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Common Fixed Point on Generalized Weak Contraction Mappings in Extended Rectangular b-Metric Spaces

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Abstract. This paper presents an existence and uniqueness of common fixed point for two functions on generalized weakly contraction mappings in extended rectangular b-metric spaces.

1. Introduction and Preliminaries

One of generalization of the metric space is introduced by Bakhtin in 1989 [5], that is b-metric space. Czerwik [6] in 1993 utilized the space for fixed point results on Banach's contraction mapping. However, many authors have utilized the space for fixed point results in various type contraction mapping [8],[18]. Furthermore, in 2017, Kamran [15] generalized b-metric space to become extended b-metric space, some authors who have used the space for fixed point results such as, [1],[3],[24]. In 2016, George et al. [10] introduced a new notion as an extension of b-metric, it is said a rectangular b-metric. By utilizing this space, some authors, such as [11],[14],[19],[21], have yielded some fixed point theorems on different types of contraction mapping. Very recently, in 2019, Mustafa et al. [21] introduced a generalization of rectangular b-metric space for some fixed point theorems. The space is called an extended rectangular b-metric space. Asim et al. [4] worked in this space for fixed point results and its applications. For more on generalized metric spaces see also [7],[20],[26].

During this time, Banach's contraction principle is mostly used for the research of fixed point of many types of contraction mapping. Many authors have generalized the Banach's contraction in various ways [12],[17],[22],[25]. Generalized weakly contraction mapping is one of the interesting study as the generalization of Banach's contraction in recent years [16],[23],[28]. Dutta et al. [9] introduced the generalized weakly contraction in the following theorem:

A mapping $T : X \to X$ where (X, d) is a complete metric space is said to be *generalized weakly contraction* if

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$$d(Tx,Ty) \leq \Psi(d(x,y)) - \varphi(d(x,y)),$$

where $\Psi,\varphi : [0,+\infty) \to [0,+\infty)$ are continuous and monotone nondecreasing functions with $\Psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Dutta [9] stated that if a function holds for those conditions, then T has a unique fixed point. If taking $\Psi(t) = kt$, where $0 < k < 1$ and $\varphi(t) = 0$, we have a Banach's contraction. If $\Psi(t) = t$, we have a weakly contraction mapping which is introduced by Alber et al. [2].

In this paper, we introduce a new type of generalized weakly contraction mapping and present a common fixed point theorem on extended rectangular b-metric spaces. We also provide some examples to illustrate and clarify the definitions or the theorems.

To begin with, we need to present some basic definitions and notations which will be used in the main results as follows:

Definition 1.([5],[6]). Let X be a non-empty set. A mapping $d_b : X\times X \to [0,+\infty)$ is said to be a b-metric, if there exists $b \geq 1$ such that d_b satisfies the following conditions:

1. $d_b(x,y) = 0$, if and only if $x = y$,

2. $d_b(x,y) = d_b(y,x)$,

3. $d_b(x,y) \leq b[d_b(x,s) + d_b(s,y)]$,

for all $x,y,s \in X$.

The pair (X,d_b) is called a b-metric space.

Definition 2.([15]). Let X be a non-empty set. A mapping $d_b : X\times X \to [0,+\infty)$ is said to be an extended b-metric, if there exists a function $b : X\times X \to [1,+\infty)$ such that d_b satisfies the following conditions:

1. $d_b(x,y) = 0$, if and only if $x = y$,

2. $d_b(x,y) = d_b(y,x)$,

3. $d_b(x,y) \leq b(x,y)[d_b(x,s) + d_b(s,y)]$,

for all $x,y,s \in X$.

The pair (X,d_b) is called an extended b-metric space.

Definition 3.([10]). Let X be a non-empty set. A mapping $d_b : X\times X \to [0,+\infty)$ is said to be a rectangular *b-metric*, if there is $b \geq 1$ such that d_b satisfies the following conditions:

1. $d_b(x,y) = 0$, if and only if $x = y$,

2. $d_b(x,y) = d_b(y,x)$,

3. $d_b(x,y) \leq b[d_b(x,s) + d_b(s,t) + d_b(t,y)]$,

for all $x,y \in X$ and $s,t \in X\backslash\{x,y\}$.

The pair (X,d_b) is called rectangular b-metric space.

Definition 4.([4]). Let X be a non-empty set. A mapping $d_b : X\times X \to [0,+\infty)$ is said to be an extended rectangular b-metric, if there exists a function $b : X\times X \to [1,+\infty)$ such that d_b satisfies the following conditions:

A1. $d_b(x,y) = 0$, if and only if $x = y$,

A2. $d_b(x,y) = d_b(y,x)$,

A3. $d_b(x,y) \leq b(x,y)[d_b(x,s) + d_b(s,t) + d_b(t,y)]$, (1)

for all $x,y \in X$ and $s,t \in X\backslash\{x,y\}$.

The pair (X,d_b) is called an extended rectangular b-metric space.

Example 5. Let $X = [0,+\infty)$ *and* $d_b(x,y) = \frac{(x-y)^p}{3^p}$, $p > 1$ *with* $b(x,y) = 4^{(x-y)^2}$.

It is obvious for condition A1 and A2. For condition A3, we consider from Jensen inequality, thus we have

$$d_b(x,y) = \frac{(x-y)^p}{3^p} \leq \frac{1}{3}[(x-s)^p + (s-t)^p + (t-y)^p] \leq$$
$$4^{(x-y)^2}[(x-s)^p + (s-t)^p + (t-y)^p] = b(x,y)[d_b(x,s) + d_b(s,t) + d_b(t,y)]$$

for all $x,y,s,t \in X$.

So, this shows that $d_b(x,y) = (x-y)^p$ *is an extended rectangular b-metric with* $b(x,y) = 4^{(x-y)^2}$*. However, in general that* $d_b(x,y)$ *is not continuous.* □

Example 6.(see [6]). *Let* $X = \{0,2\} \cup \left\{\frac{1}{n} | n \in N\right\}$ *and choose a function* $b(x,y) = 4^{(x-y)^2}$, *for* $x, y \in X$. *Define* $d_b : X \times X \to [0,+\infty)$ *a function as follows:*

$$d_b(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y, x, y \in \{0,2\} \text{ or } x, y \in \left\{\frac{1}{n} | n \in \mathbb{N}\right\} \\ y, & x \in \{0,2\}, y \in \left\{\frac{1}{n} | n \in \mathbb{N}\right\} \\ x, & x \in \left\{\frac{1}{n} | n \in N\right\}, y \in \{0,2\} \end{cases} \tag{2}$$

It is easy to show that d_b is a complete extended rectangular b-metric on X with $b(x,y) = 4^{(x-y)^2}$.

We choose a sequence $\{1/n\}_{n \in N}$, it is easy to show that sequence $\{1/n\}_{n \in N}$ converges to 0 and 2 and is not a Cauchy sequence. Although $\lim_{n\to+\infty} \frac{1}{n} = 0$, yet $1 = \lim_{n\to+\infty} d_b\left(\frac{1}{n}, \frac{1}{3}\right) \neq d\left(0, \frac{1}{3}\right) = \frac{1}{3}$, hence d_b is not continuous. (X, d_b) is not metrizable [27], because there exists a sequence in (X, d_b) which is not convergent to a unique point in X. This fact implies that (X, d_b) is not a Hausdorff space. □

Definition of compatible of two functions f and g in metric space is introduced by Jungck [13], however, we need the definition in extended rectangular b-metric as follows:

Definition 7. Let (X, d_b) be an extended rectangular b-metric space and $\{x_n\}$ a sequence in X.

a. $\{x_n\}$ is called *convergent* to $x \in X$if and only if $d_b(x_n, x) \to 0$, as $n \to +\infty$.
b. $\{x_n\}$ is called *Cauchy* if and only if $d_b(x_n, x_m) \to 0$, as $n, m \to +\infty$

In addition, X is called *complete*, if every Cauchy sequence is then convergent in X.

Definition 8. Let (X, d_b) be an extended rectangular b-metric space and $\{x_n\}$ be a sequence in X. Self-mapping f and g on X is said to be compatible, if $d_b(fx_n, u) \to 0$ and $d_b(gx_n, u) \to 0$, then $d_b(fgx_n, gfx_n) \to 0$, as $n \to +\infty$.

Definition 9.(see [13]) Let X be a nonempty set and $T_1, T_2 : X \to X$ be self-mappings. If $T_1x = T_2x = y$ for some $x \in X$, then y is called a *point of coincidence* of $T_1$and T_2, and x is called a *coincidence point* of T_1 and T_2.

Definition 10.(see [13]) Let X be a nonempty set and $T_1, T_2 : X \to X$ be self-mappings. $\{T_1, T_2\}$ is called *weakly compatible*, for every $x \in X$, if $T_1x = T_2x$ then $T_2T_1x = T_1T_2x$.

Definition 11. Let (X, d_b) and (Y, ρ_b) be the extended rectangular b-metric spaces. A mapping $f : X \to Y$ is said to be continuous on X if and only if every sequence $\{x_n\}$ that is convergent to x, then the sequence $\{fx_n\}$ is convergent to fx.

In the main results we need a Hausdorff property, so that if a sequence in (X, d_b) is convergent, then it has a unique limit point in X.

2. Main Results

In this section, we prove an existence and uniqueness of common fixed point for new type of generalized weak contraction mapping and using *compatible* property of two functions, for further theorem we use a rational contraction type and *weakly compatible* of two functions.

Theorem 12. *Let* (X, d_b) *be a Hausdorff and complete extended rectangular b-metric space. Let* $f, g : X \to X$ *be continuous mappings such that* $f(X) \subseteq g(X)$ *and satisfy the following conditions:*

$$b(x,y)\Psi(d_b(fx,fy)) \leq \Psi\left(\lambda\left(d_b(fx,gx) + d_b(fy,gy)\right) + \frac{\gamma d_b(gx,gy)}{b(gx,gy)}\right) - \beta\varphi\left(d_b(fx,gy)\, d_b(fy,gx)\right), \tag{3}$$

where $\beta \geq 0$, $0 < \lambda, \gamma < 1$, $\frac{\lambda+\gamma}{1-\lambda} < 1$, $\Psi, \varphi : [0,+\infty) \to [0,+\infty]$ *are continuous and* Ψ *is nondecreasing with* $\Psi(t) = \varphi(t) = 0$ *if and only if* $t = 0$.

If f *and* g *are compatible and* $\lim_{n\to+\infty} b\left(f^2x_n, gfx_n\right) < \frac{1}{\lambda}$, *then* f, g *have a unique common fixed point in* X.

Proof. Let $x_0 \in X$ and since $f(X) \subseteq g(X)$, then we can define a sequence $\{y_n\}$, where $y_n = fx_n = gx_{n+1.}$
Since Ψ is nondecreasing, using (3) we have,

$$b(x_n, x_{n+1})\Psi(d_b(fx_n, fx_{n+1})) = b(x_n, x_{n+1})\Psi(d_b(y_n, y_{n+1})) \leq$$
$$\leq \Psi\left(\lambda(d_b(fx_n, gx_n) + d_b(fx_{n+1}, gx_{n+1})) + \frac{\gamma d_b(gx_n, gx_{n+1})}{b(gx_n, gx_{n+1})}\right) -$$
$$\beta\varphi\left(d_b(fx_n, gx_{n+1})\, d_b(fx_{n+1}, gx_n)\right) =$$
$$\Psi\left(\lambda(d_b(y_n, y_{n-1}) + d_b(y_{n+1}, y_n)) + \frac{\gamma d_b(y_{n-1}, y_n)}{b(y_{n-1}, y_n)}\right) -$$
$$\beta\varphi\left(d_b(y_n, y_n)\, d_b(y_{n+1}, y_{n-1})\right) \leq$$
$$\Psi(\lambda(d_b(y_n, y_{n-1}) + d_b(y_{n+1}, y_n)) + \gamma d_b(y_{n-1}, y_n))\ . \tag{4}$$

Thus we have,

$$\Psi(d_b(y_n, y_{n+1})) \leq \frac{\Psi(\lambda(d_b(y_n, y_{n-1}) + d_b(y_{n+1}, y_n)) + \gamma d_b(y_{n-1}, y_n))}{b(x_n, x_{n+1})} \leq$$
$$\Psi(\lambda(d_b(y_n, y_{n-1}) + d_b(y_{n+1}, y_n)) + \gamma d_b(y_{n-1}, y_n)) \tag{5}$$

Since Ψ is a nondecreasing, we have

$$d_b(y_n, y_{n+1}) \leq \lambda(d_b(y_n, y_{n-1}) + d_b(y_{n+1}, y_n)) + \gamma d_b(y_{n-1}, y_n)\ . \tag{6}$$

Then we get $d_b(y_n, y_{n+1}) \leq \frac{\lambda+\gamma}{1-\lambda} d_b(y_{n-1}, y_n)$.
Let $\alpha = \frac{\lambda+\gamma}{1-\lambda}$, then we have $d_b(y_n, y_{n+1}) \leq \alpha d_b(y_{n-1}, y_n)$, and by using recursively we get

$$d_b(y_n, y_{n+1}) \leq \alpha^n d_b(y_0, y_1)\ . \tag{7}$$

Since $0 < \alpha < 1$, and using (7), we have

$$d_b(y_n, y_{n+1}) \to 0, \text{ as } n \to +\infty \tag{8}$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. By using (3), we have

$$b(x_m, x_n)\Psi(d_b(y_m, y_n)) = b(x_m, x_n)\Psi(d_b(fx_m, fx_n)) \leq$$
$$\Psi\left(\lambda(d_b(fx_m, gx_m) + d_b(fx_n, gx_n)) + \frac{\gamma d_b(gx_m, gx_n)}{b(gx_m, gx_n)}\right) -$$
$$\beta\varphi(d_b(fx_m, gx_n)\, d_b(fx_n, gx_m)) \leq \Psi(\lambda(d_b(fx_m, gx_m) + d_b(fx_n, gx_n)) +$$
$$\left.\frac{\gamma d_b(gx_m, gx_n)}{b(gx_m, gx_n)}\right) \leq \Psi\left(\lambda(d_b(y_m, y_{m-1}) + d_b(y_n, y_{n-1})) + \frac{\gamma d_b(y_{m-1}, y_{n-1})}{b(y_{m-1}, y_{n-1})}\right) \leq$$
$$\Psi(\lambda(d_b(y_m, y_{m-1}) + d_b(y_n, y_{n-1})) +$$
$$\left.\gamma\left(\frac{d_b(y_{m-1}, y_m) + d_b(y_m, y_n) + d_b(y_n, y_{n-1})}{b(y_{m-1}, y_{n-1})}\right)\right) \leq$$
$$\Psi(\lambda(d_b(y_m, y_{m-1}) + d_b(y_n, y_{n-1})) + \gamma d_b(y_{m-1}, y_m) + d_b(y_m, y_n) + d_b(y_n, y_{n-1}))\ . \tag{9}$$

Thus we get

$$\Psi(d_b(y_m, y_n)) \leq \frac{1}{b(x_m, x_n)}(\Psi(\lambda(d_b(y_m, y_{m-1}) + d_b(y_n, y_{n-1})) +$$
$$\gamma[d_b(y_{m-1}, y_m) + d_b(y_m, y_n) + d_b(y_n, y_{n-1})])) \leq$$
$$\Psi(\lambda(d_b(y_m, y_{m-1}) + d_b(y_n, y_{n-1})) + \gamma[d_b(y_{m-1}, y_m) + d_b(y_m, y_n) + d_b(y_n, y_{n-1})])\ . \tag{10}$$

Since Ψ is nondecreasing, we get

$$d_b(y_m, y_n) \leq \lambda\left(d_b(y_m, y_{m-1}) + d_b(y_n, y_{n-1})\right) + \gamma[d_b(y_{m-1}, y_m) + d_b(y_n, y_{n-1})] \tag{11}$$

by using (7) and (8) we get

$$d_b(y_m, y_n) \to 0, \text{ as } n, m \to +\infty \ . \tag{12}$$

Hence, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u^* \in X$ such that $d_b(y_n, u^*) \to 0$ as $n \to +\infty$, that is

$$d_b(fx_n, u^*) \to 0 \text{ and } d_b(gx_n, u^*) \to 0, \text{ as } n \to +\infty \ . \tag{13}$$

Since f, g are compatible, then from (13) we have

$$d_b(gfx_n, fgx_n) \to 0, \text{ as } n \to +\infty \ . \tag{14}$$

Since f and g are continuous mapping, then from (13) we have

$$d_b\left(f^2x_n, fu^*\right) \to 0, d_b(fgx_n, fu^*) \to 0, d_b(gfx_n, gu^*) \to 0, \text{ as } n \to +\infty \ . \tag{15}$$

$$\begin{aligned}
&\Psi(d_b\left(f^2x_n, fx_n\right)) \leq \\
&\frac{1}{b(fx_n, x_n)}\{\Psi\left(\lambda\left(d_b\left(f^2x_n, gfx_n\right) + d_b(fx_n, gx_n)\right) + \frac{\gamma d_b(gfx_n, gx_n)}{b(gfx_n, gx_n)}\right) - \\
&\beta\varphi\left(d_b\left(f^2x_n, gx_n\right) d_b(fx_n, gfx_n)\right)\} \leq \frac{1}{b(fx_n, x_n)} \\
&\times \Psi\left(\lambda\left(b\left(f^2x_n, gfx_n\right)\left[d_b\left(f^2x_n, fu^*\right) + d_b(fu^*, fgx_n) + d_b(fgx_n, gfx_n)\right] + \right.\right. \\
&\left.\left. d_b(y_n, y_{n-1})\right) + \frac{\gamma d_b(gfx_n, gx_n)}{b(gfx_n, gx_n)}\right) - \beta\varphi\left(d_b\left(f^2x_n, gx_n\right) d_b(fx_n, gfx_n)\right) \leq \\
&\Psi\left(\lambda\left(b\left(f^2x_n, gfx_n\right)\left[d_b\left(f^2x_n, fu^*\right) + d_b(fu^*, fgx_n) + d_b(fgx_n, gfx_n)\right] + \right.\right. \\
&\left.\left. d_b(y_n, y_{n-1})\right) + \frac{\gamma d_b(gfx_n, gx_n)}{b(gfx_n, gx_n)}\right) \leq \\
&\left\{\Psi\left(\lambda\left(b\left(f^2x_n, gfx_n\right)\left[d_b\left(f^2x_n, fu^*\right) + d_b(fu^*, fgx_n) + d_b(fgx_n, gfx_n)\right] + \right.\right.\right. \\
&\hspace{5cm} \left.\left.\left. \alpha^{n-1} d_b(y_0, y_1)\right) + \gamma d_b(gfx_n, gx_n)\right)\right\}.
\end{aligned} \tag{16}$$

Since Ψ is nondecreasing, we get

$$\begin{aligned}
&d_b\left(f^2x_n, fx_n\right) \leq \\
&\left(\lambda\left(b\left(f^2x_n, gfx_n\right)\left[d_b\left(f^2x_n, fu^*\right) + d_b(fu^*, fgx_n) + d_b(fgx_n, gfx_n)\right] + \right.\right. \\
&\hspace{5cm} \left.\left. \alpha^{n-1} d_b(y_0, y_1)\right) + \gamma d_b(gfx_n, gx_n)\right).
\end{aligned} \tag{17}$$

Since $\lim_{n\to+\infty} b\left(f^2x_n, gfx_n\right) < \frac{1}{\lambda}$, using (14), (15), (17), and for $n \to +\infty$, we get

$$\lim_{n\to+\infty} d_b\left(f^2x_n, fx_n\right) = 0 \ , \tag{18}$$

$$d_b(fu^*, u^*) \leq b(fu^*, u^*)\left[d_b\left(fu^*, f^2x_n\right) + d_b\left(f^2x_n, fx_n\right) + d_b(fx_n, u^*)\right] \ . \tag{19}$$

By using (13), (15), (18), by using (19) and for $n \to +\infty$, we obtain

$$d_b(fu^*, u^*) = 0\,. \tag{20}$$

Thus we have $fu^* = u^*$.

$$d_b(u^*, gu^*) = d_b(fu^*, gu^*) \leq b(fu^*, gu^*)\left[d_b(fu^*, fgx_n) + d_b(fgx_n, gfx_n) + d_b(gfx_n, gu^*)\right]. \tag{21}$$

By using (14), (15) and taking $n \to +\infty$ in (21), we get

$$d_b(u^*, gu^*) = 0\,. \tag{22}$$

Hence, we get $fu^* = gu^* = u^*$, this implies that u^* is common fixed point of f and g.

Next, we show uniqueness of common fixed point of of f and g. Suppose that t^* is another of common fixed point of of f and g, this means $t^* = ft^* = gt^*$.

$$\begin{aligned}\Psi(d_b(u^*, t^*)) = \Psi(d_b(fu^*, ft^*)) \leq \frac{1}{b(u^*, t^*)}[\Psi(\lambda(d_b(fu^*, gu^*) + d_b(ft^*, gt^*)) \\ + \frac{\gamma d_b(gu^*, gt^*)}{b(gu^*, gt^*)}\Bigg) - \beta\varphi(d_b(fu^*, gt^*)\, d_b(ft^*, gu^*)] \leq \\ \Psi\left(\lambda(d_b(fu^*, gu^*) + d_b(ft^*, gt^*)) + \frac{\gamma d_b(gu^*, gt^*)}{b(gu^*, gt^*)}\right) \leq \\ \Psi(\lambda(d_b(fu^*, gu^*) + d_b(ft^*, gt^*)) + \gamma d_b(gu^*, gt^*))\ .\end{aligned} \tag{23}$$

Since Ψ is nondecreasing, we have

$$\begin{aligned}d_b(u^*, t^*) \leq \lambda(d_b(fu^*, gu^*) + d_b(ft^*, gt^*)) + \gamma d_b(gu^*, gt^*) = \\ \lambda(d_b(u^*, u^*) + d_b(t^*, t^*)) + \gamma d_b(u^*, t^*)\ .\end{aligned}$$

Thus we get $(1-\gamma)\, d_b(u^*, t^*) \leq 0$. Since $1-\gamma > 0$, we have $d_b(u^*, t^*) = 0$. This implies that $u^* = t^*$. □

Example 13. Let $X = [0,1]$, define $d_b(x,y) = \frac{1}{9}(x-y)^2$ and $b(x,y) = 4^{(x-y)^2}$ on $X \times X$. Define f, g to be self-mappings on X as follows: $f(x) = \frac{x}{2^4}$and $g(x) = x$, and define $\Psi(t) = \frac{9}{2}t$, $\varphi(t) = \frac{t}{2^9}$ for $t \in [0, +\infty)$ and take $\lambda = \frac{1}{4}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2^9}$.

In fact, it is clear that $d_b(x,y) = \frac{1}{9}(x-y)^2$ is an extended rectangular b-metric with $b(x,y) = 4^{(x-y)^2}$.

Let $x_0 = \frac{1}{2}$, by using $y_n = f(x_n) = g(x_{n+1})$, for $n = 0, 1, 2, \ldots.$, we have

$$y_0 = f(x_0) = f\left(\frac{1}{2}\right) = \frac{1}{2^5} = g\left(\frac{1}{2^5}\right) = g(x_1)\,,$$

$$y_1 = f(x_1) = f\left(\frac{1}{2^5}\right) = \frac{1}{2^9} = g\left(\frac{1}{2^9}\right) = g(x_2)\,,$$

$$y_2 = f(x_2) = f\left(\frac{1}{2^9}\right) = \frac{1}{2^{13}} = g\left(\frac{1}{2^{13}}\right) = g(x_3)\ldots$$

In general, we have sequences $\{x_n\}$ and $\{y_n\}$ where $x_n = \frac{1}{2^{4n+5}}$and $y_n = \frac{1}{2^{4n+5}}$.

$$\lim_{n\to+\infty} b\left(f^2x_n, gfx_n\right) = \lim_{n\to+\infty} b\left(f(f(\frac{1}{2^{4n+5}})), gf(\frac{1}{2^{4n+5}})\right) =$$

$$\lim_{n\to+\infty} b\left((f(\frac{1}{2^{4n+9}})), g((\frac{1}{2^{4n+9}}))\right) = \lim_{n\to+\infty} b\left(\frac{1}{2^{4n+13}}, \frac{1}{2^{4n+13}}\right) =$$

$$\lim_{n\to+\infty} 4^{\left(\frac{1}{2^{4n+13}} - \frac{1}{2^{4n+13}}\right)^2} = 1 < \frac{1}{\lambda} = 4 \ .$$

Next, we show the condition

$$b(x,y)\,\Psi(d_b(fx,fy)) \leq \Psi\left(\lambda\left(d_b(fx,gx) + d_b(fy,gy)\right) + \frac{\gamma d_b(gx,gy)}{b(gx,gy)}\right) -$$

$$\beta\varphi\left(d_b(fx,gy)\,d_b(fy,gx)\right)$$

holds.

Consider that

$$\Psi\left(\lambda\left(d_b\left(\frac{x}{2^4},x\right) + d_b\left(\frac{y}{2^4},y\right)\right) + \frac{\gamma d_b(x,y)}{b(x,y)}\right) - \beta\varphi\left(d_b\left(\frac{x}{2^4},y\right) d_b\left(\frac{y}{2^4},x\right)\right) =$$

$$\Psi\left(\frac{1}{4}\left(\frac{1}{9}\left(\frac{x}{2^4} - x\right)^2 + \frac{1}{9}\left(\frac{y}{2^4} - y\right)^2\right) + \frac{1}{9}\frac{\gamma(x-y)^2}{4^{(x-y)^2}}\right) -$$

$$-\beta\varphi\left(\left(\frac{x}{16} - y\right)^2\left(\frac{y}{16} - x\right)^2\right) =$$

$$\frac{9}{2}\left(\frac{1}{4}\left(\frac{1}{9}\left(\frac{x}{2^4} - x\right)^2 + \frac{1}{9}\left(\frac{y}{2^4} - y\right)^2\right) + \frac{1}{9}\frac{(x-y)^2}{2^9 4^{(x-y)^2}}\right) -$$

$$\frac{\frac{1}{81}\left(\frac{x}{2^4} - y\right)^2\left(\frac{y}{2^4} - x\right)^2}{2^{10}} = \frac{1}{8}\left(\left(\left(\frac{x}{2^4} - x\right)^2 + \left(\frac{y}{2^4} - y\right)^2\right) + \frac{4(x-y)^2}{2^9 4^{(x-y)^2}}\right) -$$

$$\frac{\frac{1}{81}\left(\frac{x}{2^4} - y\right)^2\left(\frac{y}{2^4} - x\right)^2}{2^{10}}$$

Since $x, y \in X = [0,1]$, we have

$$= \frac{1}{8}\left(\left(\left(\frac{x}{2^4} - x\right)^2 + \left(\frac{y}{2^4} - y\right)^2\right) + \frac{4(x-y)^2}{2^9 4^{(x-y)^2}}\right) - \frac{\frac{1}{81}\left(\frac{x}{2^4} - y\right)^2\left(\frac{y}{2^4} - x\right)^2}{2^{10}}$$

Since $x, y \in [0,1]$, we have $\left(\frac{x}{2^4} - y\right)^2\left(\frac{y}{2^4} - x\right)^2 \leq \left(\frac{x}{2^4} - y\right)^2$. So we get

$$\geq \frac{1}{8}\left(\left(\left(\frac{x}{2^4} - x\right)^2 + \left(\frac{y}{2^4} - y\right)^2\right) + \frac{4(x-y)^2}{2^9 4^{(x-y)^2}}\right) - \frac{\frac{1}{81}\left(\frac{x}{2^4} - y\right)^2}{2^{10}}$$

$$= \frac{1}{8}\left(\left(\left(\frac{15x}{2^4}\right)^2 + \left(\frac{15y}{2^4}\right)^2\right) + \frac{4(x-y)^2}{2^9 4^{(x-y)^2}}\right) - \frac{\frac{1}{81}\left(\left(\frac{x}{2^4}\right)^2 - \frac{xy}{2^3} + y^2\right)}{2^{10}}$$

$$\geq \frac{1}{8}\left(\left(\left(\frac{15x}{2^4}\right)^2 + \left(\frac{15y}{2^4}\right)^2\right) + \frac{4(x-y)^2}{2^9 4^{(x-y)^2}}\right) - \frac{\frac{1}{81}\left(\left(\frac{x}{2^4}\right)^2 + y^2\right)}{2^{10}}$$

$$\geq \frac{1}{8}\left(\left(\left(\frac{15x}{2^4}\right)^2 + \left(\frac{15y}{2^4}\right)^2\right) + \frac{4(x-y)^2}{2^9 4^{(x-y)^2}}\right) - \frac{\left(\left(\frac{x}{2^4}\right)^2 + y^2\right)}{2^{10}}$$

$$\geq \frac{1}{8}\left(\left(\frac{15x}{2^4}\right)^2 + \left(\frac{15y}{2^4}\right)^2\right) + \frac{4}{2^9}\frac{(x-y)^2}{4} - \frac{\left(\left(\frac{x}{2^4}\right)^2 + y^2\right)}{2^{10}}$$

$$= \frac{225x^2 + 225y^2}{2^{11}} + \frac{4}{2^9}\frac{(x-y)^2}{4} - \frac{\left(\left(\frac{x}{2^4}\right)^2 + y^2\right)}{2^{10}}$$

$$= \frac{225x^2 + 225y^2}{2^{11}} + \frac{4}{2^9}\frac{(x-y)^2}{4} - \frac{\left(x^2 + y^2\right)}{2^{10}}$$

$$= \frac{225x^2 + 225y^2}{2^{11}} + \frac{4(x-y)^2}{2^{11}} - \frac{2\left(x^2 + y^2\right)}{2^{11}}$$

$$\geq \frac{225x^2 + 225y^2}{2^{11}} + \frac{(x-y)^2}{2^{11}} - \frac{2\left(x^2 + y^2\right)}{2^{11}}$$

$$= \frac{224x^2 + 224y^2 - 2xy}{2^{11}} = \frac{224\left(x^2 + y^2 - \frac{2}{224}xy\right)}{2^{11}}$$

$$\geq \frac{224\left(x^2 + y^2 - 2xy\right)}{2^{11}} = \frac{224(x-y)^2}{2^{11}} .$$

It is clear that for $x, y \in [0,1]$ we have $\frac{224}{2^{11}} = \frac{448}{2^{12}} \geq \frac{4^{(x-y)^2}}{2^{12}}$, so we have

$$\frac{224(x-y)^2}{2^{11}} = \frac{448(x-y)^2}{2^{12}} \geq \frac{4^{(x-y)^2}}{2^{12}}(x-y)^2 = 4^{(x-y)^2}\frac{\left(\frac{x}{2^4} - \frac{y}{2^4}\right)^2}{4} =$$

$$4^{(x-y)^2}\frac{(d_b\left(\frac{x}{2^4}, \frac{y}{2^4}\right))}{4} = 4^{(x-y)^2}\frac{(d_b(fx, fy))}{4} = b(x,y)\Psi(d_b(fx, fy)).$$

Thus we have a condition:

$$b(x,y)\Psi(d_b(fx,fy)) \leq \Psi(\lambda(d_b(fx,gx) + d_b(fy,gy))) +$$

$$\frac{\gamma d_b(gx,gy)}{b(gx,gy)} - \beta\varphi(d_b(fx,gy)d_b(fy,gx)) .$$

Hence, based on Theorem 12, this implies that f and g have unique common fixed point, that is $x = 0$. □

Corollary 14. *Let (X, d_b) be a Hausdorff and complete extended rectangular b-metric space. Let $f, g : X \to X$ be the continuous mapping, such that $f(X) \subseteq g(X)$, $g(X)$ be closed and satisfying the following conditions*

$$b(x,y)d_b(fx,fy) \leq \lambda(d_b(fx,gx) + d_b(fy,gy)) + \frac{\gamma d_b(gx,gy)}{b(gx,gy)} ,$$

where $0 < \lambda, \gamma < 1$, $\frac{\lambda+\gamma}{1-\lambda} < 1$. If $\{f, g\}$ is compatible and $\lim_{n\to+\infty} b\left(f^2x_n, gfx_n\right) < \frac{1}{\lambda}$, then f, g have a unique common fixed point in X.

Proof. By taking $\Psi(t) = t$, $\beta = 0$ in Theorem 12. □

In the next theorem, we will use weakly compatible property of two functions.

Theorem 15. *Let (X, d_b) be a Hausdorff complete extended rectangular b-metric space . Let $f, g : X \to X$ be the mappings such that $f(X) \subseteq g(X)$, $g(X)$ be closed and satisfy the following conditions:*

$$b(x,y)\,\Psi(d_b(fx,fy)) \leq \Psi\left(\lambda\left(\frac{d_b(fx,gx)+d_b(fy,gy)}{b(fx,gx)} + \frac{d_b(fx,gy)\,d_b(fy,gx)}{b(fx,gy)\left[1+d_b(fx,gy)+d_b(fy,gx)\right]}\right)\right), \quad (24)$$

where $0 < \lambda < \frac{1}{2}$ and the functions $\Psi : [0,+\infty) \to [0,+\infty)$ are continuous and nondecreasing with $\Psi(t) = 0$ if and only if $t = 0$.

If f and g are weakly compatible, then f, g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and since $f(X) \subseteq g(X)$, we can define a sequence $\{x_n\}$ and $\{y_n\}$, where $y_n = fx_n = gx_{n+1}$.

From (24) we have

$$b(x_n,x_{n+1})\,\Psi(d_b(fx_n,fx_{n+1})) = b(x_n,x_{n+1})\,\Psi(d_b(y_n,y_{n+1})) \leq$$

$$\Psi\left(\lambda\left(\frac{d_b(fx_n,gx_n)+d_b(fx_{n+1},gx_{n+1})}{d_b(fx_n,gx_n)} + \frac{d_b(fx_n,gx_{n+1})\,d_b(fx_{n+1},gx_n)}{b(fx_n,gx_{n+1})\left[1+d_b(fx_n,gx_{n+1})+d_b(fx_{n+1},gx_n)\right]}\right)\right) =$$

$$\Psi\left(\lambda\left(\frac{d_b(fx_n,gx_n)+d_b(fx_{n+1},gx_{n+1})}{d_b(fx_n,gx_n)} + \frac{d_b(y_n,y_n)\,d_b(y_{n+1},y_{n-1})}{b(y_n,y_n)\left[1+d_b(y_n,y_n)+d_b(y_{n+1},y_{n-1})\right]}\right)\right) =$$

$$\Psi\left(\lambda\left(\frac{d_b(fx_n,gx_n)+d_b(fx_{n+1},gx_{n+1})}{d_b(fx_n,gx_n)}\right)\right) .$$

Since Ψ is nondecreasing, we have

$$b(x_n,x_{n+1})\,\Psi(d_b(y_n,y_{n+1})) \leq \Psi(\lambda(d_b(fx_n,gx_n)+d_b(fx_{n+1},gx_{n+1}))) .$$

Thus we get

$$\Psi(d_b(y_n,y_{n+1})) \leq \frac{\Psi(\lambda(d_b(fx_n,gx_n)+d_b(fx_{n+1},gx_{n+1})))}{b(x_n,x_{n+1})} (\leq \Psi(\lambda(d_b(y_n,y_{n-1})+d_b(y_{n+1},y_n))).$$

Since Ψ is nondecreasing, we have

$$d_b(y_n,y_{n+1}) \leq \lambda(d_b(y_n,y_{n-1})+d_b(y_{n+1},y_n)) .$$

We get $d_b(y_n,y_{n+1}) \leq \frac{\lambda}{1-\lambda} d_b(y_{n-1},y_n)$.

Let $\alpha = \frac{\lambda}{1-\lambda}$, then we have $d_b(y_n,y_{n+1}) \leq \alpha d_b(y_{n-1},y_n)$, and recursively we get

$$d_b(y_n,y_{n+1}) \leq \alpha^n d_b(y_0,y_1) . \quad (25)$$

Since $0 < \lambda < \frac{1}{2}$ we have $0 < \alpha < 1$, using (25), we have

$$d_b(y_n,y_{n+1}) \to 0, \text{ as } n \to +\infty . \quad (26)$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. Suppose $\{y_n\}$ is not a Cauchy sequence, this means there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $m_k, n_k \in \mathbb{N}$, $n_k > m_k > k$ such that

$$d_b(y_{m_k},y_{n_k}) \geq \varepsilon , \quad (27)$$

but

$$d_b(y_{m_k-1},y_{n_k}) < \varepsilon . \quad (28)$$

Using (24) and since Ψ is nondecreasing, we have

$$b\left(x_{m_k}, x_{m_k}\right) \Psi(d_b\left(y_{m_k}, y_{n_k}\right)) = b\left(x_{m_k}, x_n\right) \Psi(d_b\left(fx_{m_k}, fx_{n_k}\right)) \leq$$
$$\Psi\left(\lambda\left(\frac{d_b\left(fx_{m_k}, gx_{m_k}\right) + d_b\left(fx_{n_k}, gx_{n_k}\right)}{b\left(fx_{m_k}, gx_{m_k}\right)} + \frac{d_b\left(fx_{m_k}, gx_{n_k}\right) d_b\left(fx_{n_k}, gx_{m_k}\right)}{b\left(fx_{m_k}, gx_{n_k}\right)\left[1 + d_b\left(fx_{m_k}, gx_{n_k}\right) d_b\left(fx_{n_k}, gx_{m_k}\right)\right]}\right)\right) \leq$$
$$\Psi\left(\lambda\left(\frac{d_b\left(fx_{m_k}, gx_{m_k}\right) + d_b\left(fx_{n_k}, gx_{n_k}\right)}{b\left(fx_{m_k}, gx_{m_k}\right)} + d_b\left(fx_{n_k}, gx_{m_k}\right)\right)\right) \leq$$

$$\Psi\left(\lambda\left(d_b\left(y_{m_k}, y_{m_k-1}\right) + d_b\left(y_{n_k}, y_{n_k-1}\right) + d_b\left(y_{n_k}, y_{m_k-1}\right)\right)\right) \ . \tag{29}$$

Thus we get

$$\Psi\left(d_b\left(y_{m_k}, y_{n_k}\right)\right) \leq \frac{\Psi\left(\lambda\left(d_b\left(y_{m_k}, y_{m_k-1}\right) + d_b\left(y_{n_k}, y_{n_k-1}\right) + d_b\left(y_{n_k}, y_{m_k-1}\right)\right)\right)}{b\left(x_m, x_n\right)} \leq$$
$$\Psi\left(\lambda\left(d_b\left(y_{m_k}, y_{m_k-1}\right) + d_b\left(y_{n_k}, y_{n_k-1}\right) + d_b\left(y_{n_k}, y_{m_k-1}\right)\right)\right).$$

Since Ψ is nondecreasing, we have

$$d_b\left(y_{m_k}, y_{n_k}\right) \leq \lambda\left(d_b\left(y_{m_k}, y_{m_k-1}\right) + d_b\left(y_{n_k}, y_{n_k-1}\right) + d_b\left(y_{n_k}, y_{m_k-1}\right)\right) \ . \tag{30}$$

So, by using (26), (28) and for $k \to +\infty$ in (30) we obtain

$$\lim_{k\to+\infty} d_b\left(y_{m_k}, y_{n_k}\right) < \lambda\varepsilon < \varepsilon \ . \tag{31}$$

Since Ψ is a nondecreasing, from (27) we have

$$\Psi(\varepsilon) \leq \Psi\left(d_b\left(y_{m_k}, y_{n_k}\right)\right) \ . \tag{32}$$

Since $\Psi$is a continuous mapping, so by using (31) and taking $k \to +\infty$ in (32), we obtain

$$\Psi(\varepsilon) \leq \lim_{k\to+\infty} \Psi\left(d_b\left(y_{m_k}, y_{n_k}\right)\right) < \Psi(\varepsilon) \ , \tag{33}$$

which is a contradiction.

Hence, we get $\{y_n\}$ is a Cauchy sequence. Since X is complete, then $d_b(y_n, u^*) \to 0$, as $n \to +\infty$, that is

$$d_b\left(fx_n, u^*\right) \to 0 \text{ and } d_b\left(gx_n, u^*\right) \to 0, \text{ for some } u^* \in fX \subseteq gX \ . \tag{34}$$

Since $gx_n \in gX$ and gX are closed, it implies that $u^* \in gX$. Therefore there exists $x^* \in X$ such that

$$u^* = gx^* \ .$$

Next, we prove that f and g have a coincidence point in X.

$$b\left(x^*, x_n\right) \Psi\left(d_b\left(fx^*, fx_n\right)\right) \leq$$
$$\Psi\left(\frac{\lambda\left(d_b\left(fx^*, gx^*\right) + d_b\left(fx_n, gx_n\right)\right)}{b\left(fx^*, gx^*\right)} + \frac{\lambda d_b\left(fx^*, gx_n\right) d_b\left(fx_n, gx^*\right)}{b\left(fx^*, gx_n\right)\left[1 + d_b\left(fx^*, gx_n\right) + d_b\left(fx_n, gx^*\right)\right]}\right) \leq$$

$$\Psi\left(\frac{\lambda\left(d_b\left(fx^*, gx^*\right) + d_b\left(fx_n, gx_n\right)\right)}{b\left(fx^*, gx^*\right)} + \lambda d_b\left(fx_n, gx^*\right)\right) \ . \tag{35}$$

Thus we get

$$\Psi\left(d_b\left(fx^*, fx_n\right)\right) \leq$$

$$\frac{1}{b(x^*,x_n)}\Psi\left(\frac{\lambda\left(d_b\left(fx^*,gx^*\right)+d_b\left(fx_n,gx_n\right)\right)}{b\left(fx^*,gx^*\right)} + \lambda d_b\left(fx_n, gx^*\right)\right) \leq$$

$$\leq \Psi\left(\frac{\lambda\left(d_b\left(fx^*, gx^*\right) + d_b\left(fx_n, gx_n\right)\right)}{b\left(fx^*, gx^*\right)} + \lambda d_b\left(fx_n, gx^*\right)\right). \tag{36}$$

Thus we have

$$d_b\left(fx^*, fx_n\right) \leq \frac{\lambda\left(d_b\left(fx^*, gx^*\right) + d_b\left(fx_n, gx_n\right)\right)}{b\left(fx^*, gx^*\right)} + \lambda d_b\left(fx_n, gx^*\right).$$

While we have,

$$d_b\left(fx^*, gx^*\right) \leq b\left(fx^*, gx^*\right)\left[d_b\left(fx^*, fx_n\right) + d_b\left(fx_n, gx_n\right) + d_b\left(gx_n, gx^*\right)\right] \leq$$

$$b\left(fx^*, gx^*\right)\left[\frac{\lambda\left(d_b\left(fx^*,gx^*\right)+d_b\left(fx_n,gx_n\right)\right)}{b\left(fx^*,gx^*\right)} + \lambda d_b\left(fx_n, gx^*\right) + d_b\left(y_n, y_{n-1}\right) + d_b\left(gx_n, u^*\right)\right]$$
$$= \lambda\left(d_b\left(fx^*, gx^*\right) + d_b y_n, y_{n-1}\right) + b\left(fx^*, gx^*\right)\left[\lambda d_b\left(fx_n, gx^*\right) + d_b\left(y_n, y_{n-1}\right) + d_b\left(gx_n, u^*\right)\right]$$

Thus for $n \to +\infty$ we get

$$d_b\left(fx^*, gx^*\right) \leq \lambda d_b\left(fx^*, gx^*\right)$$

Since $1-\lambda > 0$, we obtain $d_b\left(fx^*, gx^*\right) = 0$. Hence, we have $fx^* = gx^* = u^*$. Since $\{f, g\}$ is weakly compatible, we have $gfx^* = fgx^* = fu^* = gu^*$,

$$b\left(x^*, u^*\right)\Psi\left(d_b\left(u^*, fu^*\right)\right) = b\left(x, y\right)\Psi\left(d_b\left(fx^*, fu^*\right)\right) \leq$$

$$\Psi\left(\frac{\lambda\left(d_b\left(fx^*,gx^*\right)+d_b\left(fu^*,gu^*\right)\right)}{b\left(fx^*,gx^*\right)} + \frac{\lambda d_b\left(fx^*,gu^*\right)d_b\left(fu^*,gx^*\right)}{b\left(fx^*,gu^*\right)\left[1+d_b\left(fx^*,gu^*\right)+d_b\left(fu^*,gx^*\right)\right]}\right) =$$
$$\Psi\left(\frac{\lambda\left(d_b\left(fx^*,fx^*\right)+d_b\left(fu^*,fu^*\right)\right)}{b\left(fx^*,gx^*\right)} + \lambda d_b\left(fu^*, u^*\right)\right) = \Psi\left(\lambda d_b\left(fu^*, u^*\right)\right).$$

Thus we get

$$\Psi\left(d_b\left(u^*, fu^*\right)\right) \leq \frac{1}{b\left(x^*, u^*\right)}\Psi\left(\lambda d_b\left(fu^*, u^*\right)\right) \leq \Psi\left(\lambda d_b\left(fu^*, u^*\right)\right).$$

Since Ψ is nondecreasing, we have $d_b\left(u^*, fu^*\right) \leq \lambda d_b\left(fu^*, u^*\right)$.

Since $1-\lambda > 0$, then we have $d_b\left(u^*, fu^*\right) = 0$. Hence, $u^* = fu^* = gu^*$, this impies that f, g have a common fixed point.

Next, we have to show uniqueness of common fixed point of f and g.

Suppose that t^* is another common fixed point of f and g, this means $t^* = ft^* = gt^*$. Consider that

$$\Psi(d_b\left(u^*, t^*\right)) = \Psi(d_b\left(fu^*, ft^*\right)) \leq$$
$$\frac{1}{b(u^*,t^*)}\left[\Psi\left(\lambda\left(\frac{d_b\left(fu^*,gu^*\right)+d_b\left(ft^*,gt^*\right)}{b\left(fu^*,gu^*\right)} + \frac{d_b\left(fu^*,gt^*\right)d_b\left(ft^*,gu^*\right)}{d_b\left(fu^*,gt^*\right)\left[1+d_b\left(fu^*,gt^*\right)+d_b\left(ft^*,gu^*\right)\right]}\right)\right)\right] \leq$$
$$\frac{1}{b(u^*,t^*)}\left[\Psi\left(\lambda\left(d_b\left(fu^*, gu^*\right) + d_b\left(ft^*, gt^*\right) + d_b\left(fu^*, gt^*\right)\right)\right)\right] \leq$$
$$\Psi\left(\lambda\left(d_b\left(fu^*, gu^*\right) + d_b\left(ft^*, gt^*\right) + d_b\left(fu^*, gt^*\right)\right)\right) =$$
$$\Psi\left(\lambda\left(d_b\left(u^*, u^*\right) + d_b\left(t^*, t^*\right) + d_b\left(u^*, gt^*\right)\right)\right) = \Psi\left(\lambda d_b\left(u^*, gt^*\right)\right).$$

Since Ψ is nondecreasing, we get

$$d_b\left(u^*, t^*\right) \leq \lambda d_b\left(u^*, t^*\right).$$

Since $1-\lambda > 0$, thus we obtain $d_b\left(u^*, t^*\right) = 0$, it implies that $u^* = t^*$. □

If taking $g(x) = x$, then we have corollary as follows:

Corollary 16. *Let (X, d_b) be a Hausdorff and complete extended rectangular b-metric space. Let $f : X \to X$ be a mapping and satisfy the following conditions:*

$$b(x,y)\, d_b(fx, fy) \leq \lambda \left(\frac{d_b(fx,x) + d_b(fy,y)}{b(fx,x)} + \frac{d_b(fx,y)\, d_b(fy,x)}{b(fx,y)\,[1 + d_b(fx,y) + d_b(fy,x)]} \right), \tag{37}$$

where $0 < \lambda < \frac{1}{2}$. Then f have a unique fixed point in X.

Proof. From (24) by taking $g(x) = x$ and $\Psi(t) = t$ in Theorem 15 this implies that f has a a unique fixed point in X. □

In the last theorem, we utilize Theorem 12 on inequality of Riemann integral, as follows.

Theorem 17. *Let (X, d_b) be a Hausdorff and complete extended rectangular b-metric space and the functions $k, l : [0, +\infty) \to [0, +\infty)$ be Riemann integrable on $[0, +\infty)$ with $\int_0^{\varepsilon} k(s)\, ds > 0$ for every $\varepsilon > 0$.*

If $f : X \to X$ be a self-mapping satisfying the integral inequality condition:

$$\int_0^{d_b(fx,fy)} k(s)ds \leq \frac{1}{b(x,y)} \left(\int_0^{\lambda\left(d_b(fx,x) + d_b(fy,y)\right) + \frac{\gamma d_b(x,y)}{b(x,y)}} k(s)ds - \beta \int_0^{d_b(x,y)} l(s)ds \right),$$

where $\beta \geq 0$, $0 < \lambda, \gamma < 1$, $\frac{\lambda+\gamma}{1-\lambda} < 1$, then f has a unique fixed point.

Proof. Taking $g(x) = x$, $\Psi(t) = \int_0^t k(s)ds$ and $\varphi(s) = \int_0^t l(s)ds$, since $k(s)$ is Riemann integrable on $[0, +\infty)$, we have $\Psi(t)$ is continuous and nondecreasing on $[0, +\infty)$. Then from Theorem 12, we immediately conclude that f has a unique fixed point. □

3. Conclusions

Extended rectangular b-metric space is not a Hausdorff space in general. Therefore, in the results we need a Hausdorff property, so that there exists a unique limit point of sequence as the uniqueness of fixed point. The results of fixed point on weak contraction such as Theorem 12 and Theorem 15 have an application for existence and uniqueness of fixed point of an integral inequality such as in Theorem 17.

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