



# Geometric Inequalities for Non-Integrable Distributions in Statistical Manifolds with Constant Curvature

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**Abstract.** In this paper, we make Euler inequality, Chen first inequality and Chen-Ricci inequality for non-integrable distributions in statistical manifolds with constant curvatures. Moreover, we investigate the conditions for equality cases.

## 1. Introduction

It is well known that the curvature invariants play an important role in Riemannian geometry as well as in physics. Among them, the most and well known objects are the sectional curvature, scalar curvature, and Ricci curvatures. Establishing sharp relationships between intrinsic and extrinsic curvature invariants are one of interesting topics in submanifold theory.

In 1993, B. Chen [4] defined a new type of curvature invariants, called  $\delta$ -invariants (or Chen invariants). He proved the Chen first inequality for submanifolds of a real space form. The Chen first invariant of an  $n$ -dimensional Riemannian manifold  $M^n$  is defined by  $\delta_{M^n} = \tau - \inf K$ , where  $\tau$  and  $K$  are the scalar and sectional curvatures of  $M^n$ , respectively. Also a sharp relationship between the Ricci curvature and the squared mean curvature for any Riemannian submanifold of a real space form was proved in [5], which is known as the Chen-Ricci inequality. Many geometers studied similar problems for different classes of submanifolds in various ambient spaces.

Statistical manifolds were introduced by Amari [1] in 1985. There are many applications in information geometry, which represents one of the main tools for machine learning and evolutionary biology. Since such a manifold is endowed with a pairing of torsion-free connections, called dual connections (or conjugate connections in affine geometry [12, 14, 18]), its geometry is closely related to affine differential geometry. Moreover, a statistical structure is a generalization of a Hessian structure.

In general, the dual connections are not metric; thus one cannot define a sectional curvature with respect to them by the standard definition from Riemannian geometry. Opozda [15, 16] has proposed two different definitions.

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The interest in Chen invariants for statistical submanifolds in statistical manifolds grew in the recent years. Aydin et al., in [2], proved some geometric inequalities for the scalar curvature and the Ricci curvature associated to the dual connections of submanifolds in statistical manifolds with a constant curvature. Moreover, Aydin et al. further in [3] obtained a generalized Wintgen inequality for statistical submanifolds in statistical manifolds with a constant curvature by using the sectional curvature in [15, 16]. By virtue of the sectional curvature, Mihai et al. [10] proved Euler and Chen-Ricci inequalities for statistical submanifolds in Hessian manifolds of constant Hessian curvature. Chen et al. in [6] got a Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature. Malek and Akbari, in [9], obtained bounds for Casorati curvatures of submanifolds in Cosymplectic statistical space forms.

On the other hand, any regular submanifold of  $\mathbb{R}^m$  is locally an integral manifold of a Pfaff system or its incident, or dual, regular distribution. For any submanifold of a differentiable manifold  $M$  the situation is the same. So, in some sense, the geometric study of regular distributions or Pfaff systems is a natural generalization of the geometric study of submanifolds. The case of integrable Pfaff systems, or integrable distributions, corresponds to the study of foliations. See [13] for a systematic study of this subject.

In [11], the author considered non-integrable distributions in a Riemannian manifold. The second fundamental form was defined and the Gauss equation for non-integrable distributions was established. In [17], Wang established the Gauss, Codazzi, and Ricci equations for non-integrable distributions with respect to a semi-symmetric metric connection, a kind of semi-symmetric non-metric connections and a statistical connection. He also obtained Chen's inequality for non-integrable distributions in real space forms with respect to a semi-symmetric metric connection and a kind of semi-symmetric non-metric connection.

Motivated by the above studies, we will in the present paper make a Euler inequality, a Chen first inequality and a Chen-Ricci inequality for non-integrable distributions in the statistical manifolds with constant curvature using the sectional curvature defined in [15].

## 2. Statistical Manifolds and non-integrable distributions

Let  $(M^m, g)$  be an  $m$ -dimensional Riemannian manifold  $(M^m, g)$ . A pair  $(\bar{\nabla}, g)$  is called a statistical structure on  $M$  if  $\bar{\nabla}$  is an affine and torsion-free connection satisfying

$$(\bar{\nabla}_Z g)(X, Y) = (\bar{\nabla}_X g)(Z, Y) \quad (1)$$

for all  $X, Y, Z \in \mathcal{L}(TM)$ .

In a statistical manifold, there exists a pair of torsion-free affine connections  $\bar{\nabla}$  and  $\bar{\nabla}^*$  satisfying

$$Zg(X, Y) = g(\bar{\nabla}_Z X, Y) + g(X, \bar{\nabla}_Z^* Y), \quad (2)$$

for any  $X, Y, Z \in \mathcal{L}(TM)$ . The statistical manifold is denoted by  $(M, g, \bar{\nabla})$ . The connections  $\bar{\nabla}$  and  $\bar{\nabla}^*$  are called dual connections, and it is easily seen that  $(\bar{\nabla}^*)^* = \bar{\nabla}$ . If  $(\bar{\nabla}, g)$  is a statistical structure on  $M$ , then  $(\bar{\nabla}^*, g)$  is also a statistical structure on  $M$  [1]. For the pair connections  $\bar{\nabla}$  and  $\bar{\nabla}^*$ , we have:

$$\bar{\nabla} + \bar{\nabla}^* = 2\nabla, \quad (3)$$

where  $\nabla$  is the Levi-Civita connection on  $M^m$ . If  $\bar{\nabla} = \bar{\nabla}^*$ , then the statistical manifold simply reduces to usual Riemannian manifold and  $(M, g, \bar{\nabla})$  is called trivial statistical.

Denote by  $R, \bar{R}$  and  $\bar{R}^*$  the Riemannian curvature tensor fields of  $\nabla, \bar{\nabla}$  and  $\bar{\nabla}^*$ , respectively. A statistical structure  $(\bar{\nabla}, g)$  is said to be of constant curvature  $c \in \mathbb{R}$  if

$$\bar{R}(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}, \forall X, Y, Z \in \Gamma(TM). \quad (4)$$

A statistical structure  $(\bar{\nabla}, g)$  of constant curvature 0 is called a Hessian structure. The curvature tensor fields  $\bar{R}$  and  $\bar{R}^*$  of dual connections satisfy

$$g(\bar{R}^*(X, Y)Z, W) = -g(Z, \bar{R}(X, Y)W). \quad (5)$$

From (5) it follows immediately that if  $(\bar{\nabla}, g)$  is a statistical structure of constant curvature  $c$ , then  $(\bar{\nabla}^*, g)$  is also a statistical structure of constant curvature  $c$ . In particular, if  $(\bar{\nabla}, g)$  is Hessian, so is  $(\bar{\nabla}^*, g)$  (one can see [7] for details).

In [8], Furuhata and Hasegawa have defined the statistical curvature tensor field  $\bar{S}$  for a statistical manifold  $(M, \bar{\nabla}, g)$  as follows (see also [15]):

$$\bar{S}(X, Y)Z = \frac{1}{2}\{\bar{R}(X, Y)Z + \bar{R}^*(X, Y)Z\} \tag{6}$$

for any  $X, Y, Z \in \Gamma(TM)$ .

Let  $D \subseteq TM$  be a non-integrable distribution with constant rank  $n$  in an  $m$ -dimensional statistical manifold  $(M, g, \bar{\nabla})$ . A non-integrable distribution is a subbundle of the tangent bundle  $TM$  and there exist  $X, Y \in \Gamma(D)$  such that  $[X, Y]$  is not in  $\Gamma(D)$ , where  $\Gamma(D)$  is the space of sections of  $D$ . The distribution  $D$  inherits a metric tensor field  $g^D$  from the original  $g$  in  $M$ . Let  $D^\perp \subseteq TM$  be the orthogonal distribution to  $D$  which inherits a metric tensor field  $g^{D^\perp}$  from the  $g$  and then  $g = g^D \oplus g^{D^\perp}$ .

Let  $\pi^D : TM \rightarrow D, \pi^{D^\perp} : TM \rightarrow D^\perp$  be the projections. For  $X, Y \in \Gamma(D)$ , we define  $\nabla_X^D Y = \pi^D(\nabla_X Y)$ ,  $[X, Y]^D = \pi^D([X, Y])$  and  $[X, Y]^{D^\perp} = \pi^{D^\perp}([X, Y])$ . By [11], for  $X, Y \in \Gamma(D)$  we have

$$\begin{aligned} \nabla_X^D g^D &= 0, T(X, Y) := \nabla_X^D Y - \nabla_Y^D X - [X, Y] = -[X, Y]^{D^\perp}, \\ \nabla_X Y &= \nabla_X^D Y + B(X, Y), \end{aligned} \tag{7}$$

where  $B(X, Y) = \pi^{D^\perp}(\nabla_X Y)$  is called the second fundamental forms. Obviously,  $B(X, Y) \neq B(Y, X)$ . The formula (7) may be called the Gauss formula with respect to the Levi-Civita connection.

Similarly, the Gauss formulae with respect to the dual connections [17] are

$$\begin{aligned} \bar{\nabla}_X Y &= \bar{\nabla}_X^D Y + \bar{B}(X, Y), \\ \bar{\nabla}_X^* Y &= \bar{\nabla}_X^{D,*} Y + \bar{B}^*(X, Y), \end{aligned} \tag{8}$$

where  $\bar{B}$  and  $\bar{B}^*$  are also not symmetric  $(0, 2)$ -tensors and are also called the second fundamental forms with dual connections. Using (3), we have  $2B(X, Y) = \bar{B}(X, Y) + \bar{B}^*(X, Y)$ .

Given  $X, Y, Z \in \Gamma(D)$ , the curvature tensors  $R^D$  on  $D$  with respect to  $\nabla^D$  is defined by

$$R^D(X, Y)Z = \nabla_X^D \nabla_Y^D Z - \nabla_Y^D \nabla_X^D Z - \nabla_{[X, Y]^D}^D Z - \pi^D([X, Y]^{D^\perp}, Z). \tag{9}$$

In (9),  $R^D$  is a tensor field by adding the extra term  $-\pi^D([X, Y]^{D^\perp}, Z)$ .

Then the curvature tensors  $\bar{R}^D$  and  $\bar{R}^{D,*}$  on  $D$  with respect to the statistical connection  $\bar{\nabla}^D$  and  $\bar{\nabla}^{D,*}$  can be defined analogously.

The corresponding Gauss equations are given by [17]:

$$g(R(X, Y)Z, W) = g(R^D(X, Y)Z, W) + g(B(X, Z), B(Y, W)) - g(B(X, W), B(Y, Z)) + g(B(Z, W), [X, Y]), \tag{10}$$

$$g(\bar{R}(X, Y)Z, W) = g(\bar{R}^D(X, Y)Z, W) + g(\bar{B}(X, Z), \bar{B}^*(Y, W)) - g(\bar{B}^*(X, W), \bar{B}(Y, Z)) + g(\bar{B}^*(Z, W), [X, Y]), \tag{11}$$

and

$$g(\bar{R}^*(X, Y)Z, W) = g(\bar{R}^{D,*}(X, Y)Z, W) + g(\bar{B}^*(X, Z), \bar{B}(Y, W)) - g(\bar{B}(X, W), \bar{B}^*(Y, Z)) + g(\bar{B}(Z, W), [X, Y]). \tag{12}$$

The curvature tensor  $S^D$  for non-integrable distributions  $D$  with respect to the dual connections is defined by

$$S^D(X, Y)Z = \frac{1}{2}[\bar{R}^D(X, Y)Z + \bar{R}^{D,*}(X, Y)Z] \tag{13}$$

Let  $\{E_1, \dots, E_n\}$  and  $\{E_{n+1}, \dots, E_m\}$  be a local orthonormal basis of  $\Gamma(D)$  and  $\Gamma(D^\perp)$ , respectively. Then the mean curvature vector fields of  $D$  denoted by  $H, \bar{H}$ , and  $\bar{H}^*$  are given by

$$H = \frac{1}{n} \sum_{i=1}^n B(E_i, E_i) = \frac{1}{n} \sum_{\alpha=n+1}^m \left( \sum_{i=1}^n h_{ii}^\alpha \right) E_\alpha, h_{ij}^\alpha = g(B(E_i, E_j), E_\alpha),$$

$$\bar{H} = \frac{1}{n} \sum_{i=1}^n \bar{B}(E_i, E_i) = \frac{1}{n} \sum_{\alpha=n+1}^m \left( \sum_{i=1}^n \bar{h}_{ii}^\alpha \right) E_\alpha, \bar{h}_{ij}^\alpha = g(\bar{B}(E_i, E_j), E_\alpha),$$

and

$$\bar{H}^* = \frac{1}{n} \sum_{i=1}^n \bar{B}^*(E_i, E_i) = \frac{1}{n} \sum_{\alpha=n+1}^m \left( \sum_{i=1}^n \bar{h}_{ii}^{*\alpha} \right) E_\alpha, \bar{h}_{ij}^{*\alpha} = g(\bar{B}^*(E_i, E_j), E_\alpha)$$

for  $1 \leq i, j \leq n$  and  $n + 1 \leq \alpha \leq m$ .

We also set

$$\|B\|^2 = \sum_{i,j=1}^n g(B(E_i, E_j), B(E_i, E_j)) = \sum_{\alpha=n+1}^m \sum_{i,j=1}^n (h_{ij}^\alpha)^2,$$

$$\|\bar{B}\|^2 = \sum_{i,j=1}^n g(\bar{B}(E_i, E_j), \bar{B}(E_i, E_j)) = \sum_{\alpha=n+1}^m \sum_{i,j=1}^n (\bar{h}_{ij}^\alpha)^2,$$

and

$$\|\bar{B}^*\|^2 = \sum_{i,j=1}^n g(\bar{B}^*(E_i, E_j), \bar{B}^*(E_i, E_j)) = \sum_{\alpha=n+1}^m \sum_{i,j=1}^n (\bar{h}_{ij}^{*\alpha})^2.$$

The curve  $\gamma$  is  $\nabla$ -geodesic (resp.  $\bar{\nabla}$ -geodesic, or  $\bar{\nabla}^*$ -geodesic) if  $\nabla_\gamma \dot{\gamma} = 0$  (resp.  $\bar{\nabla}_\gamma \dot{\gamma} = 0$ , or  $\bar{\nabla}_\gamma^* \dot{\gamma} = 0$ ). We say that  $D$  is totally geodesic with respect to the Levi-Civita connection  $\nabla$  (resp. the dual connection  $\bar{\nabla}$ , or  $\bar{\nabla}^*$ ) if every  $\nabla$ -geodesic (resp.  $\bar{\nabla}$ -geodesic, or  $\bar{\nabla}^*$ -geodesic) with initial condition in  $D$  is contained in  $D$ . Similar to Theorem 19 in [11], we have the following.

**Proposition 2.1.** *A distribution  $D$  is totally geodesic with respect to  $\nabla$  (resp.  $\bar{\nabla}$ , or  $\bar{\nabla}^*$ ) if and only if the symmetric part of the second fundamental form is identically zero, i.e.,  $B(X, Y) + B(Y, X) = 0$  (resp.  $\bar{B}(X, Y) + \bar{B}(Y, X) = 0$ , or  $\bar{B}^*(X, Y) + \bar{B}^*(Y, X) = 0$ ).*

Let  $\Pi \subset D$  be a 2-plane section. Denote by  $K_0^D(\Pi)$  and  $K^D(\Pi)$  the sectional curvature on  $D$  with the induced connection  $\nabla^D$  and dual connections which are defined by, respectively,

$$K_0^D(\Pi) = \frac{1}{2} [g(R^D(E_1, E_2)E_2, E_1) + g(R^D(E_2, E_1)E_1, E_2)], \tag{14}$$

$$K^D(\Pi) = \frac{1}{2} [g(S^D(E_1, E_2)E_2, E_1) + g(S^D(E_2, E_1)E_1, E_2)], \tag{15}$$

where  $\{E_1, E_2\}$  is orthonormal basis of  $\Pi$  and  $K_0^D(\Pi), K^D(\Pi)$  are independent of the choice of  $E_1, E_2$ . For any orthonormal basis  $\{E_1, \dots, E_n\}$  of  $D$ , the scalar curvature  $\tau_0^D$  and  $\tau^D$  on  $D$  with respect to the induced connection  $\nabla^D$  and dual connections are given by, respectively,

$$\tau_0^D = \frac{1}{2} \sum_{1 \leq i, j \leq n} g(R^D(E_i, E_j)E_i, E_j), \tag{16}$$

$$\tau^D = \frac{1}{2} \sum_{1 \leq i, j \leq n} g(S^D(E_i, E_j)E_i, E_j). \quad (17)$$

Set

$$A^D = \frac{1}{2} \sum_{1 \leq i, j \leq n} g(B(E_j, E_i), [E_j, E_i]),$$

and

$$\Omega^\Pi = -\frac{1}{2} g(B(E_1, E_2) - B(E_2, E_1), [E_1, E_2]).$$

Then  $A^D$  and  $\Omega^\Pi$  are independent of the choice of the orthonormal basis.

We state the following algebraic lemmas, which will be used in the proof of the Chen first inequality.

**Lemma 2.2.** [6] Let  $n \geq 3$  be an integer and  $a_1, \dots, a_n$  be  $n$  real numbers. Then we have:

$$\sum_{1 \leq i < j \leq n} a_i a_j - a_1 a_2 \leq \frac{n-2}{2(n-1)} \left( \sum_{i=1}^n a_i \right)^2.$$

Furthermore, the equality case of the above inequality holds if and only if  $a_1 + a_2 = a_3 = \dots = a_n$ .

The following lemma will be essential for the proof of the Chen-Ricci inequality.

**Lemma 2.3.** Let  $a_1, \dots, a_n$  be  $n$  real numbers. Then we have:

$$a_1 \sum_{i=2}^n a_i \leq \frac{1}{4} \left( \sum_{i=1}^n a_i \right)^2.$$

Furthermore, the equality case of the above inequality holds if and only if  $a_1 = a_2 + \dots + a_n$ .

*Proof.* The inequality is equivalent to

$$0 \leq (a_1 - a_2 - \dots - a_n)^2,$$

with the equality holding if and only if  $a_1 = a_2 + \dots + a_n$ .  $\square$

### 3. Euler's inequality

In this section, we will prove the Euler's inequality for the non-integrable distributions in statistical manifolds with constant curvature.

**Theorem 3.1.** Let  $(M, c)$  be an  $m$ -dimensional statistical manifold with constant curvature  $c$ . Let  $D \subset TM$  be a non-integrable distribution with constant rank  $n$  and  $TM = D \oplus D^\perp$ . Then

$$2\tau^D - 4\tau_0^D \geq n(n-1)c - 2A^D - n^2(\|\bar{H}\|^2 + \|\bar{H}^*\|^2) - \frac{1}{2}(\|\bar{B}\|^2 + \|\bar{B}^*\|^2) - 4\tau_0. \quad (18)$$

where  $\tau_0$  is the scalar curvature of the Levi-civita connection  $\nabla$  on  $M$ .

*Proof.* From (4), (5) and (6), we have

$$\begin{aligned} g(S(X, Y)Z, W) &= \frac{1}{2} [g(\bar{R}(X, Y)Z, W) + g(\bar{R}^*(X, Y)Z, W)] \\ &= \frac{1}{2} [g(\bar{R}(X, Y)Z, W) - g(\bar{R}(X, Y)W, Z)] \\ &= c [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \end{aligned} \quad (19)$$

Using (11), (12), (13) and (19), we obtain

$$\begin{aligned}
 g(S^D(X, Y)Z, W) &= c[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
 &\quad + \frac{1}{2} \{ [g(\bar{B}^*(X, W), \bar{B}(Y, Z)) - g(\bar{B}(X, Z), \bar{B}^*(Y, W))] \\
 &\quad + [g(\bar{B}(X, W), \bar{B}^*(Y, Z)) - g(\bar{B}^*(X, Z), \bar{B}(Y, W))] \} \\
 &\quad - \frac{1}{2} [g(\bar{B}^*(Z, W), [X, Y]) + g(\bar{B}(Z, W), [X, Y])] \\
 &= c[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
 &\quad + \frac{1}{2} \{ [g(\bar{B}^*(X, W), \bar{B}(Y, Z)) + g(\bar{B}(X, W), \bar{B}^*(Y, Z))] \\
 &\quad - [g(\bar{B}(X, Z), \bar{B}^*(Y, W)) + g(\bar{B}^*(X, Z), \bar{B}(Y, W))] \} \\
 &\quad + g(B(Z, W), [Y, X]),
 \end{aligned} \tag{20}$$

where we used  $2B(Z, W) = \bar{B}(Z, W) + \bar{B}^*(Z, W)$ .

Let  $\{E_1, \dots, E_n\}$  and  $\{E_{n+1}, \dots, E_m\}$  be orthonormal basis of  $\Gamma(D)$  and  $\Gamma(D^\perp)$ , respectively. Putting  $X = W = E_i, Y = Z = E_j$  in (20) and taking summation, we get

$$\begin{aligned}
 2\tau^D &= \sum_{1 \leq i, j \leq n} g(S^D(E_i, E_j)E_j, E_i) \\
 &= \sum_{1 \leq i, j \leq n} \{ c[g(E_j, E_j)g(E_i, E_i) - g(E_i, E_j)g(E_j, E_i)] \\
 &\quad + \frac{1}{2} \{ [g(\bar{B}^*(E_i, E_i), \bar{B}(E_j, E_j)) + g(\bar{B}(E_i, E_i), \bar{B}^*(E_j, E_j))] \\
 &\quad - [g(\bar{B}(E_i, E_j), \bar{B}^*(E_j, E_i)) + g(\bar{B}^*(E_i, E_j), \bar{B}(E_j, E_i))] \} \\
 &\quad + g(B(E_j, E_i), [E_j, E_i]) \} \\
 &= n(n-1)c + \sum_{1 \leq i, j \leq n} g(B(E_j, E_i), [E_j, E_i]) \\
 &\quad + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i, j \leq n} [(\bar{h}_{ii}^\alpha \bar{h}_{jj}^{*\alpha} + \bar{h}_{ii}^{*\alpha} \bar{h}_{jj}^\alpha) - (\bar{h}_{ij}^\alpha \bar{h}_{ji}^{*\alpha} + \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^\alpha)] \\
 &= n(n-1)c + 2A^D + n^2 g(\bar{H}, \bar{H}^*) - \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i, j \leq n} (\bar{h}_{ij}^\alpha \bar{h}_{ji}^{*\alpha} + \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^\alpha) \\
 &= n(n-1)c + 2A^D + n^2 g(\bar{H}, \bar{H}^*) - \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i, j \leq n} [(\bar{h}_{ij}^\alpha + \bar{h}_{ij}^{*\alpha})(\bar{h}_{ji}^\alpha + \bar{h}_{ji}^{*\alpha}) - \bar{h}_{ij}^\alpha \bar{h}_{ji}^\alpha - \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^{*\alpha}] \\
 &= n(n-1)c + 2A^D + n^2 g(\bar{H}, \bar{H}^*) - 2 \sum_{\alpha=n+1}^m \sum_{1 \leq i, j \leq n} h_{ij}^\alpha h_{ji}^\alpha + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i, j \leq n} (\bar{h}_{ij}^\alpha \bar{h}_{ji}^\alpha + \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^{*\alpha}).
 \end{aligned} \tag{21}$$

where we used  $\bar{h}_{ij}^\alpha + \bar{h}_{ij}^{*\alpha} = 2h_{ij}^\alpha, \forall i, j, \alpha$ .

We denote by  $\tau_0$  the scalar curvature of the Levi-civita connection  $\nabla$  on  $M$ . Gauss equation (10) implies

$$2\tau_0^D = 2\tau_0 + n^2 \|H\|^2 - \sum_{\alpha=n+1}^m \sum_{1 \leq i, j \leq n} h_{ij}^\alpha h_{ji}^\alpha + 2A^D. \tag{22}$$

We note that

$$\begin{aligned} \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i, j \leq n} (\bar{h}_{ij}^\alpha \bar{h}_{ji}^\alpha + \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^{*\alpha}) &\geq -\frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i, j \leq n} \left[ \frac{(\bar{h}_{ij}^\alpha)^2 + (\bar{h}_{ji}^\alpha)^2}{2} + \frac{(\bar{h}_{ij}^{*\alpha})^2 + (\bar{h}_{ji}^{*\alpha})^2}{2} \right] \\ &= -\frac{1}{2} (\|\bar{B}\|^2 + \|\bar{B}^*\|^2) \end{aligned} \tag{23}$$

From (21),(22) and (23), we obtain

$$\begin{aligned} 2\tau^D - 4\tau_0^D &\geq n(n-1)c - 2A^D + n^2g(\bar{H}, \bar{H}^*) - 2n^2\|H\|^2 - \frac{1}{2}(\|\bar{B}\|^2 + \|\bar{B}^*\|^2) - 4\tau_0 \\ &= n(n-1)c - 2A^D - n^2(\|\bar{H}\|^2 + \|\bar{H}^*\|^2) - \frac{1}{2}(\|\bar{B}\|^2 + \|\bar{B}^*\|^2) - 4\tau_0. \end{aligned} \tag{24}$$

This ends the proof Theorem 3.1.  $\square$

**Corollary 3.2.** *The equality case of (18) holds if and only if  $\bar{h}_{ij}^\alpha + \bar{h}_{ji}^\alpha = 0, \bar{h}_{ij}^{*\alpha} + \bar{h}_{ji}^{*\alpha} = 0, \forall i, j = 1, \dots, n; \alpha = n + 1, \dots, m$ . i.e.,  $D$  is totally geodesic with respect to dual connections  $\bar{\nabla}$  and  $\bar{\nabla}^*$ .*

Let  $\tilde{M}(\tilde{c})$  be a Hessian manifold of constant Hessian curvature  $\tilde{c}$ . Then it is flat with respect to the dual connections  $\bar{\nabla}$  and  $\bar{\nabla}^*$ . Moreover  $\tilde{M}(\tilde{c})$  is a Riemannian space form of constant sectional curvature  $-\tilde{c} \setminus 4$  (with respect to the Levi-Civita connection  $\nabla$ ). So from Theorem 3.1, we have the following.

**Theorem 3.3.** *Let  $\tilde{M}(\tilde{c})$  be an  $m$ -dimensional Hessian manifold of constant Hessian curvature  $\tilde{c}$ . Let  $D \subset TM$  be a non-integrable distribution with constant rank  $n$  and  $TM = D \oplus D^\perp$ . Then*

$$2\tau^D - 4\tau_0^D \geq -2A^D - n^2(\|\bar{H}\|^2 + \|\bar{H}^*\|^2) - \frac{1}{2}(\|\bar{B}\|^2 + \|\bar{B}^*\|^2) + \frac{n(n-1)\tilde{c}}{2}. \tag{25}$$

#### 4. Chen first inequality

This section is devoted to establish Chen first inequality for non-integrable distributions in statistical manifolds of constant curvature.

**Theorem 4.1.** *Let  $(M, c)$  be an  $m$ -dimensional statistical manifold with constant curvature  $c$ . Let  $D \subset TM$  be a non-integrable distribution with constant rank  $n$  and  $TM = D \oplus D^\perp$ . Then*

$$\begin{aligned} \tau^D - K^D(\Pi) - 2(\tau_0^D - K_0^D(\Pi)) &\geq \frac{(n+1)(n-2)}{2}c - A^D - \Omega^D - \frac{n^2(n-2)}{4(n-1)}(\|\bar{H}\|^2 + \|\bar{H}^*\|^2) \\ &\quad - \frac{1}{4}(\|\bar{B}\|^2 + \|\bar{B}^*\|^2) - 2(\tau_0 - K_0(\Pi)), \end{aligned} \tag{26}$$

where  $\tau_0$  and  $K_0(\Pi)$  are the scalar curvature and sectional curvature of the Levi-civita connection  $\nabla$  on  $M$ , respectively.

Proof. Using (17) and (20), we have

$$\begin{aligned}
 \tau^D &= \sum_{1 \leq i < j \leq n} \left\{ c \left[ g(E_j, E_j)g(E_i, E_i) - g(E_i, E_j)g(E_j, E_i) \right] \right. \\
 &\quad + \frac{1}{2} \left\{ \left[ g(\bar{B}^*(E_i, E_i), \bar{B}(E_j, E_j)) + g(\bar{B}(E_i, E_i), \bar{B}^*(E_j, E_j)) \right] \right. \\
 &\quad \left. \left. - \left[ g(\bar{B}(E_i, E_j), \bar{B}^*(E_j, E_i)) + g(\bar{B}^*(E_i, E_j), \bar{B}(E_j, E_i)) \right] \right\} \right\} \\
 &\quad + \frac{1}{2} \sum_{1 \leq i, j \leq n} g(B(E_j, E_i), [E_j, E_i]) \\
 &= \frac{n(n-1)}{2}c + A^D + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \left[ (\bar{h}_{ii}^\alpha \bar{h}_{jj}^{*\alpha} + \bar{h}_{ii}^{*\alpha} \bar{h}_{jj}^\alpha) - (\bar{h}_{ij}^\alpha \bar{h}_{ji}^{*\alpha} + \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^\alpha) \right] \\
 &= \frac{n(n-1)}{2}c + A^D + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \left[ (\bar{h}_{ii}^\alpha + \bar{h}_{ii}^{*\alpha})(\bar{h}_{jj}^\alpha + \bar{h}_{jj}^{*\alpha}) - \bar{h}_{ii}^\alpha \bar{h}_{jj}^\alpha - \bar{h}_{ii}^{*\alpha} \bar{h}_{jj}^{*\alpha} \right. \\
 &\quad \left. - (\bar{h}_{ij}^\alpha + \bar{h}_{ij}^{*\alpha})(\bar{h}_{ji}^\alpha + \bar{h}_{ji}^{*\alpha}) + \bar{h}_{ij}^\alpha \bar{h}_{ji}^\alpha + \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^{*\alpha} \right] \\
 &= \frac{n(n-1)}{2}c + A^D + 2 \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} (h_{ii}^\alpha h_{jj}^\alpha - h_{ij}^\alpha h_{ji}^\alpha) - \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \bar{h}_{ii}^\alpha \bar{h}_{jj}^\alpha \\
 &\quad - \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \bar{h}_{ii}^{*\alpha} \bar{h}_{jj}^{*\alpha} + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \bar{h}_{ij}^\alpha \bar{h}_{ji}^\alpha + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^{*\alpha},
 \end{aligned}$$

where we used  $\bar{h}_{ij}^\alpha + \bar{h}_{ij}^{*\alpha} = 2h_{ij}^\alpha, \forall i, j, \alpha$ .

We denote by  $\tau_0$  the scalar curvature of the Levi-civita connection  $\nabla$  on  $M$ . Gauss equation (10) implies

$$\begin{aligned}
 \tau^D &= \frac{n(n-1)}{2}c + 2\tau_0^D - 2\tau_0 - A^D - \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \bar{h}_{ii}^\alpha \bar{h}_{jj}^\alpha \\
 &\quad - \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \bar{h}_{ii}^{*\alpha} \bar{h}_{jj}^{*\alpha} + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \bar{h}_{ij}^\alpha \bar{h}_{ji}^\alpha + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{1 \leq i < j \leq n} \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^{*\alpha},
 \end{aligned} \tag{27}$$

Let  $E_1, E_2$  be a basis of  $\Pi \subset D$ . By (15) and (20), we get

$$\begin{aligned}
 K^D(\Pi) &= c - \Omega^\Pi + 2 \sum_{\alpha=n+1}^n (h_{11}^\alpha h_{22}^\alpha - h_{12}^\alpha h_{21}^\alpha) - \frac{1}{2} \sum_{\alpha=n+1}^m \bar{h}_{11}^\alpha \bar{h}_{22}^\alpha \\
 &\quad - \frac{1}{2} \sum_{\alpha=n+1}^m \bar{h}_{11}^{*\alpha} \bar{h}_{22}^{*\alpha} + \frac{1}{2} \sum_{\alpha=n+1}^m \bar{h}_{12}^\alpha \bar{h}_{21}^\alpha + \frac{1}{2} \sum_{\alpha=n+1}^m \bar{h}_{12}^{*\alpha} \bar{h}_{21}^{*\alpha},
 \end{aligned} \tag{28}$$

where  $K_0(\Pi)$  is sectional curvature of the Levi-civita connection  $\nabla$  on  $M$ .

Using the Gauss equation (10) for the Levi-Civita connection, we obtain

$$K_0^D(\Pi) = K_0(\Pi) - \Omega^\Pi + \sum_{\alpha=n+1}^n (h_{11}^\alpha h_{22}^\alpha - h_{12}^\alpha h_{21}^\alpha). \tag{29}$$



From (28) and (29), we have

$$\begin{aligned}
 K^D(\Pi) &= c + \Omega^\Pi + 2K_0^D(\Pi) - 2K_0(\Pi) - \frac{1}{2} \sum_{\alpha=n+1}^m \bar{h}_{11}^\alpha \bar{h}_{22}^\alpha \\
 &\quad - \frac{1}{2} \sum_{\alpha=n+1}^m \bar{h}_{11}^{*\alpha} \bar{h}_{22}^{*\alpha} + \frac{1}{2} \sum_{\alpha=n+1}^m \bar{h}_{12}^\alpha \bar{h}_{21}^\alpha + \frac{1}{2} \sum_{\alpha=n+1}^m \bar{h}_{12}^{*\alpha} \bar{h}_{21}^{*\alpha}
 \end{aligned}
 \tag{30}$$

By subtracting (30) from (27), we obtain

$$\begin{aligned}
 \tau^D - K^D(\Pi) - 2(\tau_0^D - K_0^D(\Pi)) &= \frac{(n+1)(n-2)}{2}c - A^D - \Omega^D - 2(\tau_0 - K_0(\Pi)) \\
 &\quad - \frac{1}{2} \sum_{\alpha=n+1}^m [ \sum_{1 \leq i < j \leq n} \bar{h}_{ii}^\alpha \bar{h}_{jj}^\alpha - \bar{h}_{11}^\alpha \bar{h}_{22}^\alpha ] - \frac{1}{2} \sum_{\alpha=n+1}^m [ \sum_{1 \leq i < j \leq n} \bar{h}_{ii}^{*\alpha} \bar{h}_{jj}^{*\alpha} - \bar{h}_{11}^{*\alpha} \bar{h}_{22}^{*\alpha} ] \\
 &\quad + \frac{1}{2} \sum_{\alpha=n+1}^m [ \sum_{1 \leq i < j \leq n} \bar{h}_{ij}^\alpha \bar{h}_{ji}^\alpha - \bar{h}_{12}^\alpha \bar{h}_{21}^\alpha ] + \frac{1}{2} \sum_{\alpha=n+1}^m [ \sum_{1 \leq i < j \leq n} \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^{*\alpha} - \bar{h}_{12}^{*\alpha} \bar{h}_{21}^{*\alpha} ].
 \end{aligned}
 \tag{31}$$

Then Lemma 1.1 implies

$$\sum_{\alpha=n+1}^m [ \sum_{1 \leq i < j \leq n} \bar{h}_{ii}^\alpha \bar{h}_{jj}^\alpha - \bar{h}_{11}^\alpha \bar{h}_{22}^\alpha ] \leq \sum_{\alpha=n+1}^m \frac{n-2}{2(n-1)} (\sum_{i=1}^n \bar{h}_{ii}^\alpha)^2 = \frac{n^2(n-2)}{2(n-1)} \|\bar{H}\|^2,
 \tag{32}$$

$$\sum_{\alpha=n+1}^m [ \sum_{1 \leq i < j \leq n} \bar{h}_{ii}^{*\alpha} \bar{h}_{jj}^{*\alpha} - \bar{h}_{11}^{*\alpha} \bar{h}_{22}^{*\alpha} ] \leq \sum_{\alpha=n+1}^m \frac{n-2}{2(n-1)} (\sum_{i=1}^n \bar{h}_{ii}^{*\alpha})^2 = \frac{n^2(n-2)}{2(n-1)} \|\bar{H}^*\|^2.
 \tag{33}$$

We note that

$$\begin{aligned}
 &\sum_{\alpha=n+1}^m [ \sum_{1 \leq i < j \leq n} \bar{h}_{ij}^\alpha \bar{h}_{ji}^\alpha - \bar{h}_{12}^\alpha \bar{h}_{21}^\alpha ] \\
 &= \sum_{\alpha=n+1}^m [ \sum_{3 \leq j \leq n} \bar{h}_{1j}^\alpha \bar{h}_{j1}^\alpha + \sum_{2 \leq i < j \leq n} \bar{h}_{ij}^\alpha \bar{h}_{ji}^\alpha ] \\
 &\geq - \sum_{\alpha=n+1}^m [ \sum_{3 \leq j \leq n} \frac{(\bar{h}_{1j}^\alpha)^2 + (\bar{h}_{j1}^\alpha)^2}{2} + \sum_{2 \leq i < j \leq n} \frac{(\bar{h}_{ij}^\alpha)^2 + (\bar{h}_{ji}^\alpha)^2}{2} ] \\
 &\geq - \sum_{\alpha=n+1}^m [ \sum_{3 \leq j \leq n} \frac{(\bar{h}_{1j}^\alpha)^2 + (\bar{h}_{j1}^\alpha)^2}{2} + \sum_{2 \leq i < j \leq n} \frac{(\bar{h}_{ij}^\alpha)^2 + (\bar{h}_{ji}^\alpha)^2}{2} + \sum_{i=1}^n \frac{(\bar{h}_{ii}^\alpha)^2}{2} + \frac{(\bar{h}_{12}^\alpha)^2 + (\bar{h}_{21}^\alpha)^2}{2} ] \\
 &= - \frac{\|\bar{B}\|^2}{2}.
 \end{aligned}
 \tag{34}$$

Similarly, we get

$$\sum_{\alpha=n+1}^m [ \sum_{1 \leq i < j \leq n} \bar{h}_{ij}^{*\alpha} \bar{h}_{ji}^{*\alpha} - \bar{h}_{12}^{*\alpha} \bar{h}_{21}^{*\alpha} ] \geq - \frac{\|\bar{B}^*\|^2}{2}.
 \tag{35}$$

By summing the above relations we obtain

$$\begin{aligned}
 \tau^D - K^D(\Pi) - 2(\tau_0^D - K_0^D(\Pi)) &\geq \frac{(n+1)(n-2)}{2}c - A^D - \Omega^D - 2(\tau_0 - K_0(\Pi)) \\
 &\quad - \frac{n^2(n-2)}{4(n-1)} (\|\bar{H}\|^2 + \|\bar{H}^*\|^2) - \frac{1}{4} (\|\bar{B}\|^2 + \|\bar{B}^*\|^2).
 \end{aligned}
 \tag{36}$$

This completes the proof of Theorem 4.1.  $\square$

**Corollary 4.2.** *The equality case of (26) holds if and only if  $D$  are also totally geodesic with respect to dual connections  $\bar{\nabla}$  and  $\bar{\nabla}^*$ , and  $\bar{h}_{12}^\alpha = \bar{h}_{21}^\alpha = 0, \bar{h}_{12}^{*\alpha} = \bar{h}_{21}^{*\alpha} = 0$  for  $\alpha = n + 1, \dots, m$ .*

*Proof.* The equality case of (32) holds if and only if  $\bar{h}_{11}^\alpha + \bar{h}_{22}^\alpha = \bar{h}_{33}^\alpha = \dots = \bar{h}_{nn}^\alpha$ . The equality case of (33) holds if and only if  $\bar{h}_{11}^{*\alpha} + \bar{h}_{22}^{*\alpha} = \bar{h}_{33}^{*\alpha} = \dots = \bar{h}_{nn}^{*\alpha}$ .

The equality case of (34) holds if and only if  $\bar{h}_{1j}^\alpha + \bar{h}_{j1}^\alpha = 0$  for  $3 \leq j \leq n, \bar{h}_{ij}^\alpha + \bar{h}_{ji}^\alpha = 0$  for  $2 \leq i < j \leq n, \bar{h}_{ii}^\alpha = 0$  for  $i = 1, \dots, n$  and  $\bar{h}_{12}^\alpha = \bar{h}_{21}^\alpha = 0$ .

The equality case of (35) holds if and only if  $\bar{h}_{1j}^{*\alpha} + \bar{h}_{j1}^{*\alpha} = 0$  for  $3 \leq j \leq n, \bar{h}_{ij}^{*\alpha} + \bar{h}_{ji}^{*\alpha} = 0$  for  $2 \leq i < j \leq n, \bar{h}_{ii}^{*\alpha} = 0$  for  $i = 1, \dots, n$  and  $\bar{h}_{12}^{*\alpha} = \bar{h}_{21}^{*\alpha} = 0$ . This ends the proof of Corollary 4.2.  $\square$

For a Hessian manifold of constant Hessian curvature  $\tilde{c}$ , from Theorem 4.1 we have the following:

**Theorem 4.3.** *Let  $\tilde{M}(\tilde{c})$  be an  $m$ -dimensional statistical manifold with constant curvature  $c$ . Let  $D \subset TM$  be a non-integrable distribution with constant rank  $n$  and  $TM = D \oplus D^\perp$ . Then*

$$\begin{aligned} \tau^D - K^D(\pi) - 2(\tau_0^D - K_0^D(\pi)) &\geq \frac{(n+1)(n-2)\tilde{c}}{4} - A^D - \Omega^D \\ &\quad - \frac{n^2(n-2)}{4(n-1)} (\|\bar{H}\|^2 + \|\bar{H}^*\|^2) - \frac{1}{4} (\|\bar{B}\|^2 + \|\bar{B}^*\|^2). \end{aligned}$$

### 5. Chen-Ricci inequality

In this section, we will prove a Chen-Ricci inequality for non-integrable distributions in the statistical manifolds of constant curvature.

For each unit vector field  $X \in \Gamma(D)$ , we choose an orthonormal basis  $\{E_1, \dots, E_n\}$  of  $D$  such that  $E_1 = X$ . We define

$$\begin{aligned} Ric^D(X) &= \sum_{j=2}^n g(S^D(X, E_j)E_j, X); \\ A^D(X) &= \sum_{j=2}^n g(B(E_j, X), [E_j, X]); \\ \|\bar{B}^X\|^2 &= \sum_{j=2}^n [g(\bar{B}(E_j, X), \bar{B}(E_j, X)) + g(\bar{B}(X, E_j), \bar{B}(X, E_j))]; \\ \|\bar{B}^{*X}\|^2 &= \sum_{j=2}^n [g(\bar{B}^*(E_j, X), \bar{B}^*(E_j, X)) + g(\bar{B}^*(X, E_j), \bar{B}^*(X, E_j))]. \end{aligned}$$

**Theorem 5.1.** *Let  $(M, c)$  be an  $m$ -dimensional statistical manifold with constant curvature  $c$ . Let  $D \subset TM$  be a non-integrable distribution with constant rank  $n$  and  $TM = D \oplus D^\perp$ . Then*

$$\begin{aligned} Ric^D(X) - 2Ric_0^D(X) &\geq (n-1)c - A^D(X) - \frac{n^2}{8} (\|\bar{H}\|^2 + \|\bar{H}^*\|^2) \\ &\quad - \frac{1}{4} (\|\bar{B}\|^2 + \|\bar{B}^*\|^2) - 2Ric_0(X), \end{aligned} \tag{37}$$

where  $Ric_0(X)$  and  $Ric_0^D(X)$  are Ricci curvatures with respect to the Levi-civita connection  $\nabla$  on  $M$  and  $D$ , respectively.

Proof. From (20) and the definition of  $Ric^D(X)$ , we have

$$\begin{aligned}
 Ric^D(X) &= \sum_{j=2}^n \left\{ c \left[ g(E_j, E_j)g(X, X) - g(X, E_j)g(E_j, X) \right] + \frac{1}{2} \left\{ \left[ g(\bar{B}^*(X, X), \bar{B}(E_j, E_j)) \right. \right. \right. \\
 &\quad \left. \left. \left. + g(\bar{B}(X, X), \bar{B}^*(E_j, E_j)) \right] - \left[ g(\bar{B}(X, E_j), \bar{B}^*(E_j, X)) + g(\bar{B}^*(X, E_j), \bar{B}(E_j, X)) \right] \right\} \\
 &\quad \left. + g(B(E_j, X), [E_j, X]) \right\} \\
 &= (n-1)c + A^D(X) + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{j=2}^n \left[ \bar{h}_{11}^\alpha \bar{h}_{jj}^{*\alpha} + \bar{h}_{11}^{*\alpha} \bar{h}_{jj}^\alpha - \bar{h}_{1j}^\alpha \bar{h}_{j1}^{*\alpha} - \bar{h}_{1j}^{*\alpha} \bar{h}_{j1}^\alpha \right] \tag{38} \\
 &= (n-1)c + A^D(X) + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{j=2}^n \left[ (\bar{h}_{11}^\alpha + \bar{h}_{11}^{*\alpha})(\bar{h}_{jj}^\alpha + \bar{h}_{jj}^{*\alpha}) \right. \\
 &\quad \left. - \bar{h}_{11}^\alpha \bar{h}_{jj}^{*\alpha} - \bar{h}_{11}^{*\alpha} \bar{h}_{jj}^\alpha - (\bar{h}_{1j}^\alpha + \bar{h}_{1j}^{*\alpha})(\bar{h}_{j1}^\alpha + \bar{h}_{j1}^{*\alpha}) + \bar{h}_{1j}^\alpha \bar{h}_{j1}^\alpha + \bar{h}_{1j}^{*\alpha} \bar{h}_{j1}^{*\alpha} \right] \\
 &= (n-1)c + A^D(X) + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{j=2}^n \left[ 4(h_{11}^\alpha h_{jj}^\alpha - h_{1j}^\alpha h_{j1}^\alpha) - \bar{h}_{11}^\alpha \bar{h}_{jj}^\alpha - \bar{h}_{11}^{*\alpha} \bar{h}_{jj}^{*\alpha} + \bar{h}_{1j}^\alpha \bar{h}_{j1}^\alpha + \bar{h}_{1j}^{*\alpha} \bar{h}_{j1}^{*\alpha} \right]
 \end{aligned}$$

Also, with respect to the Levi-Civita connection, we get

$$\begin{aligned}
 Ric_0^D(X) &= \sum_{j=2}^n g(R^D(X, E_j)E_j, X) \\
 &= Ric_0(X) + \sum_{\alpha=n+1}^m \sum_{j=2}^n (h_{11}^\alpha h_{jj}^\alpha - h_{1j}^\alpha h_{j1}^\alpha) + A^D(X), \tag{39}
 \end{aligned}$$

where  $Ric_0(X)$  and  $Ric_0^D(X)$  are Ricci curvatures with respect to the Levi-civita connection on  $M$  and  $D$ , respectively.

Using (38) and (39), we get

$$\begin{aligned}
 Ric^D(X) - 2Ric_0^D(X) &= (n-1)c - A^D(X) - 2Ric_0(X) - \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{j=2}^n \bar{h}_{11}^\alpha \bar{h}_{jj}^\alpha \\
 &\quad - \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{j=2}^n \bar{h}_{11}^{*\alpha} \bar{h}_{jj}^{*\alpha} + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{j=2}^n \bar{h}_{1j}^\alpha \bar{h}_{j1}^\alpha + \frac{1}{2} \sum_{\alpha=n+1}^m \sum_{j=2}^n \bar{h}_{1j}^{*\alpha} \bar{h}_{j1}^{*\alpha}. \tag{40}
 \end{aligned}$$

Applying Lemma 2.2, we have

$$\sum_{\alpha=n+1}^m \sum_{j=2}^n \bar{h}_{11}^\alpha \bar{h}_{jj}^\alpha \leq \frac{n^2}{4} \|F\|^2, \quad \sum_{\alpha=n+1}^m \sum_{j=2}^n \bar{h}_{11}^{*\alpha} \bar{h}_{jj}^{*\alpha} \leq \frac{n^2}{4} \|F^*\|^2. \tag{41}$$

On the other hand, we note

$$\sum_{\alpha=n+1}^m \sum_{j=2}^n \bar{h}_{1j}^\alpha \bar{h}_{j1}^\alpha \geq - \sum_{\alpha=n+1}^m \sum_{j=2}^n \frac{(\bar{h}_{1j}^\alpha)^2 + (\bar{h}_{j1}^\alpha)^2}{2} = - \frac{\|\bar{B}^X\|^2}{2}, \tag{42}$$

$$\sum_{\alpha=n+1}^m \sum_{j=2}^n \bar{h}_{1j}^{*\alpha} \bar{h}_{j1}^{*\alpha} \geq - \sum_{\alpha=n+1}^m \sum_{j=2}^n \frac{(\bar{h}_{1j}^{*\alpha})^2 + (\bar{h}_{j1}^{*\alpha})^2}{2} = - \frac{\|\bar{B}^{*X}\|^2}{2}, \tag{43}$$

Using (41),(42) and (43), the formula (40) can become

$$\begin{aligned} Ric^D(X) - 2Ric_0^D(X) &\geq (n-1)c - A^D(X) - \frac{n^2}{8}(\|\bar{H}\|^2 + \|\bar{H}^*\|^2) \\ &\quad - \frac{1}{4}(\|\bar{B}^X\|^2 + \|\bar{B}^{*X}\|^2) - 2Ric_0(X). \end{aligned} \quad (44)$$

□

**Corollary 5.2.** *The equality in (37) holds if and only if for  $\alpha \in \{n+1, \dots, m\}$*

$$\begin{aligned} \bar{h}_{11}^\alpha &= \sum_{j=2}^n \bar{h}_{jj}^\alpha, \bar{h}_{11}^{*\alpha} = \sum_{j=2}^n \bar{h}_{jj}^{*\alpha}; \\ \bar{h}_{1j}^\alpha + \bar{h}_{j1}^\alpha &= 0, \bar{h}_{1j}^{*\alpha} + \bar{h}_{j1}^{*\alpha} = 0, j = 2, \dots, n. \end{aligned}$$

For a Hessian manifold of constant Hessian curvature  $\tilde{c}$ , from Theorem 4.1 we have the following:

**Theorem 5.3.** *Let  $\tilde{M}(\tilde{c})$  be an  $m$ -dimensional statistical manifold with constant curvature  $c$ . Let  $D \subset TM$  be a non-integrable distribution with constant rank  $n$  and  $TM = D \oplus D^\perp$ . Then*

$$\begin{aligned} Ric^D(X) - 2Ric_0^D(X) &\geq \frac{(n-1)\tilde{c}}{2} - A^D(X) - \frac{n^2}{8}(\|\bar{H}\|^2 + \|\bar{H}^*\|^2) \\ &\quad - \frac{1}{4}(\|\bar{B}^X\|^2 + \|\bar{B}^{*X}\|^2). \end{aligned} \quad (45)$$

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