, \xi, \phi$ are tensor fields on a smooth manifold $M^n$ of types (0,1), (1,0) and (1,1) respectively, such that

$$\eta(\xi) = -1, \quad \phi^2 E = E + \eta(E)\xi.$$  \hspace{1cm} (3)

The foregoing equations imply that

$$\phi\xi = 0, \quad \eta \circ \phi = 0.$$  \hspace{1cm} (4)

Then $M^n$ admits a Lorentzian metric $g$ of type (0,2) such that

$$g(E, \xi) = \eta(E), \quad g(\phi E, \phi F) = g(E, F) + \eta(E)\eta(F)$$ \hspace{1cm} (5)

for any vector fields $E, F$. Then the structure $(\eta, \xi, \phi, g)$ is called Lorentzian almost para-contact structure. The manifold $M^n$ equipped with a Lorentzian almost para-contact structure $(\eta, \xi, \phi, g)$ is called a Lorentzian almost para-contact manifold (briefly LAP-manifold).

If we denote $\Phi(E, F) = g(E, \phi F)$, then we obtain [7]

$$\Phi(E, F) = g(E, \phi F) = g(\phi E, F) = \Phi(F, E),$$ \hspace{1cm} (6)

where $E, F$ are any vector fields.

An LAP-manifold $M^n$ equipped with the structure $(\eta, \xi, \phi, g)$ is said to be a Lorentzian para-contact manifold (briefly LP-manifold) if

$$\Phi(E, F) = \frac{1}{2}[(\nabla_E \eta)F + (\nabla_F \eta)E],$$ \hspace{1cm} (7)

where $\Phi$ is defined by (6) and $\nabla$ indicates the covariant differentiation operator with respect to the Lorentzian metric $g$. A Lorentzian almost para-contact manifold $M^n$ is said to be a $LP$-Sasakian manifold if it satisfies

$$(\nabla_E \phi)F = \eta(F)E + g(E, F)\xi + 2\eta(E)\eta(F)\xi.$$ \hspace{1cm} (8)

Also since the vector field, $\eta$ is closed in an $LP$-Sasakian manifold we have

$$(\nabla_E \eta)F = \Phi(E, F) = g(E, \phi F), \quad \Phi(E, \xi) = 0, \quad \nabla_E \xi = \phi E.$$ \hspace{1cm} (9)

Furthermore, we find that the eigen values of $\phi$ are -1, 0 and 1. Here the multiplicity of 0 is one. Let us assume that the multiplicities of -1 and 1 are $k$ and $l$ respectively. Then we get, $\text{trace}(\phi) = l - k$. Hence, if
Using (17) and (16) in (15), we deduce

(\text{trace}(\phi))^2 = (n - 1),\text{ then either } l = 0 \text{ or } k = 0.\text{ Then the structure is called a trivial LP-Sasakian structure. Throughout this article we presume that } \text{trace}(\phi) \neq 0, \text{ i.e., } \xi \text{ is not harmonic.}

Let us presume that \( \{e_i\} \) be an orthonormal basis such that \( e_1 = \xi \).\text{ Then the well-known Ricci tensor } S \text{ and the scalar curvature } r \text{ are defined by}

\[
S(E,F) = \sum_{i=1}^{n} e_i g(R(e_i,E)F,e_i)
\]

and

\[
r = \sum_{i=1}^{n} e_i S(e_i,e_i),
\]

where we put \( e_i = g(e_i,\xi) \), that is, \( e_1 = -1, e_2 = \cdots = e_n = 1 \).

Also in an LP-Sasakian manifold \( M^n \), the subsequent relations hold ([1], [7], [10]):

\[
\eta(R(E,F)Z) = g(F,Z)\eta(E) - g(E,Z)\eta(F),
\]

(10)

\[
R(E,F)\xi = \eta(F)E - \eta(E)F,
\]

(11)

\[
R(\xi,E)F = g(E,F)\xi - \eta(F)E,
\]

(12)

\[
S(E,\xi) = (n - 1)\eta(E),
\]

(13)

\[
\nabla_\xi \eta = 0,
\]

(14)

for any vector fields \( E, F, Z \) where \( R \) is the Riemannian curvature tensor, \( S \) is the Ricci tensor and \( \nabla \) is the Levi-Civita connection associated to the metric \( g \).

It is well-known that a 3-dimensional Riemannian manifold \( M \) assumes the following curvature form

\[
R(E,F)Z = g(F,Z)QE - g(E,Z)QF + S(F,Z)E - S(E,Z)F
\]

\[
- \frac{r}{2} [g(F,Z)E - g(E,Z)F],
\]

(15)

for any vector fields \( E, F, Z \) where \( Q \) is the Ricci operator, i.e., \( g(QE,F) = S(E,F) \) and \( r \) is the scalar curvature of the manifold. Replacing \( F = Z = \xi \) in the previous equation and utilizing (11) and (13) we get (see [10])

\[
QE = \frac{1}{2} [(r - 2)E + (r - 6)\eta(E)\xi].
\]

(16)

In view of (16) the Ricci tensor is written as

\[
S(E,F) = \frac{1}{2} [(r - 2)g(E,F) + (r - 6)\eta(E)\eta(F)].
\]

(17)

Using (17) and (16) in (15), we deduce

\[
R(E,F)Z = \frac{(r - 4)}{2} [g(F,Z)E - g(E,Z)F]
\]

\[
+ \frac{(r - 6)}{2} [g(F,Z)\eta(E)\xi - g(E,Z)\eta(F)\xi]
\]

\[
+ \eta(F)\eta(Z)E - \eta(E)\eta(Z)F.
\]

(18)

We first prove the following Lemma:
Lemma 2.1. Let $M^3$ be a LP-Sasakian manifold. Then we have
\[ \xi r = -2(r - 6)\text{trace}(\phi). \]  
\[ (19) \]

Proof. The equation (16) can be rewritten as:
\[ QF = \frac{1}{2}[(r - 2)F + (r - 6)\eta(F)\xi]. \]
\[ (20) \]

Taking covariant derivative along $E$ and recalling (9) we write
\[ (\nabla_{E}Q)F = \frac{(Er)}{2}F + \frac{(Er)}{2}\eta(F)\xi + \frac{(r - 6)}{2}g(E, \phi F)\xi + \frac{(r - 6)}{2}\eta(F)\phi E. \]
\[ (21) \]

Taking inner product operation with respect to $Z$ in the foregoing equation, we obtain
\[ g((\nabla_{E}Q)F, Z) = \frac{(Er)}{2}g(F, Z) + \frac{(Er)}{2}\eta(F)\eta(Z) + \frac{(r - 6)}{2}g(E, \phi F)\eta(Z) + \frac{(r - 6)}{2}\eta(F)g(\phi E, Z). \]
\[ (22) \]

Putting $E = Z = e_i$ (where $\{e_i\}$ is an orthonormal basis for the tangent space of $M^3$ and taking $\sum i, 1 \leq i \leq 3$) in the above equation and utilizing the formula of Riemannian manifolds $\text{div}Q = \frac{1}{2}\text{grad} r$, we obtain
\[ (\xi r)\eta(F) = -2(r - 6)\eta(F)\text{trace}(\phi). \]
\[ (23) \]

Substituting $F = \xi$ in the above equation we get the desired result. This finishes the proof. \[ \Box \]

If an LP-Sasakian manifold $M^3$ is a space of constant curvature, then the manifold is said to be a space form.

Lemma 2.2. (Lemma 1.1 of [10]) A 3-dimensional LP-Sasakian manifold is a space form if and only if the scalar curvature $r = 6$.

Lemma 2.3. (Lemma 3.8 of [4]) For any vector fields $E, F$ on $M^3$, for a gradient ARS $(M, g, \gamma, m, \lambda)$, we have
\[ R(E, F)D\gamma = (\nabla_{E}Q)E - (\nabla_{E}Q)F + \{F(2\lambda + \triangle\gamma)E - E(2\lambda + \triangle\gamma)F\}, \]
\[ (24) \]

where $\triangle\gamma = \text{div} D\gamma$, $\triangle$ is the Laplacian operator.

3. ARS on 3-dimensional LP-Sasakian manifolds

We consider a 3-dimensional para-Sasakian manifold $M$ admitting an ARS defined by (1). Using Kulkarni-Nomizu product in (1) we write
\[ 2R(E, F, W, X) + 2\lambda [g(E, X)g(F, W) - g(E, W)g(F, X)] + [g(E, X)(\varepsilon_{Z}g)(F, W) + g(F, W)(\varepsilon_{Z}g)(E, X) - g(E, W)(\varepsilon_{Z}g)(F, X) - g(F, X)(\varepsilon_{Z}g)(E, W)] = 0. \]
\[ (25) \]

Contracting (25) over $E$ and $X$, we get
\[ (\varepsilon_{Z}g)(F, W) + 2S(F, W) + (4\lambda + 2\text{div} Z)g(F, W) = 0. \]
\[ (26) \]

Utilizing (17) in the above equation we obtain
\[ (\varepsilon_{Z}g)(F, W) = -(r - 2 + 4\lambda + 2\text{div} Z)g(F, W) - (r - 6)\eta(F)\eta(W) = 0. \]
\[ (27) \]
Applying $Z$ has constant divergence and executing covariant derivative along $E$, we lead

\[
(\nabla_E Zg)(F, W) = - [\{(E \omega) + 4(E\lambda)\}g(F, W) - (E\omega)\eta(F)\eta(W) - (r - 6)[g(\phi E, F)\eta(W) + g(\phi E, W)\eta(F)] = 0. \tag{28}
\]

Now we recall the formula by Yano (see, [16]):

\[
(\nabla_E Zg)(F, W) = g((\nabla_Z V)(E, F), W) - g((\nabla_Z V)(E, W), F). \tag{29}
\]

Using symmetric property of $E \nabla$, it reveals from (29) that

\[
g((\nabla_Z V)(E, F), W) = \frac{1}{2}(\nabla_E Zg)(F, W) + \frac{1}{2}(\nabla_F Zg)(E, W) - \frac{1}{2}(\nabla_W Zg)(E, F). \tag{30}
\]

Utilizing (28) in (30) we obtain

\[
2g((\nabla_Z V)(E, F), W) = - [\{(E \omega) + 4(E\lambda)\}g(F, W) - (E\omega)\eta(F)\eta(W) - (r - 6)[g(\phi E, F)\eta(W) + g(\phi E, W)\eta(F)]
\]

\[
- [(r \omega) + 4(r\lambda)]g(E, W) - (r \omega)\eta(E)\eta(W)
\]

\[
- (r - 6)[g(\phi E, W)\eta(W) + g(\phi E, F)\eta(F)]
\]

\[
+ [(W \omega) + 4(W\lambda)]g(F, W) + (W \omega)\eta(E)\eta(F)
\]

\[
- (r - 6)[g(\phi W, E)\eta(E) + g(\phi W, F)\eta(E)]. \tag{31}
\]

After substituting $E = F = e_i$ in the foregoing equation and removing $Z$ from both sides, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking $\Sigma i, 1 \leq i \leq 3$, we have

\[
(\nabla_Z \Sigma)(e_i, e_i) = 2D\lambda - (E \omega)\xi - 2(r - 6)trace(\phi)\xi, \tag{32}
\]

where $E \omega = g(D\omega, E)$, $D$ denotes the gradient operator with respect to $g$.

Now differentiating (1) and utilizing it in (29) we can easily determine

\[
g((\nabla_Z V)(E, F), W) = (\nabla_W S)(E, F) - (\nabla_E S)(F, W) - (\nabla_F S)(E, W). \tag{33}
\]

Taking $E = F = e_i$ (where $\{e_i\}$ is an orthonormal frame) in (33) and summing over $i$ we obtain

\[
(\nabla_Z \Sigma)(e_i, e_i) = 0, \tag{34}
\]

for all vector fields $Z$. Combining (32) and (34) gives

\[
-2D\lambda + (E \omega)\xi + 2(r - 6)trace(\phi)\xi = 0. \tag{35}
\]

Utilizing (19) in the previous equation, we get

\[
D\lambda = 0. \tag{36}
\]

This implies that $\lambda$ is constant. This leads to the following theorem:

**Theorem 3.1.** If the soliton vector $Z$ has constant divergence in a LP-Sasakian manifold $M^3$, then an ARS reduces to a Riemann soliton.
Now let the potential vector field \( Z \) be point-wise collinear with the characteristic vector field \( \xi \) (i.e., \( Z = b\xi \), where \( b \) is a function on \( M^3 \)) and has constant divergence. Therefore from (26) we lead

\[
g(V_{\xi}b\xi, F) + g(V_{\xi}b\xi, E) + 2S(E,F) + 4\lambda g(E,F) = 0. \tag{37}
\]

Using (9) in (37), we get

\[
(EB)\eta(F) + (FB)\eta(E) + 2S(E,F) + (4\lambda + 2\text{div}Z)g(E,F) = 0. \tag{38}
\]

Putting \( F = \xi \) in (38) yields

\[
-(EB)\eta(E) + (\xi b)\eta(E) + 2\eta(E) + (4\lambda + 2\text{div}Z)\eta(E) = 0. \tag{39}
\]

Putting \( E = \xi \) in (39) we have

\[
(\xi b) = (2\lambda + \text{div}Z - 2). \tag{40}
\]

Putting the value of \( \xi b \) in (39) gives

\[
db = -(6\lambda + 3\text{div}Z + 2)\eta. \tag{41}
\]

Operating (41) by \( d \) and utilizing Poincare lemma \( d^2 = 0 \), we infer

\[
0 = d^2b = -(6\lambda + 3\text{div}Z + 2)d\eta - 6d\lambda\eta. \tag{42}
\]

Executing wedge product of (42) with \( \eta \), we have

\[
-(6\lambda + 3\text{div}Z + 2)\eta \wedge d\eta = 0. \tag{43}
\]

Since \( \eta \wedge d\eta \not= 0 \) in a LP-Sasakian manifold \( M^3 \), therefore

\[
\lambda = -(\frac{1}{2}\text{div}Z + \frac{1}{3}). \tag{44}
\]

Using (44) in (41) gives \( db = 0 \) i.e., \( b = \text{constant} \). Also from (32) we obtain

\[
\lambda = -(\frac{1}{2}\text{div}Z + \frac{1}{3}) = \text{constant}. \tag{45}
\]

Hence we write the following:

**Theorem 3.2.** If the metric of a LP-Sasakian manifold \( M^3 \) is ARS and \( Z \) is pointwise collinear with \( \xi \) and has constant divergence, then \( Z \) is a constant multiple of \( \xi \) and the ARS reduces to a Riemann soliton.

**Corollary 3.3.** If a LP-Sasakian manifold \( M^3 \) admits an ARS of type \( (g, \xi) \), then the ARS reduces to a Riemann soliton.

4. **Gradient Almost Riemann soliton**

This section is devoted to investigate a LP-Sasakian manifold \( M^3 \) admitting gradient ARS. Now before producing the detailed proof of our main theorems, we first write the following results without proof (Since the result can be obtained directly from (21)):

**Lemma 4.1.** For a LP-Sasakian manifold \( M^3 \), we have

\[
(V_{\xi}Q)\xi = -\left(\frac{r}{2} - 3\right)\phi E, (V_{\xi}Q)E = -2r - 6\text{trace}\phi[E + \eta(E)\xi]. \tag{46}
\]
Replacing $F$ by $\xi$ in (24) and utilizing the foregoing Lemma, we obtain

$$R(E, \xi)D\gamma = \{\frac{r}{2} - 3\phi E - 2(r - 6)\text{trace}\phi[E + \eta(E)\xi]\}$$

$$+ \{\xi(2\lambda + \Delta\gamma)E - E(2\lambda + \Delta\gamma)\xi\}. \quad (47)$$

Then using (8), we infer

$$g(E, D\gamma + D(2\lambda + \Delta\gamma)\xi) = \{\frac{r}{2} - 3\phi E - 2(r - 6)\text{trace}\phi[E + \eta(E)\xi]\}$$

$$+ (\xi\eta) + E(2\lambda + \Delta\gamma)E. \quad (48)$$

Executing the inner product of the previous equation with $\xi$ gives

$$E(\gamma + (2\lambda + \Delta\gamma)) = (\xi\eta) + E(2\lambda + \Delta\gamma)\eta(E), \quad (49)$$

from which easily we obtain

$$d(\gamma + (2\lambda + \Delta\gamma)) = (\xi\eta) + E(2\lambda + \Delta\gamma)\eta, \quad (50)$$

where $d$ indicates the exterior derivative. From the previous equation we see that $\gamma + (2\lambda + \Delta\gamma)$ is invariant along the distribution $D$. In other terms, $E(\gamma + (2\lambda + \Delta\gamma)) = 0$ for any $E \in D$. Using (49) in (48), we lead

$$[(\xi\eta) + E(2\lambda + \Delta\gamma)]\eta(E)\xi - E$$

$$= (\frac{r}{2} - 3\phi E - 2(r - 6)\text{trace}\phi[E + \eta(E)\xi] \quad (51)$$

Contracting the above equation yields

$$[(\xi\eta) + E(2\lambda + \Delta\gamma)] = 0. \quad (52)$$

Utilizing (52) in (51), we get

$$\{\frac{r}{2} - 3\phi E - 4\text{trace}\phi[E + \eta(E)\xi]\} = 0. \quad (53)$$

If $\{\phi E - 4\text{trace}\phi[E + \eta(E)\xi]\} = 0$, operating $\phi$ we can easily obtain $\phi^2 E = 4\text{trace}\phi \phi(E)$, which is obviously a contradiction. Thus we have $r = 6$. Hence by Lemma 2.2, the manifold is a space form.

Hence we write the following:

**Theorem 4.2.** If a LP-Sasakian manifold $M^3$ admits a gradient ARS, then the manifold is a space form.

5. **Example**

Here we consider a known example of our paper [3]. In this article, we considers a 3-dimensional manifold $M = \{(u, v, w) \in \mathbb{R}^3,w \neq 0\}$ and The vector fields

$$e_1 = e^u \frac{\partial}{\partial v}, \quad e_2 = e^v \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial w}\right), \quad e_3 = \frac{\partial}{\partial w}$$

are linearly independent at each point of $M$ and shows that the manifold is a LP-Sasakian manifold. Further, the well-known Koszul’s formula gives

$$\nabla_{e_1} e_1 = -\delta_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_1 = -\delta_1,$$

$$\nabla_{e_3} e_2 = 0, \quad \nabla_{e_2} e_3 = -\delta_3, \quad \nabla_{e_2} e_3 = -\delta_2,$$

$$\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_3 = 0. \quad (54)$$

Also, we have obtained the expressions of the curvature tensor and the Ricci tensor, respectively, as follows:
\[ R(\delta_1, \delta_2)\delta_3 = 0, \quad R(\delta_2, \delta_3)\delta_3 = -\delta_2, \quad R(\delta_1, \delta_3)\delta_3 = -\delta_1, \]
\[ R(\delta_1, \delta_2)\delta_2 = \delta_1, \quad R(\delta_2, \delta_3)\delta_2 = -\delta_3, \quad R(\delta_1, \delta_3)\delta_2 = 0, \]
\[ R(\delta_1, \delta_2)\delta_1 = -\delta_2, \quad R(\delta_2, \delta_3)\delta_1 = 0, \quad R(\delta_1, \delta_3)\delta_1 = -\delta_3, \]
and
\[ S(\delta_1, \delta_1) = g(R(\delta_1, \delta_2)\delta_2, \delta_1) - g(R(\delta_1, \delta_3)\delta_3, \delta_1) = 2. \]

Similarly we have
\[ S(\delta_2, \delta_2) = 2, \quad S(\delta_3, \delta_3) = -2 \]
and
\[ S(\delta_i, \delta_j) = 0 (i \neq j). \]

Therefore,
\[ r = S(\delta_1, \delta_1) + S(\delta_2, \delta_2) - S(\delta_3, \delta_3) = 6. \]

From the expressions of the Ricci tensor, we find that M is an Einstein manifold.

Suppose \( f : M^3 \to \mathbb{R} \) be a smooth function such that \( f = w \). Then we can obtain
\[ Df = \frac{\partial}{\partial w} = \delta_3. \]

Using (54) we get
\[ \text{Hess} f(\delta_3, \delta_3) = 0. \]

Thus from (2) we can easily see that \( g \) is a gradient Riemann soliton with \( f = w \) and \( \lambda = -1 \). Hence the Theorem 4.2. is verified.

References