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A Study on Value Distribution of the Riemann Zeta Function

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Abstract. We consider the unique determination of Riemann zeta function as a solution of its functional equaiton under the condition sharing value. Besides, we show how the Riemann zeta function is uniquely determined by one or two sharing values of truncated multiplicity. The results in present paper extend the theorems given by Li in [17] and Gao, Li in [12]. Moreover, we generalize the results to L-functions in the Selberg class.

1. Introduction and Main Results

The Riemann zeta function is defined by the Dirichlet series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad s=\sigma+it,$$

for $\mathrm{Re}(s)=\sigma>1$, which is absolutely convergent, and admits an analytical continuation as a meromorphic function in the complex plane. The famous, as yet unproved, Riemann hypothesis states as follows:

Conjecture 1.1 (**Riemann Hypothesis**). *The nontrivial zeros of $\zeta(s)$ lie on the line $\sigma=\frac{1}{2}$.*

It is closely related to the distribution of prime numbers and plays a pivotal role in analytic number theory. The problem of value distribution of the Riemann zeta function has been studied extensively, including the distribution of the zeros of $\zeta(s)$ and, more generally, the c-values of $\zeta(s)$, i.e., the roots of the equation $\zeta(s)=c$ (see e.g. [24], [32]). Recently, the problem has been generalized from the Riemann zeta-function to certain L-function, see eg. monograph [15], [24].

On the other hand, as we know, the functional equation of $\zeta(s)$ plays an important role in some properties of Riemann zeta function, especially the zero distribution, and the unique determination of the Riemann zeta function as a solution of the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)f(s)=\pi^{-\frac{(1-s)}{2}}\Gamma\Big(\frac{1-s}{2}\Big)f(1-s)$$

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has been extensively studied and it is known to have different solutions with certain relations (see [1], [2] , [6] etc. for studies of solutions of the Riemann functional equation).

Recently, Hu and Li [14] constructed an entire function of order not greater than 1, with all zeros located exactly at those nontrivial zeros of $\zeta(s)$ on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, that is

$$h(s) = \frac{1}{2}\prod_{v=1}^{\infty}\left(1 - \frac{s - s^2}{|s_v|^2}\right),$$

where s_v ($v = 1, 2, \cdots$) are the zeros, counting multiplicities of $\zeta(s)$ on the half line $\mathrm{Re}(s) = \frac{1}{2}$, $\mathrm{Im}(s) > 0$. It is easy to show that this function $h(s)$ satisfy the equation

$$h(s) = h(1 - s).$$

Further, they defined a meromorphic function $\eta(s)$ using the same expression as that for $\zeta(s)$, that is

$$h(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\eta(s).$$

By simple calculation, it can be seen that the function $\eta(s)$ also satisfies the Riemann functional equation as $\zeta(s)$ does:

$$\eta(1 - s) = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\eta(s).$$

Moreover, we also can find that the only zeros of $\eta(s)$ in the domain $\mathrm{Re}(s) < 0$ are the poles of $\Gamma\left(\frac{s}{2}\right)$, which are the trivial zeros of $\zeta(s)$. Other zeros of $\eta(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$ in view of the construction of $h(s)$ and $\eta(s)$. In addition, the point $s = 1$ is the only pole of $\eta(s)$, which is a simple pole with residue 1.

Using the function $\eta(s)$, Hu and Li established a necessary and sufficient condition for the Riemann Hypothesis as follows.

Theorem 1.2 (**[14]**). *The Riemann hypothesis is true if and only if* $\zeta(s) \equiv \eta(s)$.

Obviously, Theorem 1.2 shows that to prove the Riemann hypothesis we now only need to prove that $\zeta(s) \equiv \eta(s)$, in other words, the famous Riemann hypothesis is transformed into the uniqueness problem of meromorphic function. Note that $\eta(s)$ and $\zeta(s)$ satisfy the same functional equation, this property means that $\eta(s)$ is a solution of the Riemann functional equation. Clearly, we need to seek the conditions that force the solutions to become unique one – the Riemann zeta function. The following uniqueness problem is posed by Hu and Li in [14] according to the properties of $\eta(s)$.

Question 1.1 (**Uniqueness problem[14]**). *Let $f(s)$ be a meromorphic function (of order ≤ 1 with finitely many poles) in the complex plane such that*
(i) $f(s)$ and $\zeta(s)$ satisfy the same functional equation;
(ii) $Z(f) \subseteq Z(\zeta)$.
Under what conditions are $f(s)$ and $\zeta(s)$ identically equal?

Here, we point out that, to uniquely determine functions, condition (i) and (ii) in the Question 1.1 are not sufficient. For example, we consider the following function $f(s)$,

$$f(s) = \frac{\pi^{\frac{s}{2}}}{s(s-1)\Gamma\left(\frac{s}{2}\right)},$$

it is easy to see that $f(s)$ satisfies (i) and (ii), but $f(s) \not\equiv \zeta(s)$. In order to have the uniqueness of $f(s)$ and $\zeta(s)$, Hu and Li [14] gave the following uniqueness theorem under an additional condition that $f(s)$ tends to 1 as $\sigma \to +\infty$.

Theorem 1.3 (**[14]**). *Let $f(s)$ be a nonconstant meromorphic function in the complex plane of order ≤ 1 with* $\lim_{\sigma\to+\infty} f(s) = 1$. *Then $f(s) \equiv \zeta(s)$ if and only if $f(s)$ satisfies the Riemann functional equation and $Z(f) \subseteq Z(\zeta)$.*

Note that if a meromorphic function $f(s)$ satisfies the Riemann functional equation, then we can see that the zero set of $f(s)$ in $\mathrm{Re}(s) < 0$ is a subset of the zeros set of $\zeta(s)$ in $\mathrm{Re}(s) < 0$. Hence, we process the Question 1.1 under the condition (i) only and an additional condition of sharing value. Two meromorphic functions f and g in the complex plane are said to share a value $c \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in $\mathbb{C}$, where $f^{-1}(c) = \{s \in \mathbb{C} : f(s) = c\}$. Moreover, f and g are said to share a value c CM (counting multiplicities) if they share the value c IM and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities. For basic terms and notations of the value distribution of Nevanlinna one may refer to monographs [13, 35, 36].

Theorem 1.4. *Suppose that $f(s)$ is a meromorphic function with finitely many poles. If $f(s)$ and $\zeta(s)$ satisfy the same functional equation and share $a \in \mathbb{C} \setminus \{0\}$ CM, then $f(s) \equiv \zeta(s)$.*

Remark 1.5. The condition that $a \neq 0$ in Theorem 1.4 cannot be dropped. To see this, let $f(s) = \frac{1}{s(1-s)}\zeta(s)$. Then $\zeta(s)$ and $f(s)$ satisfy all the conditions of Theorem 1.4 except that $a \neq 0$, but $f(s) \not\equiv \zeta(s)$.

Furthermore, Bombieri and Perelli have mentioned in [3] that two L-functions with "enough" common zeros (without counting multiplicities) are expected to be dependent in a certain sense. Recently, mathematicians have considered the problem of how an L-function in the Selberg class $\mathcal{S}$ ($\zeta(s)$ is a special case) is uniquely determined by preimages of complex values, or sharing values, and got a lot of interesting results, see e.g. [9], [19], [20], [33] etc.. Particularly, Li in [17] established the following result.

Theorem 1.6 **([17]).** *Let $f(s)$ be a meromorphic function in the complex plane such that $f(s)$ has finitely many poles, and let a, b be two distinct finite values. If $f(s)$ and a nonconstant L-function $\mathcal{L}(s) \in \mathcal{S}$ share the value a counting multiplicities (CM) and the value b ignoring multiplicities (IM), then $f(s) \equiv \mathcal{L}(s)$.*

Remark 1.7. The author shows that the number "two" in Theorem 1.6 is the best possible. For instance, the function $\mathcal{L} = \zeta$ and $f = \zeta e^g$, where g is an entire function, share 0 CM, but they are not identically equal.

Remark 1.8. Theorem 1.6 does not hold without the hypothesis "finitely poles". Consider the function $\mathcal{L} = \zeta$ and $f = \frac{2\zeta}{\zeta+1}$, it is clear that $\mathcal{L}$ and f with infinitely many poles share $0, 1$ CM, but they are not identically equal.

In fact, Theorem 1.6 gives an affirmative answer to the following question which was posed by Liao and Yang in [18] when one of the values a, b, c is ∞.

Question 1.2 **([18]).** *If $f(s)$ is meromorpic in the complex plane and $f(s)$ share two distinct values a, b CM and a value c IM with Riemann zeta function $\zeta(s)$, where $c \notin \mathbb{C} \setminus \{a, b, 0\}$, can we conclude that $f(s) \equiv \zeta(s)$?*

In 2012, Gao and Li [12] considered the uniqueness question for the Riemann zeta function in terms of the preimages of three complex values. Moreover, they gave an affirmative solution to the Question 1.2 by the following result.

Theorem 1.9 **([12]).** *Let $a, b, c \in \mathbb{C} \cup \{\infty\}$ be distinct. If a meromorphic function $f(s)$ in the complex plane and the Riemann zeta function $\zeta(s)$ share a, b CM and c IM except possibly at finitely many points, then $f(s) \equiv \zeta(s)$.*

In the present paper, we will extend Theorem 1.6 and Theorem 1.9 by sharing value with truncated multiplicity k, and the method of proof is different from that in [17] and [12].

Definition 1.10. *Let k be a positive integer number or ∞, $\overline{E}_{k)}(a, f)$ denotes the set of all distinct zeros of $f - a$ with multiplicity no greater than k. In particular, if $k = 1$, $\overline{E}_{1)}(a, f)$ denotes the set of simple zeros of $f - a$, and $k = \infty$, $\overline{E}_{\infty}(a, f)$ denotes all distinct zeros of $f - a$. We say that a meromorphic function f shares the value a of truncated multiplicity k with a meromorphic function g if $\overline{E}_{k)}(a, f) = \overline{E}_{k)}(a, g)$.*

Obviously, two meromorphic functions $f(s)$ and $g(s)$ share a IM when $\overline{E}_{\infty}(a, f) = \overline{E}_{\infty}(a, g)$, i.e., $\overline{E}(a, f) = \overline{E}(a, g)$. Hence, our results will extend Theorem 1.6 and Theorem 1.9 when $k = \infty$. In fact, we have the following results.

Theorem 1.11. *Let $f(s)$ be a meromorphic function in the complex plane such that $f(s)$ has finitely many poles, and let a, b be two distinct finite values. If $f(s)$ and $\zeta(s)$ share a CM and $\overline{E}_{k)}(b, \zeta) = \overline{E}_{k)}(b, f)$ for some integer $k \geq 1$, then $f(s) \equiv \zeta(s)$.*

Theorem 1.12. *Let $f(s)$ be a meromorphic function in the complex plane and $a, b, c \in \mathbb{C} \cup \{\infty\}$ be distinct. If $f(s)$ and $\zeta(s)$ share a, b CM and $\overline{E}_{k)}(c, \zeta) = \overline{E}_{k)}(c, f)$ for some integer $k \geq 1$, then $f(s) \equiv \zeta(s)$.*

It is evident that Theorem 1.11 and Theorem 1.12 are still valid when $k = \infty$.

2. Preliminaries

The Riemann zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it,$$

for $\mathrm{Re}(s) = \sigma > 1$, which is absolutely convergent, and admits an analytical continuation as a meromorphic function in the complex plane $\mathbb{C}$ of order 1, which has only a simple pole at $s = 1$ with residue equal to 1. It satisfies the following Riemann functional equation:

$$\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s), \tag{1}$$

where $\Gamma(s)$ is the Euler gamma function

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt, \quad \mathrm{Re}(s) > 0,$$

it can be analytically continued as a meromorphic function in the complex plane of order 1 without any zeros and with simple poles at $s = 0, -1, -2, \cdots$. There are equivalent forms of the functional equation (1). For instance, if we use the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

and

$$\pi^{\frac{1}{2}}\Gamma(s) = 2^{1-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right),$$

(1) implies

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s). \tag{2}$$

The allied function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{3}$$

is an entire function of order equal to 1 satisfying the functional equation

$$\xi(1-s) = \xi(s). \tag{4}$$

It is easy to see that $\zeta(s)$ has no zeros for $\mathrm{Re}(s) > 1$, and by the functional equation, the only zeros of $\zeta(s)$ in the domain $\mathrm{Re}(s) < 0$ are the poles of $\Gamma\big(\frac{s}{2}\big)$, which are called the trivial zeros of $\zeta(s)$. Other zeros, called nontrivial zeros, lie in the critical strip $0 < \mathrm{Re}(s) < 1$. Moreover, for any nonzero complex number a, the zeros of $\zeta(s) - a$, which we denote by $\rho_a = \beta_a + i\gamma_a$, are called the a-points of $\zeta(s)$, and their distribution has long been an interesting object of study (one may see [16], [23], [32] for the principal results and further references). The related results are also quite beautiful as the zeros distribution. Some of the most basic facts are these. First, there exists a number $n_0(a)$ such that $\zeta(s) - a$ has a zero quite close to $s = -2n$ for all integers $n \geq n_0(a)$, and there are at most finitely many other zeros in $\mathrm{Re}(s) \leq 0$ (Levinson [16] states this for $\mathrm{Re}(s) \leq -2$, but it is not difficult to see that it holds for $\mathrm{Re}(s) \leq 0$), we call these zeros of $\zeta(s) - a$ trivial a-points. The remaining zeros all lie in a strip $0 < \mathrm{Re}(s) < A$, where $A > 0$ depends on a, and we call these nontrivial a-points. For these we have

$$N_a(T) = \sum_{0<\gamma_a<T,\ \beta_a>0} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O_a(\log T)$$

if $a \neq 1$; if $a = 1$, there is an additional term $\log\frac{T}{\pi}$ on the right-hand side of the above equation (cf. [16] or [23]). The zeros of $\zeta(s) - a$ cluster near the line $\mathrm{Re}(s) = \frac{1}{2}$ (Levinson [16]), as do the zeros of $\zeta(s)$ (Titchmarsh [32]), of which there are approximately the same number in the strip $0 < \mathrm{Re}(s) < A$. Garunkštis and Steuding [11] have recently shown that for every a an infinite number of a-points are simple.

In addition, an L-function in the Selberg class, which includes the Riemann zeta function and essentially those Dirichlet series where one might expect a Riemann hypothesis. Introduced by Selberg [23], the Selberg class $\mathcal{S}$ is the set of all Dirichlet series

$$\mathcal{L}(s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

absolutely convergent for $\mathrm{Re}(s) > 1$ that satisfy the following axioms:

(1) Ramanujan hypothesis: for any $\epsilon > 0$, $a(n) \ll n^{\epsilon}$;
(2) Analytic continuation: the function $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order for some non-negative integer k;
(3) Functional equation: $\mathcal{L}(s)$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\bar{s})}, \quad \text{where} \quad \Lambda_{\mathcal{L}} := \mathcal{L}(s)Q^s\prod_{j=1}^{K}\Gamma(\lambda_j s + \mu_j)$$

with positive real numbers Q, λ_j, and complex numbers μ_j, ω with $\mathrm{Re}(\mu_j) \geq 0$ and $|\omega| = 1$;
(4) Euler product. $\mathcal{L}(s)$ satisfies

$$\mathcal{L}(s) = \prod_p \mathcal{L}_p(s),$$

where

$$\mathcal{L}_p(s) = \exp\Big(\sum_{k=1}^{\infty}\frac{b(p^k)}{p^{ks}}\Big)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$.

The extended Selberg class $\mathcal{S}^\sharp$ is defined as the set of all functions $\mathcal{L}(s)$ satisfying axioms (1) − (3). The degree of $\mathcal{L}(s)$ is defined by $d_{\mathcal{L}} = 2\sum_{j=1}^{K}\lambda_j$, where K, λ_j are the number in the axiom (3) (see [21]). As far as we know, the characteristic function of L-functions in $\mathcal{S}^\sharp$ have an asymptotic formula as follows:

$$T(r,\mathcal{L}) = \frac{d_{\mathcal{L}}}{\pi}r\log r + O(r).$$

This shows that the countion function $N(r,\mathcal{L})$ dominates the porximity function $m(r,\mathcal{L})$, which is different from the genenal meromorphic functions, one may refer such as [5], [35] for getting more results concerning the relationship between the countion function and characteristic function of meromorphic functions. We next introduce two Lemmas to show the properties of the zero distribution of L-functions in the extended Selberg class, and they will paly a crucial effect in our following proof. To do this, we state some notations as follows. The zeros of $\mathcal{L}(s)$ located at the poles of gamma-factors appearing in the functional equation are called trivial. They all lie in $\sigma \leq 0$, and may have multiplicity greater than one. Actually, it is easily seen that they are located at

$$s = -\frac{m + \mu_j}{\lambda_j} \quad \text{with} \quad m = 0, 1, 2, \cdots, \quad \text{and} \quad 1 \leq j \leq K.$$

All other zeros are said to be non-trivial (see [24]). We shall denote the trivial zeros of $\mathcal{L}(s)$ by $\rho_1, \rho_2, \rho_3 \cdots$, where $\mathrm{Re}(\rho_1) \geq \mathrm{Re}(\rho_2) \geq \mathrm{Re}(\rho_3) \geq \cdots$, and where each zero is listed as many times as its multiplicity. Moreover, we have the following facts (see [10]):

Lemma 2.1 ([10]). The trivial zeros of L-function in the extend Selberg class satisfy
(i) $|\mathrm{Im}\rho_\mathrm{n}| \leq \mathrm{B}_0$ for $n = 1, 2, 3, \cdots$, where $B_0 = \max_{1\leq j\leq K} |\mathrm{Im}\mu_\mathrm{j}|/\lambda_\mathrm{j}$;
(ii) $D_0 = \min_{\rho_n \neq \rho_m} |\rho_n - \rho_m|$ exists and $D_0 > 0$;
(iii) $\sum_{-U<\mathrm{Re}(\rho_\mathrm{n})\leq 0} 1 = (\sum_{1\leq j\leq K} \lambda_j)U + O(1) = \frac{d_{\mathcal{L}}U}{2} + O(1)$, as $U \to +\infty$;
(iv) there is a number $A_0 > 0$ such that $\mathcal{L}(s)$ has only trivial zeros in $\sigma \leq -A_0$.

Let ρ denote a zero of $\mathcal{L}(s)$, for σ_1, σ_2 and $t > 0$, define

$$N_{\mathcal{L}}(\sigma_1, \sigma_2) = \sum_{\sigma_1 < \mathrm{Re}(\rho) \leq \sigma_2} 1,$$

$$N_{\mathcal{L}}(\sigma_1, \sigma_2; t) = \sum_{\sigma_1 < \mathrm{Re}(\rho) \leq \sigma_2, |\mathrm{Im}\rho| \leq \mathrm{t}} 1.$$

Then we clearly have

$$N_{\mathcal{L}}(-U, -A_0) = N_{\mathcal{L}}(-U, -A_0; B_0) = \frac{d_{\mathcal{L}}U}{2} + O(1).$$

The following Lemma was proved by Steven M. Gonek, Jaeho Haan and Haseo Ki (see [10]).

Lemma 2.2 ([10]). Suppose that $\mathcal{L}(s)$ is in the extended Selberg class, for any fixed complex number $c \neq 0$, there exist positive constants A_1, B_1 and C_1 depending at most on K and the μ_j and λ_j, such that
(i) $N_{\mathcal{L}-c}(-U, -A_1) = N_{\mathcal{L}-c}(-U, -A_1; B_1) = \frac{d_{\mathcal{L}}U}{2} + O(1)$, as $U \to +\infty$;
(ii) each zero of $\mathcal{L}(s) - c$ in $\sigma \leq -A_1$ is within $|\rho_n|^{-C_1 \log|\rho_n|}$of a trivial zero ρ_n of $\mathcal{L}(s)$;
(iii) all the zeros of $\mathcal{L}(s) - c$ in $\sigma \leq -A_1$ are simple.

Obviously, Lemma 2.2 implies that for a nonzero complex number c, there are infinitely many simple trivial c-points of $\mathcal{L}(s)$, which is different with the case of $c = 0$.

3. Proof of the main theorem

Proof of Theorem 1.4.

We first show that the order of $f(s)$ is less than or equal to 1. In fact, In view of $f(s)$ and $\zeta(s)$ satisfy the same functional equation, we can deduce that

$$\frac{f(s)}{f(1-s)} = \frac{\zeta(s)}{\zeta(1-s)} = \frac{1}{2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)}, \tag{5}$$

it yields that

$$f(s) - a = \frac{\zeta(s)}{\zeta(1-s)} f(1-s) - a.$$

Note that $f(s)$ and $\zeta(s)$ share $a(\neq 0)$ CM and the number of the a-points of $\zeta(s)$ is $O(r\log r)$, it then follows that

$$N\left(r, \frac{1}{f(s)-a}\right) = N\left(r, \frac{1}{\zeta(s)-a}\right) = O(r\log r). \tag{6}$$

Furthermore, we have

$$\begin{aligned} N\left(r, \frac{1}{f(s)-a}\right) &= N\left(r, \frac{1}{\frac{\zeta(s)}{\zeta(1-s)} f(1-s)-a}\right) \\ &= N\left(r, \frac{1}{\frac{\zeta(s)}{\zeta(1-s)}}\right) + N\left(r, \frac{1}{f(1-s)-a\frac{\zeta(1-s)}{\zeta(s)}}\right) + O(r). \end{aligned}$$

Take notice of $N\left(r, \frac{1}{\frac{\zeta(s)}{\zeta(1-s)}}\right) = O(r)$, it implies that

$$N\left(r, \frac{1}{f(1-s)-a\frac{\zeta(1-s)}{\zeta(s)}}\right) = O(r\log r). \tag{7}$$

If $\lambda(f(s)) = \lambda(f(1-s)) > 1$, then for any $\epsilon > 0$ we have

$$T\left(r, f(1-s)\right) > r^{1+\epsilon}.$$

On the other hand,

$$T\left(r, \frac{a\zeta(1-s)}{\zeta(s)}\right) \le O(r\log r) + O(r),$$

it then follows that

$$T\left(r, \frac{a\zeta(1-s)}{\zeta(s)}\right) = o\left(T(r, f(1-s))\right). \tag{8}$$

From the Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T\left(r, f(1-s)\right) &\le N\left(r, f(1-s)\right) + N\left(r, \frac{1}{f(1-s)-a}\right) \\ &\quad + N\left(r, \frac{1}{f(1-s)-a\frac{\zeta(1-s)}{\zeta(s)}}\right) + S(r,f) \\ &\le O(r\log r) + S(r,f) = S(r,f), \end{aligned}$$

which is a contradiction, in view of (6), (7), (8) and the fact that $f(s)$ has finitely many poles. Thus, the order of the $f(s)$ is not greater than 1.

Next suppose that $\alpha_1, \alpha_2, \cdots, \alpha_m$ are all poles of $f(s)$ with the order $k_1, k_2, \cdots, k_m$ respectively, then $(s-\alpha_1)^{k_1}(s-\alpha_2)^{k_2}\cdots(s-\alpha_m)^{k_m} f(s)$ is an entire function. Set

$$F(s) = \frac{\zeta(s)-a}{Q(s)(f(s)-a)},$$

where

$$Q(s) = \frac{(s-\alpha_1)^{k_1}(s-\alpha_2)^{k_2}\cdots(s-\alpha_m)^{k_m}}{(s-1)} \tag{9}$$

is a ration function. Since $f(s)$ and $\zeta(s)$ share non-zero complex number a CM, $F(s)$ is an entire function without any zeros and poles in the complex plane, there exist an entire function $h(s)$ such that

$$F(s) = \frac{\zeta(s)-a}{Q(s)(f(s)-a)} = e^{h(s)}. \tag{10}$$

Note that the order of $\zeta(s)$ and $f(s)$ are at most 1, so $h(s)$ is a polynomial with the degree at most one. Suppose that

$$h(s) = c_1 s + c_0,$$

where c_1, c_0 are complex numbers.

In view of $f(s)$ and $\zeta(s)$ satisfy the same functional equation, this implies that the trivial zeros of $\zeta(s)$ are also the zeros of $f(s)$, except possibly at finitely many points. Hence, from Lemma 2.1, we can choose a subsequence $\{s_n\} = \{-2n\}_{n=p}^{\infty}$ of the trivial zeros of $\zeta(s)$. Then for two distinct common trivial zeros $s_n = -2n$, $s_{n+1} = -2(n+1)$ of $\zeta(s)$ and $f(s)$ we obtain

$$\frac{(s_n - 1)}{(s_n - \alpha_1)^{k_1}(s_n - \alpha_2)^{k_2} \cdots (s_n - \alpha_m)^{k_m}} = e^{c_1 s_n + c_0}, \tag{11}$$

and

$$\frac{(s_{n+1} - 1)}{(s_{n+1} - \alpha_1)^{k_1}(s_{n+1} - \alpha_2)^{k_2} \cdots (s_{n+1} - \alpha_m)^{k_m}} = e^{c_1 s_{n+1} + c_0}. \tag{12}$$

Therefore, combine (11) and (12), we get

$$\begin{aligned}&\frac{(s_n - \alpha_1)^{k_1}(s_n - \alpha_2)^{k_2} \cdots (s_n - \alpha_m)^{k_m}(s_{n+1} - 1)}{(s_{n+1} - \alpha_1)^{k_1}(s_{n+1} - \alpha_2)^{k_2} \cdots (s_{n+1} - \alpha_m)^{k_m}(s_n - 1)}\\ &= e^{c_1(s_{n+1} - s_n)} = e^{-2c_1}.\end{aligned} \tag{13}$$

Let n tend to $+\infty$, from (13) we can deduce that $c_1 = k\pi i$, here k is an integer. So we get

$$\left|e^{-2c_1 n + c_0}\right| = e^{\mathbf{Re}(c_0)} = \left|\frac{(-2n - 1)}{(-2n - \alpha_1)^{k_1}(-2n - \alpha_2)^{k_2} \cdots (-2n - \alpha_m)^{k_m}}\right|. \tag{14}$$

This implies that $k_1 + \cdots + k_m = 1$, that is, there is only a pole of $f(s)$, for convenience we also denote it as α_1. Otherwise, if $k_1 + \cdots + k_m > 1$, then we can get $e^{\mathbf{Re}(c_0)} = 0$ as $n \to +\infty$, this is impossible. Further, let n tend to $+\infty$ again, we get $\mathbf{Re}(c_0) = 0$. Denote $c_0 = ci$, where c is a real numbers. Consider the function

$$G(s) = \frac{\zeta(s) - a}{f(s) - a} = \frac{s - \alpha_1}{s - 1} e^{sk\pi i + ci}. \tag{15}$$

It follows from [[24], Theorem 1.3, p.9] that in any strip $1 < \sigma < 1 + \epsilon$, $\zeta(s)$ takes any nonzero value infinitely often, where $\epsilon > 0$ is a positive number. So, $\zeta(s) - a$ has infinitely many zeros $\xi_n = \mu_n + i\nu_n$ on the strip

$$Z = \left\{s : 1 < \mathbf{Re}(s) < \frac{7}{6},\ \ \mathbf{Im}(s) > 0\right\}.$$

In addition, in views of $\zeta(s)$ and $f(s)$ satisfy the same equation, we have

$$\zeta(s) f(1 - s) = f(s)\zeta(1 - s). \tag{16}$$

Note the assumption $f(s)$ and $\zeta(s)$ share a CM, and from (16) , we can see that for any a-points $\xi_n \in Z$, we have $f(1 - \xi_n) = \zeta(1 - \xi_n)$.

Furthermore, since there exists a number n_0 such that $\zeta(s) - a$ has a zero quite close to $s = -2n$ for all integers $n \geq n_0$, and there are at most finitely many other zeros in $\mathbf{Re}(s) \leq 0$. Therefore, we have $f(1 - \xi_n) = \zeta(1 - \xi_n) \neq a$ with finitely many exceptions. Without loss of generality, we assume that $\mu_n \to \mu_0$ and $\nu_n \to +\infty$ as $n \to +\infty$. Thus, (15) leads that

$$1 = \left|\frac{\zeta(1 - \xi_n) - a}{f(1 - \xi_n) - a}\right| = \left|\frac{1 - \mu_n - i\nu_n - \alpha_1}{1 - \mu_n - i\nu_n - 1}\right| e^{k\pi\nu_n}, \tag{17}$$

which implies that $k = 0$. Otherwise, we can see that $1 = \infty$ as $n \to +\infty$ if $k > 0$ and $1 = 0$ as $n \to +\infty$ if $k < 0$. In view of $f(s)$ and $\zeta(s)$ satisfy the same functional equation, therefore (15) also shows that all the trivial zeros of $\zeta(s)$ with finitely many exceptions are the roots of equation

$$e^{ci}\frac{s-\alpha_1}{s-1} - 1 = 0.$$

Since there are infinitely many trivial zeros of $\zeta(s)$, we must have

$$e^{ci}\frac{s-\alpha_1}{s-1} \equiv 1.$$

Thus, we get $f(s) \equiv \zeta(s)$.

`Proof of Theorem 1.11.`

Firstly, we assert that the order of $f(s)$ is equal to 1. In fact, by the Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T(r,f) &\le \overline{N}\left(r,\frac{1}{f-a}\right) + \overline{N}\left(r,\frac{1}{f-b}\right) + N(r,f) + S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{\zeta-a}\right) + \overline{N}_{k)}\left(r,\frac{1}{\zeta-b}\right) + \overline{N}_{(k+1}\left(r,\frac{1}{f-b}\right) + S(r,f) \\ &\le 2T(r,\zeta) + \frac{1}{2}T(r,f) + S(r,f), \end{aligned}$$

it then follows that

$$T(r,f) \le 4T(r,\zeta) + S(r,f).$$

Similarly, we can obtain

$$T(r,\zeta) \le 4T(r,f) + S(r,\zeta).$$

Therefore, we can deduce that $\lambda(f) = \lambda(\zeta) = 1$.

Next suppose that $\alpha_1, \alpha_2, \cdots, \alpha_m$ are all poles of $f(s)$ with the order $k_1, k_2, \cdots, k_m$ respectively, then $(s-\alpha_1)^{k_1}(s-\alpha_2)^{k_2}\cdots(s-\alpha_m)^{k_m}f(s)$ is an entire function. Set

$$F(s) = \frac{\zeta(s)-a}{Q(s)(f(s)-a)},$$

where

$$Q(s) = \frac{(s-\alpha_1)^{k_1}(s-\alpha_2)^{k_2}\cdots(s-\alpha_m)^{k_m}}{(s-1)} \tag{18}$$

is a ration function. Since $f(s)$ and $\zeta(s)$ share a non-zero complex number a CM, $F(s)$ is an entire function without any zeros and poles in the complex plane, there exists an entire function $h(s)$ such that

$$F(s) = \frac{\zeta(s)-a}{Q(s)(f(s)-a)} = e^{h(s)}. \tag{19}$$

Note that $\lambda(\zeta) = \lambda(f) = 1$, so $h(s)$ is a polynomial whose degree is at most one. Denote $h(s)$ as follows,

$$h(s) = c_1 s + c_0,$$

where c_1, c_0 are complex numbers.

Firstly, we consider the case when $k = 1$, that is $\overline{E}_{1)}(b,\zeta) = \overline{E}_{1)}(b,f)$.

From Lemma 2.2, we can see that there are infinitely many simple trivial zeros of $\zeta(s) - b$ in $\{s = \sigma + it | \sigma < -A\}$, where A is a positive real number.

If $b = 0$, it is well know that the trivial zeros of $\zeta(s)$ are $\left\{\rho_n\right\} = \left\{-2n\right\}_{n=1}^{\infty}$, and each zeros is simple. Then we can deduce the result similar to the proof of Theorem 1.4.

For $b \neq 0$, each zeros of $\zeta(s) - b$ is within the sufficiently small neighborhood of a trivial zero ρ_n of $\zeta(s)$ if $|\rho_n|$ is large enough. Without loss of generality, we denote the subset of the zeros of $\zeta(s) - b$ as $S = \{s_n\}$, where

$$s_n \in \mathrm{U}\Big(\rho_n,\ \ |\rho_n|^{-C_1 \log|\rho_n|}\Big), \quad n = 1, 2, \cdots,$$

and C_1 is a positive constant. Let $\Omega_n = \mathrm{U}\Big(\rho_n,\ \ |\rho_n|^{-C_1 \log|\rho_n|}\Big)$, then there exists some n_0 such that for $n > n_0$, $\Omega_n \bigcap \Omega_{n+1} = \emptyset$. Furthermore, we have

$$\mathbf{Re}(\rho_n) - |\rho_n|^{-C_1 \log|\rho_n|} < \mathbf{Re}(s_n) < \mathbf{Re}(\rho_n) + |\rho_n|^{-C_1 \log|\rho_n|},$$

and

$$\mathbf{Re}(\rho_{n+1}) - |\rho_{n+1}|^{-C_1 \log|\rho_{n+1}|} < \mathbf{Re}(s_{n+1}) < \mathbf{Re}(\rho_{n+1}) + |\rho_{n+1}|^{-C_1 \log|\rho_{n+1}|}.$$

It then yiels that

$$-2 - \alpha_n < \mathbf{Re}(s_{n+1} - s_n) < -2 + \alpha_n,$$

where

$$\alpha_n = |\rho_n|^{-C_1 \log|\rho_n|} + |\rho_{n+1}|^{-C_1 \log|\rho_{n+1}|}.$$

This implies that

$$\lim_{n\to+\infty} \mathbf{Re}(s_{n+1} - s_n) = -2.$$

Similarly, we can get

$$\Big|\mathbf{Im}(s_{n+1} - s_n)\Big| \leq |\rho_n|^{-C_1 \log|\rho_n|} + |\rho_{n+1}|^{-C_1 \log|\rho_{n+1}|} \longrightarrow 0,$$

as $n \to +\infty$. Hence,

$$\lim_{n\to+\infty} \Big(s_{n+1} - s_n\Big) = -2. \tag{20}$$

In addition, from (19), for two distinct simple zeros $s_n = \sigma_n + it_n$, $s_{n+1} = \sigma_{n+1} + it_{n+1} \in S$ of $\zeta(s) - b$, we have

$$\frac{\Big(s_n - 1\Big)}{\Big(s_n - \alpha_1\Big)^{k_1}\Big(s_n - \alpha_2\Big)^{k_2} \cdots \Big(s_n - \alpha_m\Big)^{k_m}} = e^{c_1 s_n + c_0}, \tag{21}$$

and

$$\frac{\Big(s_{n+1} - 1\Big)}{\Big(s_{n+1} - \alpha_1\Big)^{k_1}\Big(s_{n+1} - \alpha_2\Big)^{k_2} \cdots \Big(s_{n+1} - \alpha_m\Big)^{k_m}} = e^{c_1 s_{n+1} + c_0}. \tag{22}$$

Denote

$$s_{n+1} = s_n - h_n,$$

where

$$h_n = (\sigma_n - \sigma_{n+1}) + i(t_n - t_{n+1}),$$

it then yields that

$$e^{c_1\left(s_{n+1}-s_n\right)} = \frac{\left(s_n-\alpha_1\right)^{k_1}\left(s_n-\alpha_2\right)^{k_2}\cdots\left(s_n-\alpha_m\right)^{k_m}\left(s_{n+1}-1\right)}{\left(s_{n+1}-\alpha_1\right)^{k_1}\left(s_{n+1}-\alpha_2\right)^{k_2}\cdots\left(s_{n+1}-\alpha_m\right)^{k_m}\left(s_n-1\right)}$$

$$= \frac{\left(s_n-\alpha_1\right)^{k_1}\left(s_n-\alpha_2\right)^{k_2}\cdots\left(s_n-\alpha_m\right)^{k_m}\left(s_n-h_n-1\right)}{\left(s_n-h_n-\alpha_1\right)^{k_1}\left(s_n-h_n-\alpha_2\right)^{k_2}\cdots\left(s_n-h_n-\alpha_m\right)^{k_m}\left(s_n-1\right)} \tag{23}$$

$$= \frac{\left(1-\frac{\alpha_1}{s_n}\right)^{k_1}\left(1-\frac{\alpha_2}{s_n}\right)^{k_2}\cdots\left(1-\frac{\alpha_m}{s_n}\right)^{k_m}\left(1-\frac{h_n}{s_n}-\frac{1}{s_n}\right)}{\left(1-\frac{h_n}{s_n}-\frac{\alpha_1}{s_n}\right)^{k_1}\left(1-\frac{h_n}{s_n}-\frac{\alpha_2}{s_n}\right)^{k_2}\cdots\left(1-\frac{h_n}{s_n}-\frac{\alpha_m}{s_n}\right)^{k_m}\left(1-\frac{1}{s_n}\right)}.$$

Note that

$$\lim_{n\to+\infty}\frac{h_n}{s_n}=0,$$

then from (23), we can deduce that

$$\lim_{n\to+\infty} e^{c_1\left(s_{n+1}-s_n\right)} = 1.$$

This combine with (20), we get

$$e^{-2c_1}=1.$$

Therefore, $c_1 = k\pi i$, where k is an integer. So we get from (21)

$$\begin{aligned}\left|e^{k\pi i s_n + c_0}\right| &= e^{-k\pi t_n + \mathbf{Re}(c_0)}\\ &= \left|\frac{(\sigma_n - 1 + it_n)}{(\sigma_n-\alpha_1+it_n)^{k_1}(\sigma_n-\alpha_2+it_n)^{k_2}\cdots(\sigma_n-\alpha_m+it_n)^{k_m}}\right|.\end{aligned} \tag{24}$$

This implies that $k_1+\cdots+k_m = 1$, that is, there is only a pole of $f(s)$, for convenience we also denote it as α_1. Otherwise, note that $\lim\limits_{n\to+\infty}\sigma_n = -\infty$, and $\lim\limits_{n\to+\infty} t_n = 0$, so if $k_1+\cdots+k_m > 1$, then we can get $e^{\mathbf{Re}(c_0)} = 0$ as $n\to+\infty$, this is impossible. Further, let n tend to $+\infty$ again, we get $\mathbf{Re}(c_0) = 0$. Denote $c_0 = ci$ and $\alpha_1 = u+iv$, where u, v and c are all real numbers. Consider the function

$$G(s) = \frac{\zeta(s)-a}{f(s)-a} = \frac{s-\alpha_1}{s-1}e^{sk\pi i + ci}. \tag{25}$$

For any $s=\sigma+it$, it follows from (25) that

$$\left|G(\sigma+it)\right|^2 = \frac{(\sigma-u)^2+(t-v)^2}{(\sigma-1)^2+t^2}e^{-2k\pi t}. \tag{26}$$

It follows from [11] that for any b, there exists a positive number A such that $\zeta(s)-b$ has infinitely many simple zeros $\xi_n = \mu_n + i\nu_n$ on the strip

$$Z = \left\{s : 0 < \mathbf{Re}(s) < A,\ \ \mathbf{Im}(s) > 0\right\}.$$

Without loss of generality, we assume that $\mu_n \to \mu_0$ and $\nu_n \to +\infty$ as $n\to+\infty$. Thus, (25) and (26) leads that

$$1 = \left|\frac{\zeta(\xi_n)-a}{f(\xi_n)-a}\right|^2 = \frac{(\mu_n-u)^2+(\nu_n-v)^2}{(\mu_n-1)^2+\nu_n^2}e^{-2k\pi\nu_n}, \tag{27}$$

which implies that $k = 0$. Otherwise, we can see that $1 = \infty$ as $n \to +\infty$ if $k < 0$ and $1 = 0$ as $n \to +\infty$ if $k > 0$. Therefore, (25) also shows that all the simple zeros of $\zeta(s) - b$ in Z are zeros of

$$e^{ci}\frac{s-\alpha_1}{s-1} - 1.$$

Since there are infinitely many points in Z, we must have

$$e^{ci}\frac{s-\alpha_1}{s-1} \equiv 1,$$

hence, we get $f(s) \equiv \zeta(s)$.

For $k \geq 2$, it is easy to see that $\overline{E}_{1)}(b,\zeta) \subseteq \overline{E}_{k)}(b,\zeta)$, therefore, we can deduce the result from above process.

`Proof of Theorem 1.12.`

If one of a, b, c is ∞, then we can get the result from Theorem 1.11 immediately. For a, b, c are all not equal to ∞, analogues of the proof of Theorem 1.11, we may consider the function

$$F(s) = \frac{(\zeta(s)-a)(f(s)-b)}{(\zeta(s)-b)(f(s)-a)},$$

then $F(s)$ is an entire function without any poles and zeros. Then we can deal with it similar to the proof of Theorem 1.11 and deduce the result.

4. Further more results

Analogues of the properties of value distribution of $\zeta(s)$ are known to hold for L-functions in the extended Selberg class $S^\sharp$. The above proof therefore can be extended to them.

Theorem 4.1. *Suppose that $f(s)$ is a meromorphic function with finitely many poles, an L-function $\mathcal{L}(s) \in \mathcal{S}^\sharp$ with positive degree and $a \in \mathbb{C}\setminus\{0\}$. If $f(s)$ and $\mathcal{L}(s)$ satisfy the same functional equation and share a CM, then $f(s) \equiv \mathcal{L}(s)$.*

Theorem 4.2. *Let $f(s)$ be a meromorphic function in the complex plane such that $f(s)$ has finitely many poles, and let $a, b(\neq 0)$ be two distinct finite values. If $\mathcal{L}(s) \in \mathcal{S}^\sharp$ is an L-function with positive degree such that $f(s)$ and $\mathcal{L}(s)$ share a CM and $\overline{E}_{k)}(b,\mathcal{L}) = \overline{E}_{k)}(b, f)$ for some integer $k \geq 1$, then $f(s) \equiv \mathcal{L}(s)$.*

Theorem 4.3. *Let $f(s)$ be a meromorphic function in the complex plane, and $a, b, c(\neq 0) \in \mathbb{C}\cup\{\infty\}$ be three distinct complex numbers. If $\mathcal{L}(s) \in \mathcal{S}^\sharp$ is an L-function with positive degree such that $f(s)$ and $\mathcal{L}(s)$ share a, b CM and $\overline{E}_{k)}(c,\mathcal{L}) = \overline{E}_{k)}(c, f)$ for some integer $k \geq 1$, then $f(s) \equiv \mathcal{L}(s)$.*

For the function $\mathcal{L}(s)$ in the extended Selberg class, the degree of $\mathcal{L}(s)$ is defined by $d_{\mathcal{L}} = 2\sum_{j=1}^{K}\lambda_j$, where K, λ_j are the number in the axiom (3) as show in section 2. The following example shows that the condition that an L-function with positive degree is necessary.

Remark 4.4. *Let $\mathcal{L}(s) = 1 + \frac{2}{4^s}$, $f(s) = 1 + \frac{4^s}{2}$. Then it is easy to verify that $2^s\mathcal{L}(s) = 2^{1-s}\overline{\mathcal{L}(1-\overline{s})}$, thus, $\mathcal{L}(s) \in \mathcal{S}^\sharp$ with degree zero. Moreover, $f(s)$ and $\mathcal{L}(s)$ share $0, 1, \infty$ CM, but $f(s) \not\equiv \mathcal{L}(s)$.*

Obviously, Lemma 2.2 implies that for a nonzero complex number c, there are infinitely many trivial c-points of $\mathcal{L}(s)$. Furthermore, these trivial c-points of $\mathcal{L}(s)$ are all simple, which is different with the case of $c = 0$. As stated previously, the trivial zeros of $\mathcal{L}(s)$ may have multiplicity greater than one. Note that we can not do without the condition that infinitely many simple trivial zeros in the proof of Theorem 1.11 and Theorem 1.12 if $k = 1$. Hence, for Theorem 4.2 and Theorem 4.3 we require that $b \neq 0$ and $c \neq 0$ respectively. In fact, Theorem 4.2 and Theorem 4.3 respectively holds when $b = 0$ and $c = 0$, if $\mathcal{L}(s)$ in the extended Selberg class has infinitely simple trivial zeros, such as Riemann zeta function, Dirichlet L-functions and so on. If an L-function in the extended Selberg class with trivial zeros that multiplicity greater than one, then Theorem 4.2 and Theorem 4.3 respectively holds when $b = 0$ and $c = 0$ if $k \geq 2$ (see [34]).

In addition, it is remarkable that one of the fundamentally important higher transcendental functions of analytic number theory is the familiar general Hurwitz-Lerch zeta function $\Phi(z, s, a)$. It contains, as its special cases, not only the Riemann zeta function $\zeta(s)$, the Hurwitz (or generalized) zeta function $\zeta(s, a)$ and the Lerch zeta function $\ell_s(\xi)$, but also other important functions of analytic number theory as Polylogarithmic function $\mathrm{Li_s(z)}$ and the Lipschitz-Lerch zeta function $\phi(\xi, a, s)$, for details, one may refer to monographs, for example, [8, 27] etc. Indeed, just as the Riemann zeta function $\zeta(s)$, the Hurwitz (or generalized) zeta function $\zeta(s, a)$, the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ also can be continued meromorphically to the whole complex plane, except for a simple pole at $s = 1$ with its residue 1. For several novel properties and some other applications of the Hurwitz-Lerch zeta function, the interested reader may refer to the recent works such as [25, 26, 28] ect. for the principal results and further references by Srivastava and others. Moreover, as far as we know, many researchers also study many different generalizations and extensions of the familiar Hurwitz-Lerch zeta function $\Phi(z, s, a)$ by inserting certain additional parameters to the series representation of the Hurwitz-Lerch zeta function. For instance, Choi and Parmar [7] introduced and studied the extension of the generalized Hurwitz-Lerch zeta function $\Phi_{a,b,b',c}(z, t, r, \alpha)$ in two variables, whereafter, H. M. Srivastava, Recep Şahin and ğuz Yağci investigate an extended family of the generalized incomplete Hurwitz-Lerch zeta functions of two variables satisfy a decomposition formula in terms of $\Phi_{a,b,b',c}(z, t, r, \alpha)$ and obtain integral representations including the Mellin-Barnes contour integral representation, derivative formulas, summation formulas, series relations and recurrence relations(see [29]). Besides, Irem Kucukoğlu, Yilmaz Simsek and H. M. Srivastava (see [22]) construct Lerch-type zeta functions $\zeta_w(s, x, k, \lambda)$ which interpolate numbers $W_n^{(k)}(\lambda)$ and polynomials $W_n^{(k)}(x, \lambda)$ at negative integers, where if we set $x = 0$ and $k = 1$, then the functions reduces to $\zeta_w(s, 1, \lambda)$ which interpolate the Apostol-type numbers such as the Apostol-Euler numbers, the Apostol-Genocchi numbers etc., meanwhile, the function $\zeta_w(s, 1, \lambda)$ are closed related to the Hurwitz-Lerch zeta function $\Phi(z, s, a)$, the Polylogarithmic function $\mathrm{Li_s(z)}$, the Riemann zeta function $\zeta(s)$ and so on. For details about some properties of these Lerch-type zeta functions with other well-known families of zeta functions, some functional equations of the Lerch-type zeta functions and properties involving the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x, \lambda)$ one may refer to [22, 30, 31]. In this paper, we mainly study the uniqueness of L-functions in the extended Selberg class, which include the Riemann zeta function and essentially those Dirichlet series where one might expect a Riemann hypothesis. Naturally, we are interesting to know what happen on the subject of the Hurwitz (or generalized) zeta function, the Lerch zeta function, and even incluing the familiar (generalized) Hurwitz-Lerch zeta function under sharing-value conditions. It would be a very interesting and meaningful work if one can deal with them.

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