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Expansion of Implicit Mapping Theory to Split-Quaternionic Maps in Clifford Analysis

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Abstract. This paper presents the regularity of a split-quaternionic function and a corresponding split-Cauchy–Riemann system of a split quaternion. The properties of an inverse and an implicit mapping theory for a split-quaternionic map are investigated. In addition, the paper proposes a definition and expression for a split biregular mapping in an open set in $\mathbb{C}^2$. The obtained results are illustrated with some examples.

1. Introduction

The set of quaternions, introduced in 1843 by Hamilton [7], can be expressed as

$$H = q = x_0 + x_1 i + x_2 j + x_3 k \mid x_r \in \mathbb{R} \ \ (r = 0, 1, 2, 3)$$

with relations among i, j and k, satisfying

$$i^2 = j^2 = k^2 = -1, \quad ijk = -1.$$

From the relations among i, j and k, the set of quaternions is noncommutative division algebra. After then, the set of split-quaternions introduced by Cockle [3] in 1849, as follows:

$$\mathcal{S} = \{z = x_0 + x_1 i + x_2 j + x_3 k \mid x_r \in \mathbb{R}, \ \ r = 0, 1, 2, 3\},$$

where $i^2 = -1$, $j^2 = k^2 = 1$ and $ijk = 1$. The set of split quaternions is also non-commutative. On the other hand, it contains zero divisors, nilpotent elements, and nontrivial idempotents [17]. Recently, various results have been obtained for the properties of split quaternions and split quaternionic functions. Because split quaternions can be used to express Lorentzian rotations, split quaternionic equations [1, 6, 16, 17] have been previously solved on geometric and physical applications of split quaternions. Furthermore, using the properties of split quaternions, the modified split quaternions are derived which have a modified and associated form with other algebras. For example, Kula et al. [16] proposed dual split quaternions

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and provided the screw motion in $\mathbb{R}^3_1$, using the properties of Hamilton operators. Kim and Shon [14] provided a regular function with values in dual split quaternions and relations between a corresponding Cauchy-Riemann system and a regularity of functions with values in dual split quaternions.

Various versions of the implicit function theorem have been presented so far. The history of the implicit function theorem has been studied on algebraic geometry, real and complex power series, and differential geometry. Furthermore, some developments have been made in the implicit function theorem and inverse function theorem in terms of differentiable manifolds, Riemannian geometry, partial differential equations, and numerical analysis. Krantz and Parks [15], Dontchev and Rockafellar [5], Hurwicz and Richter [8], and Scarpello [20] presented many variants of the implicit function theorem with proofs and applications to algebra, differential geometry, functional analysis, and other branches of mathematics. Complex variable versions of the theorems were introduced by Krantz and Parks [15] and Burckel [2]. Oliveira [18] presented simple proofs of the implicit and inverse function theorems on a finite-dimensional Euclidean space by using the intermediate value theorem and mean value theorem.

We have previously investigated the corresponding Cauchy-Riemann systems and the regularity properties of a split-quaternion-valued function on a split-quaternionic variable [11–13]. Kilbas et al. [10] provided the most developments on the calculus of integrals and derivatives of any arbitrary real or complex order and fractional differential equations involving many different potentially useful operators of fractional calculus and its applications (see [22]). Srivastava [21] proposed various operators of fractional-order derivatives, as well as fractional-order integrals, provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics. Based on these results, in this paper, section 2 provided a derivative operator based on the algebraic properties of the split quaternion. The split quaternionic function was defined with a split quaternion as a variable, and the split regularity was defined by using the previously defined derivative operator. A Cauchy-Riemann system derived from the definition of split regularity was defined. The properties of the defined split regularity for the algebraic operation were investigated. Section 3 investigated the implicit split regular mapping, and the split regularity was presented and proved with respect to the inverse mapping theorem. The results of the respect to the implicit theorem were summarized and verified. In addition, examples of each theorem were presented to examine the use of the split regular function in the split quaternion system.

2. Preliminaries

Let e_r $(r = 0, 1, 2, 3)$ be the basis elements for the split quaternionic field $\mathcal{S}$ with the following noncommutative multiplication rules:

$$e_1^2 = -1,\ \ e_2^2 = e_3^2 = 1,\ \ e_j e_k = -e_k e_j,\ \ \overline{e_j} = -e_j\ \ (j \neq k, j \neq 0, k \neq 0),$$

where e_0 is the identity 1 of $\mathcal{S}$ and e_1 identifies the imaginary unit $i = \sqrt{-1}$ in the complex numbers. A split quaternion z is given by $z = \sum_{r=0}^{3} x_r e_r$, where x_r $(r = 0, 1, 2, 3)$ are real numbers. A split quaternion z can also be expressed as $z = z_1 + z_2 e_2$, where $z_1 = x_0 + e_1 x_1$ and $z_2 = x_2 + e_1 x_3$ are complex numbers in $\mathbb{C}$.

The split quaternionic conjugation z^* and modulus $N(z)$ of z in $\mathcal{S}$ are expressed as

$$z^* = \sum_{r=0}^{3} x_r \overline{e_r} = \overline{z_1} - z_2 e_2$$

and

$$N(z) = zz^* = x_0^2 + x_1^2 - x_2^2 - x_3^2 = |z_1|^2 - |z_2|^2,$$

respectively. The inverse element z^{-1} of z in $\mathcal{S}$ is given by the following:

$$z^{-1} = \frac{z^*}{N(z)}\ \ (N(z) \neq 0).$$

The differential operators are denoted by

$$D := \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial \overline{z_2}} \quad \text{and} \quad D^* = \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial \overline{z_2}},$$

where $\partial / \partial z_r$ and $\partial / \partial \overline{z_r}$ $(r = 1, 2)$ are usual differential operators used in complex analysis. Then, the Coulomb operator (see [4]) is given by

$$DD^* = D^* D = \frac{1}{4} \sum_{r=0}^{3} \frac{\partial^2}{\partial x_r^2} = \frac{\partial^2}{\partial z_1 \partial \overline{z_1}} - \frac{\partial^2}{\partial z_2 \partial \overline{z_2}}.$$

Let Ω be an open set in $\mathbb{C}^2$. Consider a function $f : \Omega \to \mathcal{S}$ denoted by

$$\begin{aligned} & f = \sum_{r=0}^{3} u_r e_r = f_1 + f_2 e_2, \\ & z = (z_1, z_2) \in \Omega \mapsto f(z) = f_1(z) + f_2(z) e_2 \in \mathcal{S}, \end{aligned}$$

where u_r $(r = 0, 1, 2, 3)$ are real-valued functions.

Definition 2.1. *Let Ω be an open set in $\mathbb{C}^2$. A function $f(z) = f_1(z) + f_2(z) e_2$ is said to be L(R)-split regular in Ω if the following two conditions are satisfied:*
(i) *$f_1(z)$ and $f_2(z)$ are continuously differentiable functions in Ω, and*
(ii) *$D^* f(z) = 0$ (or $f(z) D^* = 0$) in Ω.*

Remark 2.2. *Let Ω be an open set in $\mathbb{C} \times \{0\}$. A function $f(z)$ is said to be degenerated L(R)-split regular in Ω if the following two conditions are satisfied:*
(i) *$f(z)$ is a continuously differential function in Ω, and*
(ii) *$D^* f(z) = 0$ (or $f(z) D^* = 0$) in Ω.*

Instead of saying that the function $f(z)$ is L-split regular in $\Omega \subset \mathbb{C}^2$, we simply say that $f(z)$ is split regular in $\Omega \subset \mathbb{C}^2$.

Remark 2.3. *The equation $D^* f(z) = 0$ is expanded as follows:*

$$\begin{aligned} D^* f &= \{\frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial \overline{z_2}}\}(f_1(z) + f_2(z) e_2) \\ &= \{\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial \overline{f_2}}{\partial z_2}\} + \{\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial z_2}\} e_2. \end{aligned}$$

So, we have the following system:

$$\frac{\partial f_1}{\partial \overline{z_1}} = -\frac{\partial \overline{f_2}}{\partial z_2} \quad \textit{and} \quad \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial z_2}. \tag{1}$$

Remark 2.4. *The system* (1) *is equivalent to the following system:*

$$\begin{aligned} & \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} = 0, \quad \frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} - \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = 0, \\ & \frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} + \frac{\partial u_0}{\partial x_2} - \frac{\partial u_1}{\partial x_3} = 0, \quad \frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_0}{\partial x_3} = 0. \end{aligned} \tag{2}$$

The above systems (1) and (2) are called the split Cauchy-Riemann systems for $f(z)$ with values in $\mathcal{S}$.

Proposition 2.5. *[11] Let Ω be a bounded open set in $\mathbb{C}^2$. If two functions $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are split regular functions in Ω, where f_1, f_2, g_1 and g_2 are complex-valued functions, then*
(i) $\alpha f + \beta g$ is a split regular function in Ω, where α and β are real constants.
(ii) For a constant $c = c_1 + c_2 e_2$ in $\mathcal{S}$, a function fc is split regular functions in Ω,
(iii) fg is a split regular function in Ω when each of the components f_r and g_r $(r = 1, 2)$ of f and g, respectively, is a real-valued function.

Proof. (i) For α and β as real constants as f and g are split regular functions in Ω, the equation

$$D^*(\alpha f + \beta g) = \alpha D^* f + \beta D^* g = 0$$

is satisfied. Hence, $\alpha f + \beta g$ is a split regular function in Ω.
(ii) For a constant $c = c_1 + c_2 e_2$ in $\mathcal{S}$, the equation is expanded as follows:

$$D^*(fc) \;=\; \{\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial \overline{f_2}}{\partial z_2}\}c_1 + \{\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial z_2}\}\overline{c_2} + (\{\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial \overline{f_2}}{\partial z_2}\}c_2 + \{\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial z_2}\}\overline{c_1})e_2.$$

As f is a split regular function in Ω, the equation $D^*(fc) = 0$ is satisfied, and then, the function fc is a split regular function in Ω.
(iii) As each of the components f_r and g_r $(r = 1, 2)$ of f and g, respectively, is a real-valued function, we have $f_r = \overline{f_r}$ and $g_r = \overline{g_r}$ $(r = 1, 2)$. Hence, the equation

$$\begin{aligned} D^*(fg) \;=\;& \{\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial \overline{f_2}}{\partial z_2}\}g_1 + \{\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial z_2}\}\overline{g_2} + (\{\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial \overline{f_2}}{\partial z_2}\}g_2 + \{\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial z_2}\}\overline{g_1})e_2 \\ &+ f_1\frac{\partial g_1}{\partial \overline{z_1}} + \overline{f_1}\frac{\partial \overline{g_2}}{\partial z_2} + f_2\frac{\partial \overline{g_2}}{\partial \overline{z_1}} + \overline{f_2}\frac{\partial g_1}{\partial z_2} + (f_1\frac{\partial g_2}{\partial \overline{z_1}} + \overline{f_1}\frac{\partial \overline{g_1}}{\partial z_2} + f_2\frac{\partial \overline{g_1}}{\partial \overline{z_1}} + \overline{f_2}\frac{\partial g_2}{\partial z_2})e_2 \\ =\;& \{\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial \overline{f_2}}{\partial z_2}\}g_1 + \{\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial z_2}\}\overline{g_2} + (\{\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial \overline{f_2}}{\partial z_2}\}g_2 + \{\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial \overline{f_1}}{\partial z_2}\}\overline{g_1})e_2 \\ &+ f_1(\frac{\partial g_1}{\partial \overline{z_1}} + \frac{\partial \overline{g_2}}{\partial \overline{z_2}} + f_2\frac{\partial g_2}{\partial \overline{z_1}} + \frac{\partial \overline{g_1}}{\partial z_2} + (f_1\frac{\partial g_2}{\partial \overline{z_1}} + \frac{\partial \overline{g_1}}{\partial z_2} + f_2\frac{\partial g_1}{\partial \overline{z_1}} + \frac{\partial \overline{g_2}}{\partial z_2})e_2 \end{aligned}$$

is expanded. Thus, we obtain $D^*(fg) = 0$, and then, fg is a split regular function in Ω. □

3. Implicit Split-regular Mappings

Definition 3.1. *Let Ω be an open set in $\mathbb{C}^2$ and $f = (f_1, f_2) : \Omega \longrightarrow \mathcal{S}$ be a split-quaternionic function. The mapping f is said to be split regular if both components f_1 and f_2 are split regular in Ω.*

Definition 3.2. *Let Ω_1 and Ω_2 be domains in $\mathbb{C}^2$. Suppose a split regular mapping $f = (f_1, f_2) : \Omega_1 \longrightarrow \Omega_2$ is bijective and the inverse mapping $f^{-1} : \Omega_2 \longrightarrow \Omega_1$ of f is split regular. Then f is called a split biregular function, where the domains Ω_1 and Ω_2 are split biregular equivalents.*

Theorem 3.3 (Inverse mapping theorem). *Let Ω_1 be an open set in $\mathbb{C}^n$ and $f : \Omega_1 \rightarrow \mathbb{C}^n$ be a split regular mapping. For a point $z^0 \in \Omega_1$ and its image $w^0 = f(z^0)$, f is a split biregular mapping from an open neighborhood of z_0 onto that of w_0 if and only if*

$$\det J_f(z^0) := \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{pmatrix} \neq 0.$$

Proof. Let us refer to [9]. From the definition of a split biregular mapping, there exists an inverse mapping f^{-1} near $f(z_0)$. Then we have

$$\det J_{f^{-1}}(z^0)\det J_f(z^0) = \det J_{id}(z^0) = id.$$

Thus, we obtain $\det J_f(z^0) \neq 0$. Conversely, according to the real version of the inverse mapping theorem, there is an inverse mapping $g : f(\Omega_1) \to \Omega_1$ of the mapping $h : \Omega_1 \to f(\Omega_1)$ such that g is continuous and has real derivatives. Now, we show that g is split regular. As the identity mapping $g(f(z)) = z$ is split regular, we have $D^*g = 0$. Therefore, g is split regular in $f(\Omega_1)$. □

Example 3.4. *Consider the mapping f to be given by*

$$f(r,\theta) = \begin{pmatrix} r\cosh\theta \\ r\sinh\theta \end{pmatrix},$$

where r and θ are coordinates in a Cartesian (r,θ) plane. Then f is not one-to-one but locally bijective in $0 < \theta < \frac{\pi}{2}$. We find the equation

$$\det J_f(r,\theta) = \begin{vmatrix} \cosh\theta & r\sinh\theta \\ \sinh\theta & r\cosh\theta \end{vmatrix} = r \neq 0,$$

where $z_1 = r\cosh\theta$ and $z_2 = r\sinh\theta$. Therefore, f is locally split biregular in $0 < \theta < \frac{\pi}{2}$.

Theorem 3.5. *Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{C}^n \times \mathbb{C}^m$ be an open set and $f : \Omega \to \mathbb{C}^m$ be a split regular mapping. In addition, suppose that $(z^0, w^0) \in \Omega$ is a point with $f(z^0, w^0) = 0$ and*

$$\det J_f(z^0) := \begin{pmatrix} \frac{\partial f_1}{\partial w_1} & \cdots & \frac{\partial f_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial w_1} & \cdots & \frac{\partial f_m}{\partial w_m} \end{pmatrix} \neq 0.$$

Then there exists an open neighborhood $U = U_1 \times U_2$ of a point (z^0, w^0), where U_1 and U_2 are open neighborhoods of z^0 and w^0, respectively, and a split regular mapping $g : U_1 \to U_2$ such that

$$\{(a,b) \in U_1 \times U_2 \mid f(a,b) = 0\} = \{(a, g(a)) \mid a \in U_1\}.$$

Proof. From the inverse mapping theorem [9], a split regular mapping $\psi : \Omega \to \mathbb{C}^{n+m}$ with $\psi(z,w) = (z, f(z,w))$ induces a split biregular mapping $\phi : \Omega \to V$ which has an inverse mapping, where V is an open set in $\mathbb{C}^n$. Then the mapping ϕ is split regular in $\Omega_1 \times \{0\}$. Therefore, according to the real version of the inverse mapping theorem, there exists a split regular mapping g such that

$$\{(a,b) \in U \mid f(a,b) = 0\} = \{(a, g(a)) \mid a \in U_1\},$$

where $U_1 (\subset \Omega_1)$ and $U_2 (\subset \Omega_2)$ are open neighborhoods of z^0 and w^0, respectively. □

Example 3.6. *Let $\Omega \subset \mathbb{C}^2 \times \mathbb{C}^2$ be an open set, $f = (f_1, f_2) : \Omega \to \mathbb{C}^2$ be a split regular mapping and $(0, w) \in \Omega$. Suppose that $f(z,w) = (z_1, z_2)$, where $f_1(z,w) = z_1$ and $f_2(z,w) = z_2$. Then $\det J_f(0) \neq 0$ and there is a split regular map $g : U_1 \to U_2$ such that $g(z) = w$ with*

$$\{(z,w) \in U_1 \times U_2 \mid f(z,w) = 0\} = \{(0,w) \mid w \in U_2\}.$$

Theorem 3.7. *Let U be an open neighborhood of a point z in $\mathbb{C}^2$ and let $f : U \longrightarrow \mathcal{S}$ be a split regular mapping. The mapping $\overline{f} = (\overline{f_1}, \overline{f_2})$ is not split regular if and only if it is split biregular from U onto an open neighborhood of $f(z)$.*

Proof. Suppose that $\overline{f} = (\overline{f_1}, \overline{f_2})$ is not split regular mapping. From the inverse mapping theorem, as $f = (f_1, f_2)$ is a split regular mapping and

$$\det J_f(z) = \begin{vmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{vmatrix} \neq 0,$$

the mapping f is locally bijective and the inverse mapping f^{-1} belongs to class C^∞ in an open neighborhood of $w = f(z)$, where $w = (w_1, w_2)$, $w_1 = f_1(z)$ and $w_2 = f_2(z)$.

Now, we just need to show that f^{-1} is split regular in an open neighborhood of $f(z)$. Let g be an inverse mapping of f. Then, by [19],

$$\begin{aligned} 0 &= D^* z = \Big(\frac{\partial g(f(z))}{\partial \overline{z_1}} + e_2 \frac{\partial g(f(z))}{\partial \overline{z_2}}\Big) \\ &= \Big(\sum_{k=1}^{2} \frac{\partial g}{\partial w_k}\frac{\partial f_k}{\partial \overline{z_1}} + \frac{\partial g}{\partial \overline{w_k}}\frac{\partial \overline{f_k}}{\partial \overline{z_1}}\Big) + e_2\Big(\sum_{k=1}^{2} \frac{\partial g}{\partial w_k}\frac{\partial f_k}{\partial \overline{z_2}} + \frac{\partial g}{\partial \overline{w_k}}\frac{\partial \overline{f_k}}{\partial \overline{z_2}}\Big) \quad (k = 1, 2). \end{aligned}$$

Since f is a split regular mapping, we have

$$D^* f_k = \Big(\frac{\partial f_k}{\partial \overline{z_1}} + e_2 \frac{\partial f_k}{\partial \overline{z_2}}\Big) = 0 \quad (k = 1, 2).$$

In addition, as $\overline{f} = (\overline{f_1}, \overline{f_2})$ is not a split regular mapping, we have the following two cases:

(i) $D^*\overline{f_1} \neq 0$ and $D^*\overline{f_2} \neq 0$,

(ii) $D^*\overline{f_1} = 0$ and $D^*\overline{f_2} \neq 0$ (or $D^*\overline{f_1} \neq 0$ and $D^*\overline{f_2} = 0$).

However, as f is a split regular mapping, we have the equation $D^* f_1 = 0$, which implies

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial u_1}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_0} = -\frac{\partial u_0}{\partial x_1}.$$

But, if $D^*\overline{f_1} = 0$, then

$$\frac{\partial u_0}{\partial x_0} = -\frac{\partial u_1}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_0} = \frac{\partial u_0}{\partial x_1}.$$

Thus, it is sufficient to deal with the first case. If U is sufficiently small, then $\dfrac{\partial g}{\partial \overline{w_k}} = 0$ for all $k = 1, 2$. Therefore, f^{-1} is split regular in U.

For the converse, we assume that the mapping $\overline{f}$ is split regular in U. If g is the inverse mapping of f, then

$$\begin{aligned} id &= D^* z = \Big(\frac{\partial g(f(z))}{\partial z_1} - e_2 \frac{\partial g(f(z))}{\partial \overline{z_2}}\Big) \\ &= \Big(\sum_{k=1}^{2} \frac{\partial g}{\partial w_k}\frac{\partial f_k}{\partial z_1} + \frac{\partial g}{\partial \overline{w_k}}\frac{\partial \overline{f_k}}{\partial z_1}\Big) - e_2\Big(\sum_{k=1}^{2} \frac{\partial g}{\partial w_k}\frac{\partial f_k}{\partial \overline{z_2}} + \frac{\partial g}{\partial \overline{w_k}}\frac{\partial \overline{f_k}}{\partial \overline{z_2}}\Big) \quad (k = 1, 2). \end{aligned}$$

By the assumption, since the mapping f and $\overline{f}$ are split regular in U, we have the equations

$$\frac{\partial \overline{f}}{\partial \overline{z_k}} = 0, \quad \frac{\partial f}{\partial \overline{z_k}} = 0 \ \ (k = 1, 2),$$

respectively; thus, this is a contradiction. □

Example 3.8. *Let U be an open neighborhood of z in $\mathbb{C}^2$ and $f : U \to \mathcal{S}$ be a split regular mapping such that $f_1(z) = z_1$, $f_2(z) = z_2$ and $\overline{f} = (\overline{f_1}, \overline{f_2}) = (\overline{z_1}, \overline{z_2})$. Then f is a split regular mapping and $\overline{f}$ is not a split regular mapping in U. Therefore, from* Theorem 12, *f is split regular from U onto an open neighborhood of $f(z)$.*

Theorem 3.9. *For a point $z^0 = (z_1^0, z_2^0)$ in $\mathbb{C}^2$, let U be an open neighborhood of a point z^o. If*

$$\frac{\partial f_1}{\partial z_1} \neq 0$$

in U, then the equation $f_1(z) = 0$ is locally solvable to the point z_1 and the solution $z_1 = h(z_2)$ is degenerated split regular in an open neighborhood V of a point z_2^o.

Proof. From the implicit mapping theorem, there exists a C^1-function $h : \mathbb{C} \times \{0\} \to \mathbb{C}$ with $h(z_2) = z_1$ that solve the equation $f_1(z_1, z_2) = 0$. We show that the function h is degenerated split regular in U. The function f_1 can be differentiated with respect to $\overline{z_2}$:

$$\frac{\partial f_1}{\partial \overline{z_2}} = \frac{\partial f_1}{\partial z_1}\frac{\partial h}{\partial \overline{z_2}} + \frac{\partial f_1}{\partial \overline{z_1}}\frac{\partial \overline{h}}{\partial \overline{z_2}} + \frac{\partial f_1}{\partial \overline{z_2}} = 0.$$

As the function f_1 is split regular in U,

$$\frac{\partial f_1}{\partial \overline{z_1}} = 0\,, \quad \frac{\partial f_1}{\partial \overline{z_2}} = 0.$$

By the hypothesis $\dfrac{\partial f_1}{\partial z_1} \neq 0$ in U, we have $\dfrac{\partial h}{\partial \overline{z_2}} = 0$. Therefore, the solution $z_1 = h(z_2)$ is degenerated split regular in an open neighborhood V of the point z_2^0. □

Example 3.10. *Let f_1 be a split regular in an open neighborhood U of a point $z^0 = (z_1^0, z_2^0) = (0, 0)$ in $\mathbb{C}^2$ such that $f_1(z) = z_1 - z_2$. Then $\dfrac{\partial f_1}{\partial z_1} \neq 0$ in U and the equation $f_1(z) = 0$ is locally solvable with respect to the point z_1. In addition, the solution $z_1 = h(z_2) = z_2$ is degenerated split regular in an open neighborhood of the point z_2^0.*

4. Conclusion

This paper defined derivative operators and proposed a regular split-quaternionic mappings that have a split Cauchy-Riemann system on split quaternions, as well as an implicit mapping of a regular mapping in split quaternions. In addition, we investigated the properties of split biregular mappings and their relations with split regular mappings on an open set in $\mathbb{C}^2$. Some examples give to illustrate the obtained results.

The split quaternion is useful for expressing Lorentz rotation and transformation, and it can represent situations such as spacelike, timelike, and lightlike depending on the combination of the basis, so it can be applied in geometry and physics. In multivariable calculus, the implicit function theorem suggests a sufficient condition that the equations for variables show a sufficiently smooth functional relationship locally. In addition, it is possible to return from the primed to unprimed coordinates depending on the invertibility of the Jacobian matrix, which can be expressed by the inverse function theorem. The implicit and inverse theorems are used, among others, to demonstrate the existence of solutions of nonlinear partial differential equations and to parameterize the space of solutions. Therefore, by expressing the implicit and inverse function theorems by defining the split regularity for the split quaternionic function and presenting an example, they can be applied to situations that can be expressed as the split quaternions. It is expected that the contents of differential multivariate, partial differential equation, and numerical analysis for the number of split quadrants can be expanded.

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