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On I–Convergent Sequence Spaces Defined By Jordan Totient Operator

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Abstract. In this paper, our aim is to define some new sequence spaces $c_0^I(\Upsilon^r)$, $c^I(\Upsilon^r)$ and $\ell_\infty^I(\Upsilon^r)$ as a domain of triangular Jordan totient matrix and study some of its algebraic and topological properties. Further, we discuss some inclusion relations regarding these said sequence spaces.

1. Introduction

For a given positive integer r, Jordan totient function J_r is an arithmetic function which is a generalization of Euler totient function ϕ defined in [8]. The Jordan totient function is defined by the number of r-tuples of integers $(a_1, a_2, \ldots, a_r)$ satisfying $1 \leq a_i \leq n$, $i = 1, 2, \ldots, r$ and $\gcd(a_1, a_2, \ldots, a_r, n) = 1$. By inclusion–exclusion principle, if the unique prime decomposition of n is $n = p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}$ for $\alpha \geq 1$, then Jordan totient function can be defined as $J_r(n) = n^r \prod_{p|n} \left(1 - \frac{1}{p^r}\right)$. In case $r = 1$, the Jordan totient function reduces to Euler function ϕ. For some properties of Jordan totient function and some of its applications, we refer the reader to [1, 2, 20, 26]. Recall in [18] the regular Jordan totient matrix operator $\Upsilon^r = (\nu_{nk}^r)$ is defined as:

$$\nu_{nk}^r = \begin{cases} \frac{J_r(k)}{n^r}, & \text{if } k|n, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

In the classical summability theory the idea of the generalization of the convergence of sequences of real or complex numbers is to assign a limit of some sort to divergent sequences by considering a matrix transform of a sequence rather than the original sequence. Recently, the Jordan totient matrix Υ^r was used and considered as a compact operator on the space of all absolutely p–summable sequences ℓ_p. Afterwards, Kara et al. [3] introduced some new sequence spaces $\ell_\infty(\Upsilon^r)$, $c(\Upsilon^r)$ and $c_0(\Upsilon^r)$ as the sets of all sequences whose Υ^r–transforms of the sequence $x = (x_k)$ are in the spaces of all bounded ℓ_∞, convergent c and null c_0 sequences, respectively, that is

$$\lambda_{\Upsilon^r} = \left\{ x = (x_k) \in \omega : \left(\frac{1}{n^r} \sum_{k|n} J_r(k) x_k \right) \in \lambda \right\}, \quad \text{for } \lambda = \{\ell_\infty, c, c_0\}.$$

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Many authors have extensively developed the theory of the matrix transformations between some sequence spaces. Interested readers can refer to [10, 19, 21, 22, 28] and references therein.

Let X be a non empty set; then a family I of subsets of X is said to be ideal in X, if it has additive and hereditary properties, i.e., for $A, B \in I$ we have $A \cup B \in I$, and for any subset B of $A \in I$, we have $B \in I$. If $I \neq X$ and it contains all singletons, then I is said to be admissible in X. A Filter on X, denoted by $\mathcal{F}$ is a family of subsets of X satisfying $\emptyset \notin \mathcal{F}$, for any arbitrary subsets $A, B \in \mathcal{F}$ their intersection must be in $\mathcal{F}$ and for $A \in \mathcal{F}$ with $B \supset A$ we have $B \in \mathcal{F}$. For each ideal I there is a filter denoted by $\mathcal{F}(I)$ corresponding to I, known as filter associated with the ideal I defined as $\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}$. These definitions came as introduction to a new type of convergence that is more general than usual convergence and statistical convergence introduced by Fast [4] and Steinhaus [25] independently, this type is known as ideal convergence presented by Kostyrko et al. [16]. Afterward, the notion of ideal convergence or simply I–convergence was moreover explored from the view point of sequence spaces and connected to the theory of summability by Šalát et al. [24], Khan and Nazneen [13], Khan et al. [15, 28], Filipów and Tryba [5], and many others. For further details about the ideal convergence we refer the reader to [11, 12, 14, 27] and its references.

Throughout this paper, c_0^I, c^I and ℓ_∞^I, denote the spaces of all I–null, I–convergent, and I–bounded sequences, respectively. In this paper, we define the sequence $\Upsilon_n^r(x)$ that will be frequently used as Υ^r–transform of the sequence $x = (x_k)$, as follows:

$$\Upsilon_n^r(x) := \frac{1}{n^r} \sum J_r(k) x_k \tag{2}$$

and the inverse (x_n) of the sequence is computed in [18] as:

$$x_n = \sum_{k|n} \frac{\mu(\frac{n}{k})}{J_r(n)} k^r \Upsilon_k^r(x),$$

where μ denotes the Möbius function and it is defined as:

$$\mu(n) = \begin{cases} 0, & \text{if } p^2 | n \text{ for some prime number } p \\ 1, & \text{if } n = 1 \\ (-1)^i, & \text{if } n \text{ is a product of } i \text{ ditnict primes.} \end{cases}$$

Further, by combining the definitions of Jordan totient Matrix operator Υ^r and ideal convergence, we define some new sequence spaces $c_0^I(\Upsilon^r)$, $c^I(\Upsilon^r)$ and $\ell_\infty^I(\Upsilon^r)$ as the sets of all sequences whose Υ^r–transforms are in the spaces c_0^I, c^I and ℓ_∞^I, respectively. In addition, we study some topological and algebraic properties and present some inclusion relations for these sequence spaces.

In what follows, we recall some definitions and lemmas that are needful for this paper.

Definition 1.1. *[25] If B is a subset of $\mathbb{N}$ where $B = \{b \in B : b \leq n\}$, then the natural density of B denoted by $d(B)$ is*

$$d(B) = \lim_{n\to\infty} \frac{1}{n} |B|$$

here $|B|$ is the cardinality of set B.

Definition 1.2. *[4] A sequence $x = (x_k) \in \omega$ is said to be statistically convergent to a number $m \in \mathbb{R}$ if, for every $\epsilon > 0$, $d(\{k \in \mathbb{N} : |x_k - m| \geq \epsilon\}) = 0$, and denoted by $st-\lim x_k = m$. In case $m = 0$ then $x = (x_k) \in \omega$ is called st–null.*

Definition 1.3. *[23] If $x = (x_k)$ is a sequence in ω then x is said to be I–Cauchy if, for every $\epsilon > 0, \exists$ a number $N = N(\epsilon) \in \mathbb{N}$ such that $\{k \in \mathbb{N} : |x_k - x_N| \geq \epsilon\} \in I$.*

Definition 1.4. *[16] If I is an ideal and $x = (x_k)$ is a sequence in ω then x is said to be I–convergent to a number $m \in \mathbb{R}$ if, for every $\epsilon > 0$, we have $\{k \in \mathbb{N} : |x_k - m| \geq \epsilon\} \in I$, and we represent it by $I-\lim x_k = m$. If $m = 0$ then $(x_k) \in \omega$ is called I–null.*

Definition 1.5. *[17] If $x = (x_k)$ is a sequence in ω then x is called I–bounded if there exists $L > 0$ such that $\{k \in \mathbb{N} : |x_k| > L\} \in I$.*

Definition 1.6. *[23] Let $x = (x_k)$ and $y = (y_k)$ be two sequences, then we can say that $x_k = y_k$ for almost all k relative to I (in short a.a.k.r.I) if the set $\{k \in \mathbb{N} : x_k \neq y_k\} \in I$.*

Definition 1.7. *[23] Let S be a sequence space, then S is said to be solid (or normal), if $(\alpha_k x_k) \in S$ whenever (x_k) is a sequence in S and (α_k) is any sequence of scalar in ω with $|\alpha_k| < 1$, and $k \in \mathbb{N}$.*

Definition 1.8. *[23] Let $K = \{k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ and S be a sequence space. A K–step space of S is a sequence space*

$$\lambda_K^S = \{(x_{k_n}) \in \omega : (x_k) \in S\}.$$

A canonical pre–image of a sequence $(x_{k_i}) \in \lambda_K^S$ is a sequence $(y_k) \in \omega$ defined as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre–image of λ_K^S is a set of canonical pre–images of all elements in λ_K^S, i.e., y is in canonical pre–image of λ_K^S iff y is canonical pre–image of some element $x \in \lambda_K^S$.

Definition 1.9. *[23] If a sequence space S contains the canonical pre–images of its step space, then S is known as monotone sequence space.*

Lemma 1.10. *[23] Every solid space is monotone.*

Lemma 1.11. *[24] Let $K \in \mathcal{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

2. Main Results

Throughout this section, we suppose that the sequence $x = (x_k) \in \omega$ and $\Upsilon_n^r(x)$ are connected with the relation (2) and I is an admissible ideal of subset of $\mathbb{N}$. We define:

$$c_0^I(\Upsilon^r) := \{x = (x_k) \in \omega : \{n \in \mathbb{N} : |\Upsilon_n^r(x)| \geq \epsilon\} \in I\}, \tag{3}$$

$$c^I(\Upsilon^r) := \{x = (x_k) \in \omega : \{n \in \mathbb{N} : |\Upsilon_n^r(x) - \ell| \geq \epsilon, \text{ for some } \ell \in \mathbb{R}\} \in I\}, \tag{4}$$

$$\ell_\infty^I(\Upsilon^r) := \{x = (x_k) \in \omega : \exists\, K > 0 \text{ s.t } \{n \in \mathbb{N} : |\Upsilon_n^r(x)| > K\} \in I\}. \tag{5}$$

For convenience of our work we represent

$$m_0^I(\Upsilon^r) := c_0^I(\Upsilon^r) \cap \ell_\infty(\Upsilon^r), \tag{6}$$

and

$$m^I(\Upsilon^r) := c^I(\Upsilon^r) \cap \ell_\infty(\Upsilon^r). \tag{7}$$

Definition 2.1. *Let I be an admissible ideal of subset of $\mathbb{N}$. If for each $\epsilon > 0$ there exists a number $N = N(\epsilon) \in \mathbb{N}$ such that $\left\{n \in \mathbb{N} : |\Upsilon^r_n(x) - \Upsilon^r_N(x)| \geq \epsilon\right\} \in I$ then a sequence $x = (x_k) \in \omega$ is said to be Jordan totient I–Cauchy.*

Example 2.2. *Let us define an ideal I_f as $I_f = \{S \subseteq \mathbb{N} : S \text{ is finite}\}$, then I_f is an admissible ideal in $\mathbb{N}$ and $c^{I_f}(\Upsilon^r) = c(\Upsilon^r)$. Where $c(\Upsilon^r)$ is the space of all convergent sequences derived by Jordan totient matrix operator Υ^r presented in [3].*

Example 2.3. *Let I_d be a non–trivial ideal defined as $I_d = \{B \subseteq \mathbb{N} : d(B) = 0\}$. In this case, $c^{I_d}(\Upsilon^r) = S(\Upsilon^r)$. Here $S(\Upsilon^r)$ is the space of all statistically convergent sequences derived by Jordan totient matrix operator Υ^r we define as follows:*

$$S(\Upsilon^r) := \left\{x = (x_k) \in \omega : d\Big(\{n \in \mathbb{N} : |\Upsilon^r_n(x) - \ell| \geq \epsilon\}\Big) = 0, \text{ for some } \ell \in \mathbb{R}\right\}. \tag{8}$$

Remark 2.4. *Jordan convergent sequence is of course Jordan st–convergent since the natural density of all finite subsets of $\mathbb{N}$ is zero. But, the converse is not true. For example, let $x = (x_k) \in \omega$ be a sequence and*

$$\Upsilon^r_n(x) = \begin{cases} k, & \text{if } k \text{ is square of a natural number } n, \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$\Upsilon^r_n(x) = \{1, 0, 0, 4, 0, 0, 0, 0, 9, 0, \dots\}$$

and let $m = 0$. Then

$$\{n \in \mathbb{N} : |\Upsilon^r_n(x) - m| \geq \epsilon\} \subseteq \{1, 4, 9, 16, \dots, i^2, \dots\}.$$

we have

$$d(\{n \in \mathbb{N} : |\Upsilon^r_n(x) - m| \geq \epsilon\}) = 0.$$

This implies that the sequence $\Upsilon^r_n(x)$ is statistically convergent to zero, but the sequence $\Upsilon^r_n(x)$ is not convergent to $m = 0$.

Theorem 2.5. *The sequence spaces $c^I(\Upsilon^r)$, $c^I_0(\Upsilon^r)$, $\ell^I_\infty(\Upsilon^r)$, $m^I_0(\Upsilon^r)$ and $m^I(\Upsilon^r)$ are linear spaces over $\mathbb{R}$.*

Proof. Suppose that $x = (x_k)$, $y = (y_k)$ be two arbitrary sequences in $c^I(\Upsilon^r)$ and a, b be scalars. Now, since $x, y \in c^I(\Upsilon^r)$, then for any $\epsilon > 0$, there exist $m_1, m_2 \in \mathbb{R}$, such that

$$\left\{n \in \mathbb{N} : \left|\Upsilon^r_n(x) - m_1\right| \geq \frac{\epsilon}{2}\right\} \in I,$$

and

$$\left\{n \in \mathbb{N} : \left|\Upsilon^r_n(y) - m_2\right| \geq \frac{\epsilon}{2}\right\} \in I.$$

Let

$$B_1 = \left\{n \in \mathbb{N} : \left|\Upsilon^r_n(x) - m_1\right| < \frac{\epsilon}{2|a|}\right\} \in \mathcal{F}(I),$$

$$B_2 = \left\{n \in \mathbb{N} : \left|\Upsilon^r_n(y) - m_2\right| < \frac{\epsilon}{2|b|}\right\} \in \mathcal{F}(I),$$

such that $B_1^c, B_2^c \in I$. Then

$$\begin{aligned} B_3 &= \left\{n \in \mathbb{N} : \left|\Upsilon_n^r(ax+by) - (am_1 + bm_2)\right| < \epsilon\right\} \\ &\supseteq \left\{\left\{n \in \mathbb{N} : \left|\Upsilon_n^r(x) - m_1\right| < \frac{\epsilon}{2|a|}\right\} \cap \left\{n \in \mathbb{N} : \left|\Upsilon_n^r(y) - m_2\right| < \frac{\epsilon}{2|b|}\right\}\right\} \end{aligned} \tag{9}$$

Hence, the sets on HRS of equation (9) is related to $\mathcal{F}(I)$. By definition of filter associated with ideal, we can say that the complement of the set on LHS of (9) belongs to I. This gives a result that $(ax+by) \in c^I(\Upsilon^r)$. Hence $c^I(\Upsilon^r)$ is a linear space. The remaining part of the theorem can be prove on the similar manner. □

Theorem 2.6. *The spaces $X(\Upsilon^r)$ are normed spaces with the norm*

$$\|x\|_{X(\Upsilon^r)} = \sup_n |\Upsilon_n^r(x)| \text{ where } X \in \left\{c^I, c_0^I, \ell_\infty^I, \ell_\infty\right\}. \tag{10}$$

Proof of the above theorem is easy and hence omitted.

Theorem 2.7. *A sequence $x = (x_k) \in \omega$ is Jordan totient I–convergent if and only if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$, such that*

$$\left\{n \in \mathbb{N} : \left|\Upsilon_n^r(x) - \Upsilon_N^r(x)\right| < \epsilon\right\} \in \mathcal{F}(I). \tag{11}$$

Proof. Consider that the sequence $x = (x_k) \in \omega$ is Jordan totient I–convergent to some number $m \in \mathbb{R}$, then for any $\epsilon > 0$, we have

$$B_\epsilon = \left\{n \in \mathbb{N} : \left|\Upsilon_n^r(x) - m\right| < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I).$$

Take a number $N = N(\epsilon) \in B_\epsilon$. Then we have

$$\left|\Upsilon_n^r(x) - \Upsilon_N^r(x)\right| \leq \left|\Upsilon_n^r(x) - m\right| + \left|m - \Upsilon_N^r(x)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \in B_\epsilon$. Hence (11) holds.

Conversely, let (11) holds for all $\epsilon > 0$. Then

$$B'_\epsilon = \{n \in \mathbb{N} : \Upsilon_n^r(x) \in J_\epsilon\} \in \mathcal{F}(I), \text{ for all } \epsilon > 0.$$

Where $J_\epsilon = [\Upsilon_n^r(x) - \epsilon, \Upsilon_n^r(x) + \epsilon]$. By taking $\epsilon > 0$, we have $B'_\epsilon \in \mathcal{F}(I)$ and $B'_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. Hence $B'_\epsilon \cap B'_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \emptyset,$$

that is,

$$\{n \in \mathbb{N} : \Upsilon_n^r(x) \in J\} \in \mathcal{F}(I)$$

and thus

$$\text{diam } (J) \leq \frac{1}{2} \text{ diam } (J_\epsilon),$$

here the notation of diam is using for the length of interval. In the same procedure by using induction we obtain a sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that

$$\text{diam}\ (I_n) \leq \frac{1}{2}\ \text{diam}\ (I_{n-1}),\ \text{for}\ n = (2, 3, \dots)$$

and

$$\{n \in \mathbb{N} : \Upsilon^r_n(x) \in I_n\} \in \mathcal{F}(I).$$

Then there exists a number $m \in \bigcap_{n\in\mathbb{N}} I_n$ and it is a pattern process to verify that $m = I\text{–}\lim \Upsilon^r_n(x)$ observing that $x = (x_k) \in \omega$ is Jordan totient I–convergent. Hence the proof. □

Theorem 2.8. *The inclusions $c^I_0(\Upsilon^r) \subset c^I(\Upsilon^r) \subset \ell^I_\infty(\Upsilon^r)$ are strict.*

Proof. Its easy to show that $c^I_0(\Upsilon^r) \subset c^I(\Upsilon^r)$. Consider $x = (x_k) \in \omega$ such that $\Upsilon^r_n(x) = 2$. It is obvious that $\Upsilon^r_n(x) \in c^I$ but $\Upsilon^r_n(x) \notin c^I_0$, this implies, $x \in c^I(\Upsilon^r)\backslash c^I_0(\Upsilon^r)$. Next, let $x = (x_k) \in c^I(\Upsilon^r)$. Then there exists $m \in \mathbb{R}$ such that $I\text{–}\lim \Upsilon^r_n(x) = m$, that is,

$$\left\{n \in \mathbb{N} : \left|\Upsilon^r_n(x) - m\right| \geq \epsilon\right\} \in I.$$

We can write

$$\left|\Upsilon^r_n(x)\right| = \left|\Upsilon^r_n(x) - m + m\right| \leq \left|\Upsilon^r_n(x) - m\right| + |m|\,.$$

Now it is easy to conclude that $(x_k) \in \ell^I_\infty(\Upsilon^r)$. Moreover, we show the strictness of $c^I(\Upsilon^r) \subset \ell^I_\infty(\Upsilon^r)$ by constructing the following example.

Example 2.9. *Let $x = (x_k) \in \omega$ be a sequence such that*

$$\Upsilon^r_n(x) = \begin{cases} \sqrt{n}, & \textit{if } n \textit{ is square} \\ 1, & \textit{if } n \textit{ is odd non–square} \\ 0, & \textit{if } n \textit{ is even non–square.} \end{cases}$$

Then $\Upsilon^r_n(x) \in \ell^I_\infty$, but $\Upsilon^r_n(x) \notin c^I$ hence we concludes that which $x \in \ell^I_\infty(\Upsilon^r_n)\backslash c^I(\Upsilon^r)$.

Thus, we get that $c^I_0(\Upsilon^r) \subset c^I(\Upsilon^r) \subset \ell^I_\infty(\Upsilon^r)$ are strict. □

Remark 2.10. *Jordan bounded sequence is of course Jordan I–bounded as $\emptyset \in I$, I is the ideal. But, the converse need not to be true. For instance, let $x = (x_k) \in \omega$ be a sequence such that*

$$\Upsilon^r_n(x) = \begin{cases} \frac{n^2}{n+1}, & \textit{if } n \textit{ is prime} \\ 0, & \textit{otherwise.} \end{cases}$$

It is clear that the $\Upsilon^r_n(x)$ is not a bounded sequence. But, $\{n \in \mathbb{N} : \left|\Upsilon^r_n(x)\right| > 1\} \in I$. Hence (x_k) is Jordan I–bounded.

Theorem 2.11. *Let $c^I(\Upsilon^r), c^I_0(\Upsilon^r), \ell_\infty(\Upsilon^r)$ are sequence spaces then*

(i) *$c^I(\Upsilon^r)$ and $\ell_\infty(\Upsilon^r)$, overlap but neither one contains the other,*

(ii) *$c^I_0(\Upsilon^r)$ and $\ell_\infty(\Upsilon^r)$, overlap but neither one contains the other.*

Proof. **(i)** We start the proof by showing that $c^I(\Upsilon^r)$ and $\ell_\infty(\Upsilon^r)$ are not disjoint. Suppose $x = (x_k) \in \omega$ be a sequence such that $\Upsilon^r_n(x) = \frac{1}{n}$ for $n \in \mathbb{N}$. Then $x \in c^I(\Upsilon^r)$ but $x \in \ell_\infty(\Upsilon^r)$. Now, define the sequence $x = (x_k) \in \omega$ with

$$\Upsilon^r_n(x) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is square} \\ 0, & \text{otherwise.} \end{cases}$$

Thus $x \in c^I(\Upsilon^r)$ but $x \notin \ell_\infty(\Upsilon^r)$. Next, choose the sequence $x = (x_k) \in \omega$ such that

$$\Upsilon^r_n(x) = \begin{cases} n, & \text{if is even} \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \in \ell_\infty(\Upsilon^r)$ but $x \notin c^I(\Upsilon^r)$.

(ii) The proof of second part, of theorem is similar with part (i) and hence omitted.
□

Theorem 2.12. *The sets $m^I(\Upsilon^r)$ and $m^I_0(\Upsilon^r)$ are closed subspaces of $\ell_\infty(\Upsilon^r)$.*

Proof. Choose the Cauchy sequence $(x_i^{(j)})$ in $m^I(\Upsilon^r) \subset \ell_\infty(\Upsilon^r)$. Then $(x_i^{(j)})$ convergent in $\ell_\infty(\Upsilon^r)$ and we have $\lim_{j\to\infty} \Upsilon^{rj}_n(x) = \Upsilon^r_n(x)$. Now, let $I\text{–}\lim \Upsilon^{rj}_n(x) = \ell_j$ for each $j \in \mathbb{N}$. Then we must show that

(i) (ℓ_j) is convergent to a number say ℓ.

(ii) $I\text{–}\lim \Upsilon^r_n(x) = \ell$.

(i) We know that $(x_i^{(j)})$ is a Cauchy sequence, then for any $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\left|\Upsilon^{rj}_n(x) - \Upsilon^{rk}_n(x)\right| < \frac{\epsilon}{3}, \text{ for all } j,k \geq n_0. \tag{12}$$

Now let B_j and B_k be tow sets in I as follow:

$$B_j = \left\{n \in \mathbb{N} : |\Upsilon^{rj}_n(x) - \ell_j| \geq \frac{\epsilon}{3}\right\} \tag{13}$$

and

$$B_k = \left\{n \in \mathbb{N} : |\Upsilon^{rk}_n(x) - \ell_k| \geq \frac{\epsilon}{3}\right\}. \tag{14}$$

Now suppose that $j,k \geq n_0$ and $n \notin B_j \cap B_k$. Then by using (12), (13) and (14) we have

$$|\ell_j - \ell_k| \leq |\Upsilon^{rj}_n(x) - \ell_j| + |\Upsilon^{rk}_n(x) - \ell_k| + |\Upsilon^{rj}_n(x) - \Upsilon^{rk}_n(x)| < \epsilon.$$

Thus (ℓ_j) is a Cauchy sequence in $\mathbb{R}$ and thus convergent to ℓ, hence, $\lim_{j\to\infty} \ell_j = \ell$.

(ii) Let $\delta > 0$ be a given number, then we can chose m_0 as

$$|\ell_j - \ell| < \frac{\delta}{3}, \text{ for each } j > m_0. \tag{15}$$

Since $(x_i^{(j)}) \to x_i$ as $j \to \infty$. Thus

$$|\Upsilon^{rj}_n(x) - \Upsilon^r_n(x)| < \frac{\delta}{3}, \text{ for each } j > m_0. \tag{16}$$

Since (Υ^{rk}_n) is I–convergent to ℓ_k, there exists $A \in I$ such that for each $n \notin A$, we have

$$|\Upsilon^{rk}_n(x) - \ell_k| < \frac{\delta}{3}. \tag{17}$$

Without loss of generality, let $k > m_0$, then with aid of (15), (16) and (17) for all $n \notin A$, we have

$$|\Upsilon^r_n(x) - \ell| \leq |\Upsilon^r_n(x) - \Upsilon^{rk}_n(x)| + |\Upsilon^{rk}_n(x) - \ell_k| + |\ell_k - \ell| < \delta.$$

This implies that (x_i) is Jordan totient I–convergent to ℓ. Thus $m^I(\Upsilon^r)$ is a closed subspace of $\ell_\infty(\Upsilon^r)$. Similarly the second result can be obtained.

□

Theorem 2.13. *Let $c^I(\Upsilon^r), c_0^I(\Upsilon^r)$, and $\ell_\infty^I(\Upsilon^r)$ are sequence spaces with norm given by* (10). *Then $c^I(\Upsilon^r), c_0^I(\Upsilon^r)$, and $\ell_\infty^I(\Upsilon^r)$ are BK–spaces.*

Proof. It is obvious that the sequence spaces c^I, c_0^I, and ℓ_∞^I are BK–spaces with their sup–norm. Also we know that the Jordan matrix is a triangular matrix. Now by considering these two facts and Wilansky's theorem [29], we can conclude that the stated sequence spaces are BK–spaces. Hence the proof. □

Since $m^I(\Upsilon^r) \subset \ell_\infty(\Upsilon^r)$ and $m_0^I(\Upsilon^r) \subset \ell_\infty(\Upsilon^r)$ are strict spaces, then by using the Theorem 2.12, we obtained the following result.

Theorem 2.14. *The space $m^I(\Upsilon^r)$ and $m_0^I(\Upsilon^r)$ are no where dense subset of $\ell_\infty(\Upsilon^r)$.*

Theorem 2.15. *The spaces $c_0^I(\Upsilon^r)$ and $m_0^I(\Upsilon^r)$ are solid and monotone.*

Proof. Let $x = (x_k) \in c_0^I(\Upsilon^r)$, then for $\epsilon > 0$, the set

$$\left\{n \in \mathbb{N} : \left|\Upsilon_n^r(x)\right| \geq \epsilon\right\} \tag{18}$$

belongs to I. Let $\alpha = (\alpha_k)$ be a sequence of scalars with $|\alpha| \leq 1$ for all $k \in \mathbb{N}$. Then,

$$\left|\Upsilon_n^r(\alpha x)\right| = \left|\alpha \Upsilon_n^r(x)\right| \leq |\alpha| \left|\Upsilon_n^r(x)\right| \leq \left|\Upsilon_n^r(x)\right|, \text{ for all } n \in \mathbb{N}.$$

with aid of this inequality and from (18) we have

$$\{n \in \mathbb{N} : |\Upsilon_n^r(\alpha x)| \geq \epsilon\} \subseteq \{n \in \mathbb{N} : |\Upsilon_n^r(x)| \geq \epsilon\} \in I.$$

This implies that

$$\{n \in \mathbb{N} : |\Upsilon_n^r(\alpha x)| \geq \epsilon\} \in I.$$

Thus, $(\alpha x_k) \in c_0^I(\Upsilon^r)$, and hence $c_0^I(\Upsilon^r)$ is solid. Finally in view of Lemma 1.10 we obtained that $c_0^I(\Upsilon^r)$ is monotone. □

The proof for $m_0^I(\Upsilon^r)$ has same procedure and hence omitted.

Example 2.16. *If I is neither maximal nor $I = I_f$, then show by an example that the spaces $c^I(\Upsilon^r)$ and $m^I(\Upsilon^r)$ are neither monotone nor solid.*
Consider $I = I_f$ and $K = \{n \in \mathbb{N} : n$ is odd $\}$, define the K–step space E_K of E as follow:

$$E_K = \{(x_k) \in \omega : (x_k) \in E\}.$$

Now define the sequence $(z_k) \in E_K$ such that

$$\Upsilon_n^r(z) = \begin{cases} \Upsilon_n^r(x), & \text{if } n \in K \\ 0, & \text{otherwise.} \end{cases}$$

Now chose (x_k) such that $\Upsilon_n^r(x) = 3$, for all $n \in \mathbb{N}$. Then $(x_k) \in E(\Upsilon^r)$, but its K–step space pre–image is not in $E(\Upsilon^r)$, where $E = c^I$ and m^I. This implies that $E(\Upsilon^r)$ are not monotone, and hence by Lemma 1.10 the spaces $E(\Upsilon^r)$ are not solid.

Theorem 2.17. *Suppose $x = (x_k) \in \omega$ and let I be a non–trivial admissible ideal in $\mathbb{N}$. If $z = (z_k) \in c^I(\Upsilon^r)$ is a sequence, such that $\Upsilon_n^r(x) = \Upsilon_n^r(z)$ for a.a.n.r.I, then $x \in c^I(\Upsilon^r)$.*

Proof. Consider that $\Upsilon_n^r(x) = \Upsilon_n^r(z)$ for $a.a.n.r.I$, that is,

$$\{n \in \mathbb{N} : \Upsilon_n^r(x) \neq \Upsilon_n^r(z)\} \in I.$$

And let (z_k) be a sequence which is Jordan totient I–convergent to ℓ. Then for any $\epsilon > 0$, we have

$$\{n \in \mathbb{N} : |\Upsilon_n^r(z) - \ell| \geq \epsilon\} \in I.$$

Since I is an admissible ideal, then the result follows from the following inclusion

$$\{n \in \mathbb{N} : |\Upsilon_n^r(x) - \ell| \geq \epsilon\} \subseteq \{\{n \in \mathbb{N} : \Upsilon_n^r(x) \neq \Upsilon_n^r(z)\} \cup \{n \in \mathbb{N} : |\Upsilon_n^r(z) - \ell| \geq \epsilon\}\}.$$

□

Conclusion

In this research work we introduced and studied some new sequence spaces, $S(\Upsilon^r)$, $c_0^I(\Upsilon^r)$, $c^I(\Upsilon^r)$ and $\ell_\infty^I(\Upsilon^r)$, that is, we presented some new I–convergent sequence spaces derived by triangular Jordan totient matrix operator Υ^r. Also we explore some topological and algebraic properties for these spaces. Moreover we investigate some inclusion relations related to these sequence spaces. The results obtained in this paper yields novel tools to arrange and solve some problem of sequence convergence in numerous field of science and engineering.

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